Obtaining and using Kato inequalities for (convection-)diffusion problems

Boris Andreianov
University of Tours, France

based upon joint works with
Mohamed Maliki (Mohammedia), Noureddine Igbida (Limoges),
Nathaël Alibaud and Matthieu Brassart (Besançon),
Adama Ouédraogo (Bobo-Dioulasso)

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Kato inequalities.

Formal uniqueness argument
for scalar convection-diffusion PDEs.

Classical Kato inequalities

Context: estimating $|u - \hat{u}|$ for two solutions of a PDE like

$$\partial_t u + \mathrm{div} F(u) + (-\Delta)^s \phi(u) = 0,$$

with ϕ continuous, non-decreasing (possibly degenerate on intervals).
Examples: Burgers equation, GPME/FDE, their non-local analogues

Central tool: the Kato inequality: [T. Kato '72]

$$\Delta|W| \geq \mathrm{sign}(W)\,\Delta W \text{ in } \mathcal{D}'(\Omega)$$

$$\text{if } W \in L^1_{loc}(\Omega) \text{ and } \Delta W \in L^1_{loc}(\Omega).$$

Generalization [Brézis '84]

$$\Delta S(W) \geq S'(W)\,\Delta W \text{ in } \mathcal{D}'(\Omega)$$

for S Lipschitz with non-decreasing, piecewise continuous S'.

Idea of the Kato argument in the case $W \in H^1_{loc}(\Omega)$:

$\Delta S(W) = \mathrm{div}\big(S'(W)\nabla W\big)$ equals (formally) $S'(W)\Delta W + S''(W)|\nabla W|^2$,

the latter term is ≥ 0.

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The formal "Kato-based" uniqueness argument, and missing details

A local example: consider $\partial_t u + \mathrm{div} F(u) = \Delta u^+$, for $(t,x) \in \mathbb{R}_+ \times \Omega$, with either $\Omega =$ the whole space or Ω a bounded domain, with BCs.

- The associated **localized L^1 contraction ("Kato") ineq.** reads :

$$\forall \xi \in \mathcal{D}([0,T) \times \Omega)$$

$$-\int_0^T\int_\Omega |u-\hat{u}|\partial_t\xi - \int_0^T\int_\Omega \mathrm{sign}(u-\hat{u})\big(F(u)-F(\hat{u})\big)\cdot\nabla\xi \le \int_0^T\int_\Omega |u^+ - \hat{u}^+|\Delta\xi + \int_\Omega |u_0-\hat{u}_0|\xi(0,.)$$

- L^1 contraction follows if $\xi(t,x) \equiv \mathbf{1}_{[0,T)}(t)$ can be taken hereabove

Two difficilties addressed in the talk:

- **Justifying such extended "Kato inequalities"**, (formal: plug $\mathrm{sign}(u-\hat{u})$ as test function, use chain rules & Kato)
- **Exploiting "Kato ineq." via appropriate sequences** $(\xi_n)_n$, $\xi_n \to 1$ (while controlling the contributions of $\nabla\xi_n$, $\Delta\xi_n$)

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A brief overview of the talk, with highlights

1. Kato inequalities for the Laplacian, generalizations. Formal uniqueness argument and missing details
2. Entropy inequalities and doubling of variables
3. Parabolic dissipation in entropy inequalities (focus on non-local diffusion case, link to kinetic formulation)
4. Up-to-the-boundary Kato inequalities
5. Conservation laws in the whole space: a complex picture, counterexamples to uniqueness
6. Convection-diffusion case: dual problems and weighted estimates
7. Uniqueness of L^∞ solutions of stationary PM/FD equations

Entropy inequalities.

Kruzhkov doubling of variables.

Getting "Kato ineqality" via the Kruzhkov-Carrillo approach

Long way to "Kato ineq.": [Kruzhkov '70], [Carrillo '99]

- Select a set of obvious solutions, $\hat{u}(t,x) \equiv k = const$ for $k \in \mathbb{R}$
- Postulate, via the definition of "entropy solution", that u fulfills the localized L^1 contraction ("Kato") ineq. w.r.t. all such $\hat{u}$
- Deduce, via the doubling of variables hint, that "Kato ineq." holds for any couple $u, \hat{u}$ of entropy solutions

$\rightsquigarrow$ remarkable success for pure hyperbolic case $\partial_t u + \mathrm{div} F(u) = 0$

NB: "Kato" easily exploited due to finite speed of propagation:

$$\int_{-R-LT}^{R+LT} |u - \hat{u}|(T,.) \leq \int_{-R}^{R} |u_0 - \hat{u}_0|, \quad \text{with } L := Lip(F)$$

... but, what if F is non-Lipschitz ?

Difficulties in presence of (local or non-local) duffision:

- issues with literature, failure of straightforward variable doubling
- key observation: "parabolic dissipation" should enter entropy ineq.
- the resulting Kato inequalities more difficult to exploit than in the hyperbolic case, due to the infinite speed of propagation

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Keeping parabolic dissipation.

Kinetic formalism,
local and non-local diffusion cases.

The failure of straightforward Kruzhkov-like approach

Straightforward entropy inequality:

$$\partial_t |u-k| + \operatorname{div} \operatorname{sign}(u-\hat{u})\big(F(u)-F(\hat{u})\big) \leq \Delta|\phi(u)-\phi(k)|$$

(use Kato on ΔW, $W=\phi(u)-\phi(k)$ in addition to Kruzhkov tricks)

Difficulty:

While doubling variables (take $k=\hat{u}(s,y)$), there arise "cross-terms"

$$2\,\nabla_x\phi(u(t,x))\,\nabla_y\phi(\hat{u}(s,y))\,\delta_{u(t,x)=\hat{u}(s,y)}$$

(formal expression). These are uncontrolled.

Precised entropy inequalities: [Carrillo '99]

Keep track of the remainder in classical Kato inequality for ΔW:

$$\liminf_{\alpha\to 0} \frac{1}{\alpha} \mathbf{1}_{|u(x)-k|<\alpha} |\nabla\phi(u)|^2.$$

Then the control of the previously uncontrolled terms boils down to

$$2AB \leq A^2+B^2, \qquad A=\nabla_x\phi(u(t,x)), \quad B=\nabla_y\phi(\hat{u}(s,y))$$

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Keeping track of the parabolic dissipation

Carrillo's way of keeping parabolic dissipation:
Use Kruzhkov (or semi-Kruzhkov/Serre) singular entropies, keep

$$\liminf_{\alpha\to 0} \frac{1}{\alpha} \mathbf{1}_{|u(x)-k|<\alpha} |\nabla \phi(u)|^2.$$

Bendahmane-Karlsen's way of keeping parabolic dissipation:
Use smooth but general convex entropies η, just "keep everything"

$$\eta''(u(t,x))\phi'(u)|\nabla u|^2.$$

More tricky variables' doubling: [Bendahmane, Karlsen '05]

Alibaud's way of keeping dissipation for fractional diffusion:
Cut Levi-Khintchine representation formula into regular/singular parts:

$$(-\Delta)^s w = \ v.p. \int_{\mathbb{R}^N} \frac{w(x+z)-w(x)}{|z|^{N+2s}}\, dz = \int_{|z|>r} \ldots + v.p. \int_{|z|<r} \ldots.$$

keep $\mathrm{sign}(u-k)(-\Delta)^s_{>r} w$ in regular part; use (fractional) Kato to make appear $(-\Delta)^s_{<r}|w-k|$. Tricky variables' doubling: [Alibaud '07]

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Dissipation and kinetic formulation, the basics

A sharp way of keeping dissipation for fractional diffusion:

Carefully write $\mathrm{sign}(u-k)(-\Delta)^s\phi(u)$ (singularity is not a problem).

$\rightsquigarrow$ bypass cutting + variables doubling, if used with kinetic formulation

Kinetic formulation in a nutshell:

Given a function $u(t,x)$, one introduces the auxiliary quantity

$$\chi(t,x;\xi) = \chi(\xi, u(t,x)) = \begin{cases} 1, & 0 < \xi < u \\ -1, & u < \xi < 0 \\ 0, & \text{otherwise} \end{cases}$$

Key property: for $\eta(.) \in Lip$, there holds $\eta(u) = \int_{\mathbb{R}} \eta'(\xi)\chi(\xi,u)\,d\xi$.

Kinetic formulation for scalar conservation law $u_t + \mathrm{div}\, f(u) = 0$:

$$\partial_t \chi(\xi,u) + f'(\xi)\cdot\nabla_x\chi(\xi,u) = \partial_\xi m$$

where $m = m(t,x;\xi)$ is some finite nonnegative measure responsible for the dissipation of entropy (we need not know m).

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The local degenerate parabolic case

Extension to local degenerate convection-diffusion equation:

$$u_t + \operatorname{div} f(u) - \Delta\phi(u) = 0$$

(and even to anisotropic diffusion case: [Chen, Perthame '03]).

The kinetic formulation takes the form

$$\partial_t \chi(\xi, u) + f'(\xi) \cdot \nabla_x \chi(\xi, u) - \phi'(\xi)\Delta[\chi(\xi, u)] \;=\; \partial_\xi (\, m + n\,)$$

where m, n are finite nonnegative measures.

Moreover, the parabolic dissipation measure n is explicitly given by

$$n(.;\xi) \;:=\; \phi'(\xi)|\nabla u(.)|^2 \,\delta_0(u(.) - \xi) \;\; \text{(formal)}$$

Reflects both Carrillo's and the Bendahmane-Karlsen's approaches.

Outcome: Full well-posedness for $u_t + \operatorname{div} f(u) - \Delta\phi(u) = 0$, L^1 data.
Focus: kinetic formulation techniques [Perthame '02] $\rightsquigarrow$ "Kato ineq" !

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Kinetic dissipation measure of fractional Laplacian, case $\phi = \mathrm{Id}$

Kinetic formulation with $(-\Delta)^s$ **diffusion**: [Alibaud, A., Ouédraogo '20]
Starting from [Karlsen, Ulusoy '11], for smooth entropies one has

$$\int_{\mathbb{R}^N} \eta'(u(t,x))\,(u(t,x+z)-u(t,x))\,\frac{const}{|z|^{N+s}}\,dz.$$

NB: Elementary Taylor's identity

$$\forall a,b \;\; \eta'(a)(b-a) = \eta(b) - \eta(a) - \int_{\mathbb{R}} \eta''(\xi)|b-\xi|\mathbf{1}_{\mathrm{conv}\{a,b\}}(\xi)\,d\xi.$$

With singular (Kruzhkov) entropies ($\eta''(\xi) = 2\delta_0(\xi - k)$), we guess the dissipation measure suitable for the fractional Laplacian:

$$n_s(t,x,\xi) := \int_{\mathbb{R}^N} |u(t,x+z)-\xi|\mathbf{1}_{\mathrm{conv}\{u(t,x),u(t,x+z)\}}(\xi)\,\frac{const}{|z|^{N+2s}}\,dz.$$

NB The formula makes sense, rigorously, unlike in the local case.

The kinetic formulation with fractional laplacian takes the form

$$\partial_t\chi(\xi,u) + f'(\xi)\cdot\nabla_x\chi(\xi,u) + (-\Delta)^s[\chi(\xi,u)] = \partial_\xi(m+n_s)$$

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Controlling cross-terms with fractional dissipation

From tedious case-by-case observation, we have:

Lemma

There holds

$$\forall a,b,c,d \in \mathbb{R} \qquad F(a,b,c,d) \leq G(a,b,c,d),$$

$$F(a,b,c,d) := sign(a-b)sign(c-d)\int_{\mathbb{R}} \mathbf{1}_{conv\{a,b\}}(\xi)\mathbf{1}_{conv\{c,d\}}(\xi)\,d\xi$$

$$G(a,b,c,d) := \int_{\mathbb{R}} \Big(|b-\xi|\delta(\xi-c)\mathbf{1}_{conv\{a,b\}}(\xi) + |d-\xi|\delta(\xi-a)\mathbf{1}_{conv\{c,d\}}(\xi)\Big)\,d\xi.$$

Here, F represents cross-terms (like $2AB$ in Carrillo's local case), while G represents the fractional dissipation terms made explicit in the kinetic formulation (like A^2+B^2 for the local case).

$\rightsquigarrow$ "Kato inequality" recovered from this fractional kinetic formulation

$\rightsquigarrow$ we're half-way to uniqueness ?

Up-to-the-boundary Kato inequalities.

Getting Kato up to the boundary ($\Rightarrow$ uniqueness)

A set of approaches for bounded domain, various BC's:

• Get / use up-to-the-boundary entropy inequalities and doubling.
[Carrillo '99], special case with zero Dirichlet BC, half-entropies
[Otto '96], [Vovelle '02], based on "weak traces" which always exist

• Use local "Kato ineq.", then let $\xi_n \to 1$, $\nabla \xi_n \to -\delta|_{\partial\Omega}\nu$ generating
a sign-definite boundary term $\mathrm{sign}(u - \hat{u})(F(u) - F(\hat{u})) \cdot \nu$
due to existence of strong traces of the normal flux $F(u) \cdot \nu$.
Ok for the hyperbolic case: [Bardos, LeRoux, N'edélec '78] with BV,
[Vasseur '01], [Burger, Karlsen, Frid '09], [A., Sbihi '15] beyond BV
NOT Ok for parabolic case, strong traces of $\nabla\phi(u) \cdot \nu$ need not exist

• Approaches mixing strong and weak traces.

· Strong trace for $F(u)$, weak trace for $\nabla w \cdot \nu$, tricky ξ_n [A., Igbida '07]

· "weak-strong uniqueness" approach:
if there is a dense set of "trace-regular solutions",
use comparison of a trace-regular and a general solution.
Ok for some parabolic problems. [A., Bouhsiss '04], [A., Gazibo '13].

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• Use local "Kato ineq.", then let $\xi_n \to 1$, $\nabla\xi_n \to -\delta|_{\partial\Omega}\nu$ generating a sign-definite boundary term $\mathrm{sign}(u-\hat{u})\big(F(u)-F(\hat{u})\big)\cdot\nu$ due to existence of strong traces of the normal flux $F(u)\cdot\nu$.
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if there is a dense set of "trace-regular solutions",
use comparison of a trace-regular and a general solution.
Ok for some parabolic problems. [A., Bouhsiss '04], [A., Gazibo '13].

Getting Kato up to the boundary ($\Rightarrow$ uniqueness)

A set of approaches for bounded domain, various BC's:

• Get / use up-to-the-boundary entropy inequalities and doubling.
[Carrillo '99], special case with zero Dirichlet BC, half-entropies
[Otto '96], [Vovelle '02], based on "weak traces" which always exist

• Use local "Kato ineq.", then let $\xi_n \to 1$, $\nabla \xi_n \to -\delta|_{\partial\Omega}\nu$ generating
a sign-definite boundary term $\mathrm{sign}(u-\hat{u})\big(F(u)-F(\hat{u})\big)\cdot\nu$
due to existence of strong traces of the normal flux $F(u)\cdot\nu$.
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An example of tricky test functions

From [A., Igbida '07], for the case $F(u) = \tilde{F}(\phi(u))$:
test functions ξ_n with $\nabla \xi_n$ supported in $\frac{1}{n}$-neighbourhood Ω_n of $\partial\Omega$.

Explicit test functions:

Just take $\xi_n^o(x) := n \min\{\frac{1}{n}, \mathrm{dist}(x, \partial\Omega)\}$.
Requires regularity of $\partial\Omega$...

Auxiliary PDE for test functions:

Take ξ_n^o for prescribing BCs on $\partial\Omega_n$, solve $-\Delta \xi_n = 0$ in Ω_n.
$\rightsquigarrow$ quite irregular domains (even cracks) can be covered.

A general trend:

"competition" between explicit choices of simple test functions
and test functions obtained by solving some auxiliary PDE problems
(like Holmgren's method, with solving a "dual equation").
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Conservation laws in the whole space:

a complex picture,
Panov's non-uniqueness example

Hyperbolic case, the whole space...

$$\partial_t u + \operatorname{div} F(u) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^N$$

"Kato inequality" is proved by Kruzhkov for L^∞_{loc} solutions.
If F is Lipschitz, uniqueness (even a localized one) follows.

In the whole space, with non-Lipschitz flux F... uniqueness ?

- [Bénilan '72] uniqueness if F is $(1-\frac{1}{N})$ (locally) Hölder. Uniqueness for $N=1$ for merely continuous F.
- [Panov '91],[Kruzhkov, Panov '94], non-uniqueness example in L^∞
- [Kruzhkov, Panov '94],[Bénilan, Kruzhkov '96] anisotropic conditions on "cumulative" Hölder continuity of flux components

$$F_i \in C^{\alpha_i}_{loc}, \qquad \alpha_1 + \cdots + \alpha_N \geq N-1.$$

Techniques: explicit test functions, use of moduli of continuity.
Link to Panov's counterexample ($N=2$, $\alpha_1+\alpha_2 < 1 = 2-1$).

- [Bénilan, Kruzhkov '96], [A., Bénilan, Kruzhkov '00]

an unusual sufficient condition for uniqueness, a tricky proof:

$$L^1 \text{ data and solution}, \quad (N-1) \text{ flux components are monotone.}$$

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Convection-diffusion in the whole space.

Dual problems and weighted estimates

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“Kato inequality” proved in local and non-local cases.

“Cumulative Hölder” assumption, local diffusion:

[Maliki, Touré '03] With explicit test functions, uniqueness in L^∞ for

$$F \in C_{loc}^{0,1-\frac{1}{N}} \text{ (or the anisotropic condition) and } \phi \in C_{loc}^{0,1-\frac{2}{N}}$$

Key properties: $|\nabla \xi_n| \leq C|\xi_n|$, $|\Delta \xi_n| \leq C|\xi_n|$.

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Removing the Hölder restriction on ϕ:

[A., Maliki '10] With $\xi_n \to 1$ obtained from truncated fundamental solution of $(-\Delta)$, uniqueness in L^∞ with $F \in C_{loc}^{0,1-\frac{1}{N}}$ (isotropic)

Key techniques: moduli of continuity, weighted integrals, Jensen ineq.

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[A., Brassart '20] Analogous results hold for fractional diffusion.

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Refinement in the case of (locally) Lipschitz F, ϕ

• [Alibaud '07] initiated the analysis of the fractional case, using "finite-infinite speed of propagation" hint.
Analogue of Kruzhkov localized estimate accounting for diffusion

$$\int_{-R}^{R} |u-\hat{u}|(T,.) \le \int_{-R-LT}^{R+LT} |u_0-\hat{u}_0|(.) \star K(T,.)$$

• [Endal, Jakobsen '14],[Alibaud, Endal, Jakobsen'19] obtained weighted estimates via a systematic duality approach:

construct ξ_n solving a "dual equation" of Hamilton-Jacobi kind
with, e.g., $\xi(T,.) = \xi_T(.)$, e.g., $= \mathbf{1}_{[-R,R]}$

Outcome: time-dependent weighted propagation estimates

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Uniqueness of L^{∞} solutions of stationary PM/FD equations

Three arguments for uniqueness of L^∞ solutions

Byproduct of one of the above results: [A., Maliki '21], for

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How Kato ineq. imply uniqueness of L^∞ solutions?
We bring three different answers, all covering the desired L^∞ setting

- Keller-Osserman technique ([Brézis '84],[Gallouët, Morel '87]) with some refinements, uniqueness in $L^1_{loc}(\mathbb{R}^N)$
- Weigted $L^1(\mathbb{R}^N, \rho(.))$ setting with exponentially decaying weights $\rho(x) = exp(-C|x|)$, C depend on ϕ; use of Kato with $S(.) \neq |.|$
- Weigted $L^1(\mathbb{R}^N, \rho(.))$ setting with ρ superharmonic, typically $\rho(x) = \frac{1}{\max\{R,|x|\}^{N-2}}$, results close to [Bénilan, Crandall '81], extendable to weak solutions of FDE/PME evolution problem

Common techniques:
extensive use of modulus of continuity ω of ϕ and its inverse Ω, Fenchel-Legendre transform Ω^*; Jensen inequality

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