Singular solutions for fractional parabolic boundary value problems

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Overview

- General class of operators L
- Large L-harmonic functions
- Spectral decompositions
- Semigroup theories
- Representation
- Time-fractional problem

The Fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$

- Fourier transform $\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\hat{u}(\xi)$
- Singular integral $(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x)-u(x+y)}{|y|^{n+2s}}\, dy.$
- Spectral $(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta}u(x) - u(x)\right) \frac{dt}{t^{1+s}}$
- Dirichlet-to-Neumann map (Caffarelli–Silvestre, CPDE 2007)

$$(-\Delta)^s u(x) = -\lim_{t\to 0^+} t^{1-2s} U_t(x,t),$$

where

$$\begin{cases} \mathrm{div}\,(t^{1-2s}\nabla U(x,t)) = 0, & (x,t) \in \mathbb{R}^n \times (0,+\infty), \\ U(x,0) = u(x), & x \in \mathbb{R}^n. \end{cases}$$

Some Fractional Laplacians in Ω

- E.g. RFL "restricted", SFL "spectral", CFL "censored", ...
- $\mathrm{L} \sim (-\Delta)^s_{*\mathrm{FL}}$ in $\Omega \Subset \mathbb{R}^n$
- Typically $\mathrm{L}u(x) = \int_\Omega (u(x) - u(y))\mathcal{J}(x,y)\,dy + \kappa(x)u(x)$
- (H_1) L^{-1} has (symmetric) Green's kernel (for $x \neq y$ in Ω)

$$\mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{n-2s}}\left(1 \wedge \frac{\delta(x)^\gamma}{|x-y|^\gamma}\right)\left(1 \wedge \frac{\delta(y)^\gamma}{|x-y|^\gamma}\right),$$

  where $\delta(x) := \mathrm{dist}(x, \Omega^c)$
- Interior regularity: $2s \in (0,2)$
- Boundary regularity: $\gamma \in (0,1]$
- Boundary blow-up possible if: $\gamma - (2s-1) > 0$
- Better regularity if $\gamma < 2s$

RFL: Restricted Fractional Laplacian

$$\mathrm{L}u(x) = \int_\Omega (u(x) - u(y))\mathcal{J}(x,y)\,dy + \kappa(x)u(x)$$

$$(\mathrm{H}_1) \quad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma}{|x-y|^\gamma}\right)\left(1 \wedge \frac{\delta(y)^\gamma}{|x-y|^\gamma}\right)$$

- Restriction: $u|_{\Omega^c} = 0$
- $(-\Delta)^s_{\mathrm{RFL}} u(x) := \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy$

$$\Longrightarrow \mathcal{J}(x,y) = \frac{1}{|x-y|^{n+2s}}, \quad \kappa(x) = \int_{\Omega^c} \frac{dy}{|x-y|^{n+2s}} \asymp \delta(x)^{-2s}$$

- $(-\Delta)^s_{\mathrm{RFL}} x^s_+ = 0$ on $\mathbb{R}_+ \Longrightarrow \gamma = s$
- $\gamma - (2s-1) = 1 - s > 0$ for any $s \in (0,1)$

SFL: Spectral Fractional Laplacian

$$\mathrm{L}u(x) = \int_\Omega (u(x) - u(y)) \mathcal{J}(x,y)\, dy + \kappa(x) u(x)$$

$$(\mathrm{H}_1) \quad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma}{|x-y|^\gamma}\right) \left(1 \wedge \frac{\delta(y)^\gamma}{|x-y|^\gamma}\right)$$

- Spectrally: $(-\Delta)^s_{\mathrm{SFL}} \varphi_j := \mu_j^s \varphi_j \quad$ for $\begin{cases} -\Delta \varphi_j = \mu_j \varphi_j & \text{in } \Omega \\ \varphi_j = 0 & \text{on } \partial\Omega. \end{cases}$
- In terms of $\mathcal{K} =$ (heat kernel of $-\Delta$ in Ω),

$$\mathcal{J}(x,y) = \frac{1}{|\Gamma(-s)|} \int_0^\infty \mathcal{K}(t,x,y) \frac{dt}{t^{1+s}} \asymp \frac{1}{|x-y|^{n+2s}} \left(1 \wedge \frac{\delta(x)\delta(y)}{|x-y|^2}\right)$$

$$\kappa(x) = \frac{1}{|\Gamma(-s)|} \int_0^\infty \left(1 - \int_\Omega \mathcal{K}(t,x,y)\, dy\right) \frac{dt}{t^{1+s}} \asymp \delta(x)^{-2s}.$$

- Classical Hopf lemma + boundary regularity $\implies \gamma = 1$
- $\gamma - (2s - 1) = 2 - 2s > 0$ for any $s \in (0,1)$

CFL: Censored (Regional) Fractional Laplacian

$$\mathrm{L}u(x) = \int_\Omega (u(x) - u(y))\mathcal{J}(x,y)\,dy + \kappa(x)u(x)$$

$$(\mathrm{H}_1) \quad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma}{|x-y|^\gamma}\right) \left(1 \wedge \frac{\delta(y)^\gamma}{|x-y|^\gamma}\right)$$

- Censorship: $(-\Delta)^s_{\mathrm{CFL}} u(x) = \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy$

$$\mathcal{J}(x,y) = \frac{1}{|x-y|^{n+2s}}, \qquad \kappa(x) \equiv 0.$$

- Need boundary trace $\implies s \in (\frac{1}{2}, 1)$
- $(-\Delta)^s_{\mathrm{CFL}} x^{2s-1} = 0$ on $\mathbb{R}_+ \implies \gamma = 2s - 1$
- $\gamma - (2s-1) = 0$ for any $s \in (\frac{1}{2}, 1)$

Large L-harmonic functions

$$\begin{cases} \mathrm{L}u = 0 & \text{in } \Omega, \\ u \asymp \delta^{-b} & \text{on } \partial\Omega, \end{cases} \qquad \text{e.g.} \qquad (-\Delta)^s_{\mathrm{RFL}} (1-|x|^2)_+^{-(1-s)} = 0 \text{ in } B_1.$$

- Abatangelo–Gómez-Castro–Vázquez (2019)
- Precise boundary blow-up rate

$$b := \gamma - (2s-1) = \begin{cases} 1-s, & \text{RFL } (\gamma = s), \\ 2-2s, & \text{SFL } (\gamma = 1), \\ 0, & \text{CFL or } -\Delta \ (\gamma = 2s-1). \end{cases}$$

- (H$_4$) Martin (boundary Poisson) kernel $\mathbb{M} = D_\gamma \mathbb{G}$ exists

$$D_\gamma \mathbb{G}(\zeta, x) := \lim_{\Omega \ni y \to \zeta} \frac{\mathbb{G}(y,x)}{\delta(y)^\gamma} \asymp \frac{\delta(x)^\gamma}{|x-\zeta|^{n-2s+2\gamma}}, \ x \in \Omega, \ \zeta \in \partial\Omega,$$

$$(\mathrm{H}_1) \quad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma \delta(y)^\gamma}{|x-y|^{2\gamma}}\right)$$

Large L-harmonic functions

$$\begin{cases} \mathrm{L}u = 0 & \text{in } \Omega, \\ \delta^b u \asymp \mathcal{M}[1]^{-1} u = h & \text{on } \partial\Omega, \end{cases} \qquad b := \gamma - (2s-1).$$

- Martin operator $\mathcal{M} : L^\infty(\partial\Omega) \longrightarrow \delta^{-b} L^\infty(\Omega)$

$$h \longmapsto \mathcal{M}[h] = \int_{\partial\Omega} \mathbb{M}(\zeta, \cdot) h(\zeta)\, d\mathcal{H}^{n-1}_\zeta$$

  is continuous since

$$\delta(x)^b \mathcal{M}[1](x) \asymp \int_{\partial\Omega} \frac{\delta(x)^{1-2s+2\gamma}}{|x-\zeta|^{n-2s+2\gamma}}\, d\mathcal{H}^{n-1}_\zeta$$
$$\asymp \int_{\mathbb{R}^{n-1}} \frac{d\zeta'}{|e_n - \zeta'|^{n-2s+2\gamma}} \asymp 1 \qquad \text{as } x \to \partial\Omega$$

- $\mathrm{L}\big[\mathcal{M}[h]\big](x) = 0$ since $\mathbb{M}(\zeta, x) = D_\gamma \mathbb{G}(\zeta, x)$
- $\mathcal{M}[1]^{-1}\mathcal{M}[h] \to h$ towards $\partial\Omega$, for $h \in C(\partial\Omega)$

$$(\mathrm{H}_4) \quad \exists\, \mathbb{M}(\zeta, x) \asymp \frac{\delta(x)^\gamma}{|x-\zeta|^{n-2s+2\gamma}}, \qquad x \in \Omega,\ \zeta \in \partial\Omega.$$

Assumptions on L

- (H$_1$) symmetric Green's kernel $\mathbb{G}$

$$\mathbb{G}(x,y)=\mathbb{G}(y,x)\asymp\frac{1}{|x-y|^{n-2s}}\left(1\wedge\frac{\delta(x)^\gamma}{|x-y|^\gamma}\right)\left(1\wedge\frac{\delta(y)^\gamma}{|x-y|^\gamma}\right),$$

- (H$_2$) boundary regularization of Green's operator $\mathcal{G}=\mathrm{L}^{-1}$

$$\mathcal{G}:\delta^\gamma L^\infty(\Omega)\to\delta^\gamma C(\overline{\Omega})$$

- (H$_3$) sub-markovian semigroup $\mathcal{S}(t)=e^{-t\mathrm{L}}$

$$0\le u_0\le 1\quad\Longrightarrow\quad 0\le\mathcal{S}(t)[u_0]\le 1$$

- (H$_4$) Martin's kernel $\mathbb{M}=D_\gamma\mathbb{G}$

$$\exists\,\mathbb{M}(\zeta,x)\asymp\frac{\delta(x)^\gamma}{|x-\zeta|^{n-2s+2\gamma}},\quad x\in\Omega,\ \zeta\in\partial\Omega.$$

Existence-uniqueness with boundary singularity

Theorem (C.–Gómez-Castro–Vázquez, preprint)

For $\mathrm{L} \sim (-\Delta)^s_{*FL}$ *with*

- (H_1) *symmetric Green's kernel* $\mathbb{G}$
- (H_2) $\mathcal{G}$ *regularizing up to* $\delta^\gamma C(\overline{\Omega})$
- (H_3) *sub-markovian semigroup* $\mathcal{S}(t) = e^{-t\mathrm{L}}$
- (H_4) *Martin's kernel* $\mathbb{M} = D_\gamma \mathbb{G}$,

$$\begin{cases} u_t + \mathrm{L}u = f(t,x) & \text{for } x \in \Omega,\ t \in (0,T), \\ \mathcal{M}[1]^{-1}u = h(t,\zeta) & \text{for } \zeta \in \partial\Omega,\ t \in (0,T), \\ u(t,x) = 0 & \text{for } x \in \Omega^c,\ t \in (0,T), \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \end{cases}$$

has a unique weak-dual solution for $u_0 \in L^1(\Omega, \delta^\gamma)$, $f \in L^1(0,T; L^1(\Omega,\delta^\gamma))$, $h \in L^1((0,T)\times\partial\Omega)$.

- Turn boundary singularity ON/OFF as you prescribe h !

Continuity of $\mathcal{G} : f \longmapsto \int_\Omega \mathbb{G}(\cdot, y) f(y)\, dy$

$$
\begin{aligned}
L^1(\Omega; \mathcal{G}(\delta^\alpha)) &\longrightarrow L^1(\Omega, \delta^\alpha), && \text{for } \alpha > -1-\gamma, \\
L^1(\Omega) &\longrightarrow L^p(\Omega), && \text{for } p \in [1, \tfrac{n}{n-2s}), \\
L^2(\Omega) &\longrightarrow \mathbb{H}^2_{\mathrm{L}}(\Omega), && \\
L^{p_0}(\Omega) &\longrightarrow L^{p_1}(\Omega) && \text{for } p_0 \in (1, \tfrac{n}{2s}) \text{ and } \tfrac{1}{p_1} = \tfrac{1}{p_0} - \tfrac{2s}{n}, \\
L^q(\Omega) &\longrightarrow L^\infty(\Omega), && \text{for } q \in (\tfrac{n}{2s}, +\infty), \\
\delta^\alpha L^\infty(\Omega) &\longrightarrow \mathcal{G}(\delta^\alpha) L^\infty(\Omega), && \text{for } \alpha > -1-\gamma, \\
\delta^\gamma L^\infty(\Omega) &\longrightarrow \delta^\gamma C(\overline{\Omega}), && \text{(by (H}_2\text{))},
\end{aligned}
$$

where (Abatangelo–Gómez-Castro–Vázquez, 2019)

$$
\mathcal{G}(\delta^\alpha) \asymp \begin{cases} \delta^{\alpha+2s} & \text{for } \alpha + 2s < \gamma, \\ \delta^\gamma (1 + |\log \delta|) & \text{for } \alpha + 2s = \gamma, \\ \delta^\gamma & \text{for } \alpha + 2s > \gamma. \end{cases}
$$

$$
(\mathrm{H}_1) \quad \mathbb{G}(x, y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma}{|x-y|^\gamma}\right) \left(1 \wedge \frac{\delta(y)^\gamma}{|x-y|^\gamma}\right)
$$

The standard eigenvalue problem

$$\begin{cases} \mathrm{L}\varphi - \lambda\varphi = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{in } \mathbb{R}^n \setminus \Omega \text{ or } \partial\Omega \end{cases}$$

- Bonforte–Figalli–Vázquez (CVPDE 2018)
- $(\mathrm{H}_1) \implies 0 \leq \mathbb{G}(x,y) \lesssim |x-y|^{-(n-2s)}$
- $\mathcal{G} : L^2(\Omega) \xrightarrow{\text{compact}} L^2(\Omega)$ (Riesz–Fréchet–Kolmogorov)
- Discrete standard spectrum Σ

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow +\infty$$

- Standard eigenfunctions $\varphi_j = \lambda_j \mathcal{G}(\varphi_j) \implies$

$$\varphi_1 \asymp \delta^\gamma, \qquad |\varphi_j| \lesssim \delta^\gamma$$

- Energy space

$$\mathbb{H}^2_{\mathrm{L}}(\Omega) = \left\{ v \in L^2(\Omega) : \sum_{j\geq 1} \lambda_j^2 \langle v, \varphi_j \rangle^2 < +\infty \right\}$$

Eigenfunction estimates $\varphi_1 \asymp \delta^\gamma$, $|\varphi_j| \lesssim \delta^\gamma$

Behavior of eigenfunctions $\varphi_j = \lambda_j \mathcal{G}(\varphi_j)$?

- Upper bound for φ_j, $\forall j \geq 1$

$$\varphi_j = \lambda_j^k \mathcal{G}^k(\varphi_j) \in \delta^\gamma C(\overline{\Omega}) \qquad \text{(finite large } k\text{)}$$

- Lower bound for φ_1 (Hopf)

$$(\mathrm{H}_1) \quad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma}{|x-y|^\gamma}\right) \left(1 \wedge \frac{\delta(y)^\gamma}{|x-y|^\gamma}\right) \gtrsim \delta(x)^\gamma \delta(y)^\gamma$$

$$\varphi_1(x) \gtrsim \lambda_1 \int_\Omega \delta(x)^\gamma \delta(y)^\gamma \varphi_1(y)\, dy \gtrsim \delta(x)^\gamma$$

$\Longrightarrow$ Largest space for eigen-decomposition: $L^1(\Omega; \delta^\gamma)$

Semigroup theories

$$\mathcal{S}(t) : L^2(\Omega) \to L^2(\Omega), \qquad \mathcal{S}(t)[u_0] = \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle u_0, \varphi_k \rangle \varphi_k$$

defines a continuous non-expansive semigroup in $L^p(\Omega)$, $1 \leq p \leq \infty$:

- $p = 2$ by formula, $\|\mathcal{S}(t)[u_0]\|_{L^2(\Omega)} \leq e^{-\lambda_1 t} \|u_0\|_{L^2(\Omega)}$
- $p = \infty$ by submarkovian property, $|\mathcal{S}(t)[u_0]| \leq \mathcal{S}(t)[|u_0|] \leq \|u_0\|_{L^\infty(\Omega)}$
- $p = 1$ by duality, $\int_\Omega |\mathcal{S}(t)[u_0]| \leq \int_\Omega |u_0| \mathcal{S}(t)[1] \leq \int_\Omega |u_0|$
- $1 < p < \infty$ by Riesz–Thorin interpolation

$\mathcal{S}(t)$ extends to C_0 semigroup in $L^1(\Omega, \delta^\gamma)$ via Hille–Yosida:

- $A = -\mathcal{G}^{-1} : D(A) = \mathcal{G}(L^1(\Omega, \delta^\gamma)) \to X = L^1(\Omega, \delta^\gamma)$
- A closed, $\overline{D(A)} = X$
- $\|(\lambda I - A)^{-1}\| \leq 1/\lambda$

$$\|\mathcal{S}(t)[u_0]\delta^\gamma\|_{L^1(\Omega)} \lesssim e^{-\lambda_1 t} \|u_0 \delta^\gamma\|_{L^1(\Omega)}$$

Formal expansions

- Green's kernel

$$\mathbb{G}(x,y) = \sum_{k=1}^{\infty} \lambda_k^{-1} \varphi_k(x)\varphi_k(y)$$

- Green's operator

$$\mathcal{G}[f](x) = \sum_{k=1}^{\infty} \lambda_k^{-1} \langle f, \varphi_k \rangle \varphi_k(x) = \int_\Omega \mathbb{G}(x,y) f(y)\, dy$$

- Heat kernel

$$\mathbb{S}(t,x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \varphi_k(x)\varphi_k(y)$$

- Heat semigroup

$$\mathcal{S}(t)[f](t,x) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle f, \varphi_k \rangle \varphi_k(x) = \int_\Omega \mathbb{S}(t,x,y) f(y)\, dy$$

- Relations

$$\mathcal{G}[f] = \int_0^\infty \mathcal{S}(t)[f]\, dt, \qquad \mathbb{G}(x,y) = \int_0^\infty \mathbb{S}(t,x,y)\, dt.$$

One-sided Weyl's law

Theorem (C.–Gómez-Castro–Vázquez, preprint)

$$\lambda_k \gtrsim k^{\frac{2s}{n}}.$$

- Classically, for $\mathrm{L} = -\Delta$, $\lambda_k = (c + o(1))k^{\frac{2}{n}}$; known for SFL, RFL but new for CFL
- "$\gtrsim$" is important for convergence in eigen-decompositions and existence of heat kernel
- Need Sobolev's inequality $C_S^2 \|u\|^2_{L^{\frac{2n}{n-2s}}(\Omega)} \le \int_\Omega u \mathrm{L} u$ (Bonforte–Sire–Vázquez, DCDS 2015)

Proof using Cheng–Li (CMH 1981) without existence of heat kernel.

$u(t) = \mathcal{S}(t)[u_0]$ (later $u_0 \to \delta_y$ so $u(t) \to \mathbb{S}(t, \cdot, y)$) satisfies

$$\partial_t \|u(t)\|_{L^2(\Omega)} \overset{\text{Sobolev}}{\le} -2C_S \|u(t)\|^2_{L^{\frac{2n}{n-2s}}(\Omega)} \overset{\text{Hölder}}{\le} -2C_S \|u(t)\|^{\frac{n+2s}{n}}_{L^2(\Omega)},$$

$$\text{while} \qquad \int_\Omega \|u(t)\|^2_{L^2(\Omega)} \, dy \to \text{“}\iint_{\Omega\times\Omega} \mathbb{S}(t,x,y)^2 \, dx \, dy\text{”} = \sum_{k=1}^\infty e^{-2\lambda_k t} \ge k e^{-2\lambda_k t}. \qquad \square$$

Heat kernel: existence

Theorem (C.–Gómez-Castro–Vázquez, preprint)

Assume (H_1), (H_2), (H_3). *For* $t > 0$, $x, y \in \Omega$, $\zeta \in \partial\Omega$,

- $\mathcal{S}(t) : L^2(\Omega) \to \delta^\gamma C(\overline{\Omega})$, *i.e. ultracontractive + boundary regularity*
- $\mathcal{S}(t) : M(\Omega, \delta^\gamma) \to L^2(\Omega)$
- $\displaystyle\int_\Omega \mathbb{S}(t, x, y)\, dx \le 1$
- $0 \le \mathbb{S}(t, x, y) \le C(t)\delta(x)^\gamma \delta(y)^\gamma$, *i.e. intrinsic ultracontractive*
- $0 \le D_\gamma \mathbb{S}(t, \zeta, y) \le C(t)\delta(y)^\gamma$.

Proof using Fernández-Real–Ros-Oton (RACSAM 2016).

$$\left\| \frac{\mathcal{S}(t)[u_0]}{\delta^\gamma} \right\|_{C(\overline{\Omega})} \le \sum_{k=1}^\infty e^{-\lambda_k t} |\langle u_0, \varphi_k \rangle| \left\| \frac{\varphi_k}{\delta^\gamma} \right\|_{C(\overline{\Omega})} \lesssim \sum_{k=1}^\infty e^{-\lambda_k t} \lambda_k^w |\langle u_0, \varphi_k \rangle| \overset{\text{Weyl}}{\lesssim} \|u_0\|_{L^2(\Omega)} .$$

Duality $\implies$ $\mathcal{S}(t)[\delta_y/\delta(y)^\gamma] \in \delta^\gamma C(\overline{\Omega})$, $D_\gamma \mathcal{S}(t)[\delta_y/\delta(y)^\gamma] \in C(\partial\Omega)$. Estimates by concentration. □

Heat kernel: estimates

Theorem (C.–Gómez-Castro–Vázquez, preprint)

Assume (H_1), (H_2), (H_3). *For* $t > 0$, $x, y \in \Omega$, $\zeta \in \partial\Omega$,

- $0 \le \mathbb{S}(t, x, y) \lesssim t^{-\frac{n}{2s}}$
- *If* $\gamma < 2s$, *then* $\quad 0 \le \mathbb{S}(t, x, y) \le t^{-\frac{n}{2s-\gamma}} \delta(x)^\gamma \delta(y)^\gamma, \quad 0 \le D_\gamma \mathbb{S}(t, \zeta, y) \le t^{-\frac{n}{2s-\gamma}} \delta(y)^\gamma$
- *As* $t \to +\infty$, $\quad \mathbb{S}(t, x, y) = (1 + o(1)) e^{-\lambda_1 t} \varphi_1(x) \varphi_1(y), \quad D_\gamma \mathbb{S}(t, \zeta, y) = (1 + o(1)) e^{-\lambda_1 t} \varphi_1(y)$.

Enough for a theory, but far from optimal without extra hypotheses!

Proof using Davies' book (1989): (log.) Sob. ineq. $\iff$ ultracontractive.

- Sob. with $\frac{2n}{n-2s} \iff \|u(t)\|_{L^\infty(\Omega)} \lesssim t^{-\frac{n}{4s}} \|u_0\|_{L^2(\Omega)} \iff \|u(t)\|_{L^2(\Omega)} \lesssim t^{-\frac{n}{4s}} \|u_0\|_{L^1(\Omega)}$
- $L^2(\Omega, \varphi_1^{-2}\, dx)$, $\overline{\mathrm{L}} = \varphi_1 \mathrm{L}(\varphi_1^{-1} \cdot)$, $\overline{\mathbb{S}}(t, x, y) = \frac{\mathbb{S}(t,x,y)}{\varphi_1(x)\varphi_1(y)}$, dual HLS $\implies$ Sob., $\frac{2n}{n-(2s-\gamma)}$. $\square$

Heat kernel: a glance on sharp estimates

- SFL ($\gamma = 1$, s-independent) by Song (PTRF 2004)

$$t^{-\frac{n}{2}} e^{-\frac{c(T)|x-y|^2}{t}} \left(1 \wedge \frac{\delta(x)\delta(y)}{t}\right) \overset{T}{\lesssim} \mathbb{S}(t,x,y) \lesssim t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{6t}} \left(1 \wedge \frac{\delta(x)\delta(y)}{t}\right).$$

- CFL ($\gamma = 2s - 1$) by Chen–Kim–Song (PTRF 2009)
- RFL ($\gamma = s$) by Chen–Kim–Song (JEMS 2010), Bogdan–Grzywny–Ryznar (AP 2010)

$$\mathbb{S}(t,x,y) \overset{T}{\asymp} \begin{cases} t^{-\frac{n}{2s}} \left(1 \wedge \frac{t^{\frac{1}{2s}}}{|x-y|}\right)^{n+2s} \left(1 \wedge \frac{\delta(x)}{t^{\frac{1}{2s}}}\right)^{\gamma} \left(1 \wedge \frac{\delta(y)}{t^{\frac{1}{2s}}}\right)^{\gamma}, & t < T, \\ e^{-\lambda_1 t} \delta(x)^\gamma \delta(y)^\gamma, & t \geq T. \end{cases}$$

For large t and $\gamma < 2s$, our estimate

$$0 \leq \mathbb{S}(t,x,y) \lesssim t^{-\frac{n}{2s}} \wedge \left(t^{-\frac{n}{2s-\gamma}} \delta(x)^\gamma \delta(y)^\gamma\right)$$

is much worse than the sharp one for CFL/RFL:

$$\mathbb{S}(t,x,y) \asymp t^{-\frac{n}{2s}} \wedge \left(t^{-\frac{n+2\gamma}{2s}} \delta(x)^\gamma \delta(y)^\gamma\right).$$

Linear theory 1: semigroup

$$\begin{cases} u_t + \mathrm{L}u = f(t,x) & \text{for } x \in \Omega,\ t \in (0,T), \\ \mathcal{M}[1]^{-1}u = h(t,\zeta) & \text{for } \zeta \in \partial\Omega,\ t \in (0,T), \\ u(t,x) = 0 & \text{for } x \in \Omega^c,\ t \in (0,T), \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \end{cases}$$

- Solution operator $u(t,x) = \mathcal{H}[u_0, f, h](t,x)$
- Linearity $\mathcal{H}[u_0, f, h] = \mathcal{H}[u_0, 0, 0] + \mathcal{H}[0, f, 0] + \mathcal{H}[0, 0, h]$
- Case $u_0 \neq 0,\ f = 0,\ h = 0,$

$$\mathcal{H}[u_0, 0, 0](t,x) = \mathcal{S}(t)[u_0](x) = \int_\Omega \mathbb{S}(t,x,y)u_0(y)\,dy$$

- $u_0 \in L^1(\Omega, \delta^\gamma) \implies \mathcal{H}[u_0, 0, 0](t, \cdot) \in L^1(\Omega, \delta^\gamma).$

Linear theory 2: Duhamel

$$\begin{cases} u_t + \mathrm{L}u = f(t,x) & \text{for } x \in \Omega,\ t \in (0,T), \\ \mathcal{M}[1]^{-1}u = h(t,\zeta) & \text{for } \zeta \in \partial\Omega,\ t \in (0,T), \\ u(t,x) = 0 & \text{for } x \in \Omega^c,\ t \in (0,T), \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \end{cases}$$

- $u_0 = 0,\ f \neq 0,\ h = 0,$

$$\mathcal{H}[0,f,0](t,x) = \int_0^t \mathcal{S}(t-\sigma)[f(\sigma,\cdot)](x)\,d\sigma = \int_0^t \int_\Omega \mathbb{S}(t-\sigma,x,y) f(\sigma,y)\,dy\,d\sigma.$$

- $f \in L^1(0,T;L^1(\Omega,\delta^\gamma)) \implies \mathcal{H}[0,f,0](t,\cdot) \in L^1(\Omega,\delta^\gamma)$:

$$\int_\Omega |\mathcal{H}[0,f,0](t,x)|\varphi_1(x)\,dx \le \int_0^t e^{-\lambda_1(t-\sigma)} \int_\Omega |f(\sigma,y)|\varphi_1(y)\,dy\,d\sigma.$$

- $\phi \in \delta^\gamma L^\infty((0,T)\times\Omega) \implies \mathcal{H}[0,\phi,0] \in \delta^\gamma L^\infty((0,T)\times\Omega),\ D_\gamma \mathcal{H}[0,\phi,0] \in L^\infty((0,T)\times\Omega)$:

$$\mathcal{H}[0,\varphi_1,0] = \left(\int_0^t e^{-\lambda_1 \sigma}\,d\sigma\right)\varphi_1, \qquad \mathcal{H}[0,\phi_\pm,0] \le \left\| \frac{\phi_\pm}{\varphi_1} \right\|_{L^\infty((0,T)\times\Omega)} \mathcal{H}[0,\varphi_1,0].$$

Towards linear theory 3: review of elliptic problem

$h \neq 0$? Compare

$$\begin{cases} \mathrm{L} v_j = f_j & \text{in } \Omega, \\ \mathcal{M}[1]^{-1} v_j = 0 & \text{on } \partial\Omega, \\ v_j = 0 & \text{in } \Omega^c. \end{cases}$$

$$\begin{cases} \mathrm{L} v = 0 & \text{in } \Omega, \\ \mathcal{M}[1]^{-1} v = h & \text{on } \partial\Omega, \\ v = 0 & \text{in } \Omega^c. \end{cases}$$

$$v_j(x) = \mathcal{G}[f_j](x) = \int_\Omega \mathbb{G}(x,y) f_j(y)\, dx$$

$$v(x) = \mathcal{M}[h](x) = \int_{\partial\Omega} D_\gamma \mathbb{G}(x,\zeta) h(\zeta)\, d\mathcal{H}^{n-1}$$

Approximate singular boundary data from (narrow) tubular neighborhoods in interior:

$$f_j(\zeta - \delta\nu) = \frac{|\partial\Omega| \chi_{\{1/j<\delta<2/j\}}}{|\,\{1/j < \delta < 2/j\}\,|} \frac{h(\zeta)}{\delta^\gamma} \quad \Longrightarrow \quad \mathcal{G}[f_j] \overset{L^1(\Omega,\delta^\gamma)}{\longrightarrow} \mathcal{M}[h] \quad \text{as } j \to \infty$$

Limit taken via weak-dual formulation (Brezis, handwritten manuscript)

$$\int_\Omega v_j \psi = \int_\Omega f_j \mathcal{G}[\psi], \quad \forall \psi \in \delta^\gamma L^\infty(\Omega)$$

$$\int_\Omega v\psi = \int_{\partial\Omega} h D_\gamma \mathcal{G}[\psi], \quad \forall \psi \in \delta^\gamma L^\infty(\Omega)$$

Let $\varphi = \mathcal{G}[\psi] \implies$ integrating by parts

$$\int_\Omega v \mathrm{L}[\varphi] = \int_\Omega \mathrm{L}[v]\varphi + \int_{\partial\Omega} (\mathcal{M}[1]^{-1} v) D_\gamma \varphi$$

Linear theory 3: concentrating data towards boundary

$h \neq 0$? Compare

$$\begin{cases} (u_j)_t + \mathrm{L}u_j = f_j(t,x) & \text{in } (0,T)\times\Omega, \\ \mathcal{M}[1]^{-1}u_j = 0 & \text{in } (0,T)\times\partial\Omega, \\ u_j = 0 & \text{in } (0,T)\times\Omega^c, \\ u_j(0,\cdot) = u_0 & \text{in } \Omega. \end{cases}$$

$$u_j(t,x) = \mathcal{H}[0,f_j,0](t,x)$$

$$= \int_0^t \int_\Omega \mathbb{S}(t-\sigma,x,y) f_j(\sigma,y)\, dy\, d\sigma$$

$$\begin{cases} u_t + \mathrm{L}u = 0 & \text{in } (0,T)\times\Omega, \\ \mathcal{M}[1]^{-1}u = h(t,\zeta) & \text{in } (0,T)\times\partial\Omega, \\ u = 0 & \text{in } (0,T)\times\Omega^c, \\ u(0,\cdot) = u_0 & \text{in } \Omega. \end{cases}$$

$$u(t,x) = \mathcal{H}[0,0,h](t,x)$$

$$= \int_0^t \int_{\partial\Omega} D_\gamma \mathbb{S}(t-\sigma,\zeta,y) h(\sigma,\zeta)\, d\mathcal{H}^{n-1}_\zeta\, d\sigma\ ?$$

Approximate singular boundary data from (narrow) tubular neighborhoods in interior:

$$f_j(\zeta - \delta\nu) = \frac{|\partial\Omega| \chi_{\{1/j<\delta<2/j\}}}{|\{1/j<\delta<2/j\}|} \frac{h(\zeta)}{\delta^\gamma} \quad \Longrightarrow \quad \mathcal{H}[0,f_j,0] \to \mathcal{H}[0,0,h] \quad \text{as } j\to\infty.$$

Justify the guessed formula by weak-dual formulation and compactness ?

Linear theory 3: weak-dual formulation

- Weak formulation: integrate $u_t + \mathrm{L}u = f$ against φ on $(0,T)\times\Omega$ by parts,

$$\int_\Omega u(T,x)\varphi(T,x)\,dx + \int_0^T\int_\Omega u(t,x)(-\varphi_t(t,x)+\mathrm{L}\varphi(t,x))\,dx\,dt$$
$$= \int_\Omega u_0(x)\varphi(0,x)\,dx + \int_0^T\int_\Omega f(t,x)\varphi(t,x)\,dx\,dt + \int_0^T\int_{\partial\Omega} h(t,\zeta)D_\gamma\varphi(t,\zeta)\,d\mathcal{H}^{n-1}_\zeta\,dt.$$

- Weak-dual: $\forall\phi\in\delta^\gamma L^\infty((0,T)\times\Omega)$, put $\varphi(t,x)=\mathcal{H}[0,\phi,0](T-t,x)\in\delta^\gamma L^\infty((0,T)\times\Omega)$,

$$\int_0^T\int_\Omega u(t,x)\phi(T-t,x)\,dx\,dt = \int_\Omega u_0(x)\mathcal{H}[0,\phi,0](T,x)\,dx + \int_0^T\int_\Omega f(t,x)\mathcal{H}[0,\phi,0](T-t,x)\,dx\,dt$$
$$+\int_0^T\int_{\partial\Omega} h(t,\zeta)D_\gamma\mathcal{H}[0,\phi,0](T-t,\zeta)\,d\mathcal{H}^{n-1}_\zeta\,dt.$$

- $\phi=\varphi_1 \implies \displaystyle\int_0^T\int_\Omega |u|\delta^\gamma\,dx\,dt \lesssim \int_\Omega |u_0|\delta^\gamma\,dx + \int_0^T\int_\Omega |f|\delta^\gamma\,dx\,dt + \int_0^T\int_{\partial\Omega}|h|\,d\mathcal{H}^{n-1}_\zeta\,dt.$

Linear theory 3: compactness

$$\int_0^T \int_\Omega u(t,x)\phi(T-t,x)\,dx\,dt = \int_\Omega u_0(x)\mathcal{H}[0,\phi,0](T,x)\,dx + \int_0^T \int_\Omega f(t,x)\mathcal{H}[0,\phi,0](T-t,x)\,dx\,dt$$
$$+ \int_0^T \int_{\partial\Omega} h(t,\zeta) D_\gamma \mathcal{H}[0,\phi,0](T-t,\zeta)\,d\mathcal{H}^{n-1}_\zeta\,dt, \quad \forall \phi \in \delta^\gamma L^\infty((0,T)\times\Omega).$$

- $\phi = \chi_A(x)\varphi_1$, $\phi = \chi_{[t_0,t_1]}(t)\varphi_1 \implies$

$$\int_{t_0}^{t_1} \int_A |\mathcal{H}[0,f,0]|\delta^\gamma\,dx\,dt \le \omega(|t_1-t_0|,|A|) \int_0^T \int_\Omega |f|\delta^\gamma\,dx\,dt.$$

- Equi-integrability of $u_j = \mathcal{H}[0,f_j,0]$ in $L^1(0,T;L^1(\Omega,\delta^\gamma))$, provided $\|f_j\delta^\gamma\|_{L^1((0,T)\times\Omega)} \le C_0$
- Dunford–Pettis: $\{u_j\}$ weakly precompact, i.e. $u_j \rightharpoonup u$ in $L^1(0,T;L^1(\Omega,\delta^\gamma))$

Final formula

Theorem (C.–Gómez-Castro–Vázquez, preprint)

Assume (H_1), (H_2), (H_3), (H_4). *Then*

$$\mathcal{H} : L^1(\Omega, \delta^\gamma) \times L^1(0,T; L^1(\Omega, \delta^\gamma)) \times L^1((0,T) \times \partial\Omega) \longrightarrow L^1(0,T; L^1(\Omega, \delta^\gamma))$$

is continuous and

$$\mathcal{H}[u_0, f, h](t,x) = \mathcal{S}(t)[u_0](x) + \int_0^t \mathcal{S}(t-\sigma)[f(\sigma,\cdot)](x)\, d\sigma + \int_0^t \int_{\partial\Omega} D_\gamma \mathbb{S}(t-\sigma,\zeta,x) h(\sigma,\zeta)\, d\mathcal{H}_\zeta^{n-1}\, d\sigma,$$

is the unique weak-dual solution of

$$\begin{cases} u_t + \mathrm{L}u = f(t,x) & \text{for } x \in \Omega,\ t \in (0,T), \\ \mathcal{M}[1]^{-1} u = h(t,\zeta) & \text{for } \zeta \in \partial\Omega,\ t \in (0,T), \\ u(t,x) = 0 & \text{for } x \in \Omega^c,\ t \in (0,T), \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \end{cases}$$

Proof. Split into positive/negative parts, approximate by smooth functions. □

Time-independent interior data

- Let $f(t,x) \equiv f(x)$.
- $\mathcal{G}[f]$ is stationary solution with initial data $u_0 = \mathcal{G}[f]$ and interior data f

$$\mathcal{G}[f](x) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle f, \varphi_k \rangle \varphi_k(x)$$

$$\mathcal{H}[0,f,0](t,x) = \sum_{k=1}^{\infty} \frac{1 - e^{-\lambda_k t}}{\lambda_k} \langle f, \varphi_k \rangle \varphi_k(x)$$

$$\mathcal{H}\big[\mathcal{G}[f],0,0\big](t,x) = \sum_{k=1}^{\infty} \frac{e^{-\lambda_k t}}{\lambda_k} \langle f, \varphi_k \rangle \varphi_k(x)$$

$$\implies \qquad \mathcal{H}\big[\mathcal{G}[f],f,0\big](t,x) = \mathcal{G}[f](x)$$

- $\mathcal{H}\big[\mathcal{G}[f],0,0\big] = \mathcal{S}(t)\big[\mathcal{G}[f]\big] \lesssim \delta^\gamma \implies$
  - $\mathcal{H}[0,f,0](t,x)$ and $\mathcal{G}[f](x)$ have same behavior near $\partial\Omega$ for $t > 0$
- Range of exponents as in elliptic theory (Abatangelo–Gómez-Castro–Vázquez, preprint)

$$\mathcal{H}[0,\delta^\alpha,0](t,\cdot) \asymp \mathcal{G}[\delta^\alpha] \asymp \delta^{(\alpha+2s)\wedge\gamma}, \qquad \text{for } \delta \sim 0^+, \quad \alpha > -1-\gamma \,(\text{log. if } \alpha = \gamma - 2s)$$

Time-independent boundary data

- Let $h(t,x) \equiv h(x)$.
- $\mathcal{M}[h]$ is stationary solution with initial data $u_0 = \mathcal{M}[h]$ and boundary data h

$$\mathcal{M}[h](x) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \left(\int_{\partial\Omega} D_\gamma \varphi_k(\zeta) h(\zeta)\, d\mathcal{H}_\zeta^{n-1} \right) \varphi_k(x)$$

$$\mathcal{H}[0,0,h](t,x) = \sum_{k=1}^{\infty} \frac{1 - e^{-\lambda_k t}}{\lambda_k} \left(\int_{\partial\Omega} D_\gamma \varphi_k(\zeta) h(\zeta)\, d\mathcal{H}_\zeta^{n-1} \right) \varphi_k(x)$$

$$\mathcal{H}\big[\mathcal{M}[h],0,0\big](t,x) = \sum_{k=1}^{\infty} \frac{e^{-\lambda_k t}}{\lambda_k} \left(\int_{\partial\Omega} D_\gamma \varphi_k(\zeta) h(\zeta)\, d\mathcal{H}_\zeta^{n-1} \right) \varphi_k(x)$$

$$\Longrightarrow \qquad \mathcal{H}\big[\mathcal{M}[h],0,h\big](t,x) = \mathcal{M}[h](x)$$

- $\mathcal{H}\big[\mathcal{M}[h],0,0\big] = \mathcal{S}(t)\big[\mathcal{M}[h]\big] \lesssim \delta^\gamma \implies$
  - $\mathcal{M}[1]^{-1}\mathcal{M}[h](t,x) \equiv h(x)$ on $\partial\Omega$
  - $\mathcal{M}[1]^{-1}\mathcal{H}[0,0,h](t,x) \equiv h(x)$ on $\partial\Omega$
- Recover prescribed boundary singularity

Convergence to elliptic problem

$$\begin{cases} u_t + \mathrm{L}u = f(x) & \text{for } x \in \Omega,\ t \in (0,T), \\ \mathcal{M}[1]^{-1}u = h(\zeta) & \text{for } \zeta \in \partial\Omega,\ t \in (0,T), \\ u(t,x) = 0 & \text{for } x \in \Omega^c,\ t \in (0,T), \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \end{cases} \quad \xrightarrow[t\to+\infty]{?} \quad \begin{cases} \mathrm{L}u = f(x) & \text{for } x \in \Omega, \\ \mathcal{M}[1]^{-1}u = h(\zeta) & \text{for } \zeta \in \partial\Omega, \\ u(x) = 0 & \text{for } x \in \Omega^c. \end{cases}$$

Theorem (C.–Gómez-Castro–Vázquez, preprint)

Assume (H_1), (H_2), (H_3), (H_4)*. If* $f(t,x) \equiv f(x) \in L^1(\Omega, \delta^\gamma)$ *and* $h(t,\zeta) \equiv h(\zeta) \in L^1(\partial\Omega)$*, then*

$$\mathcal{H}[u_0, f, h](t,\cdot) \to \mathcal{G}[f] + \mathcal{M}[h] \qquad \text{in } L^1(\Omega,\delta^\gamma) \qquad \text{as } t \to +\infty.$$

Proof.

By $\mathcal{H}[\mathcal{G}[f], f, 0] = \mathcal{G}[f]$, $\mathcal{H}[\mathcal{M}[h], 0, h] = \mathcal{M}[h]$,

$$\mathcal{H}[u_0, f, h] - \mathcal{G}[f] - \mathcal{M}[h] = \mathcal{H}[u_0 - \mathcal{G}[f] - \mathcal{M}[h], 0, 0] = \mathcal{S}(t)[u_0 - \mathcal{G}[f] - \mathcal{M}[h]]$$

$$\left\| \left(\mathcal{S}(t)[u_0 - \mathcal{G}[f] - \mathcal{M}[h]]\right)\delta^\gamma \right\|_{L^1(\Omega)} \lesssim e^{-\lambda_1 t} \left\| \left(u_0 - \mathcal{G}[f] - \mathcal{M}[h]\right)\delta^\gamma \right\|_{L^1(\Omega)} \to 0. \qquad \square$$

Time-fractional problem

$$\begin{cases} {}^{\bullet}\partial_t^\alpha u(t,x) + \mathrm{L}u(t,x) = f(t,x), & x \in \Omega, t \in (0,T) \\ u(t,x) = 0 & x \in \Omega^c, t > 0 \\ \lim\limits_{x\to\zeta} \dfrac{u(t,x)}{u^\star(x)} = h(t,\zeta), & \zeta \in \partial\Omega, t > 0 \end{cases} \tag{$\mathrm{P}_\bullet$}$$

- Caputo derivative

$$^{C}\partial_t^\alpha u(t,x) = \tfrac{1}{\Gamma(1-\alpha)} \int_0^t \frac{(\partial_t u)(\tau,x)}{(t-\tau)^\alpha}\,\mathrm{d}\tau$$

- Riemann–Liouville derivative

$$^{R}\partial_t^\alpha u(t,x) = \tfrac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \left(\int_0^t \frac{u(\tau,x)}{(t-\tau)^\alpha} d\tau \right).$$

Time-fractional ODE: motivation

- Eigendecomposition $u(t,x) = \sum_{j=1}^{\infty} u_j(t)\varphi_j(x)$,

$$^{\bullet}\partial_t^\alpha u + \mathrm{L}u = f \qquad \Longrightarrow \qquad ^{\bullet}\partial_t^\alpha u_j + \lambda_j u_j = f_j$$

- Laplace transform

$$\mathcal{L}\left[{}^{C}\partial_t^\alpha w\right](s) = s^\alpha\,\mathcal{L}[w] - s^{\alpha-1}w(0), \qquad \alpha \in (0,1).$$

$$\mathcal{L}[{}^{R}\partial_t^\alpha w](s) = s^\alpha\,\mathcal{L}[w](s) - \lim_{h\to 0^+}[{}^{R}\partial_t^{\alpha-1}w(h)], \qquad \alpha \in (0,1).$$

$${}^{R}\partial_t^{\alpha-1}w(t) = \frac{1}{\Gamma(\alpha)}\int_0^t w(\xi)(t-\xi)^{-\alpha}\,\mathrm{d}\xi.$$

Time-fractional ODE: solution

$$\begin{cases} {}^C\partial_t^\alpha u(t) + \lambda u(t) = f(t), \quad t > 0, \\ u(0) = u_0. \end{cases} \tag{ODE$_C$}$$

$$u(t) = u_0 E_\alpha(-\lambda t^\alpha) + \int_0^t P_\alpha(t-\tau;\lambda) f(\tau)\,\mathrm{d}\tau.$$

$$\begin{cases} {}^R\partial_t^\alpha v(t) + \lambda v(t) = g(t), \quad t > 0, \\ \lim\limits_{h\to 0^+} {}^R\partial_t^{\alpha-1} v(h) = v_0. \end{cases} \tag{ODE$_R$}$$

$$v(t) = v_0 P_\alpha(t;\lambda) + \int_0^t P_\alpha(t-\tau;\lambda) g(\tau)\,\mathrm{d}\tau.$$

where

- $E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}$ is the (bounded) Mittag-Leffler function, $E_\alpha = E_{\alpha,1}$
- $P_\alpha(t;\lambda) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) = \alpha t^{\alpha-1} E'_\alpha(-\lambda t^\alpha) = -\frac{1}{\lambda}\frac{d}{dt} E_\alpha(-\lambda t^\alpha)$.

Duality

$$\int_0^T \varphi(t)\left({}^C\partial_t^\alpha[u]\right)(t)\,\mathrm{d}t = \frac{1}{\Gamma(1-\alpha)}\int_0^T \varphi(t)\int_0^t (t-\sigma)^{-\alpha}u'(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}t$$
$$= \frac{1}{\Gamma(1-\alpha)}\int_0^T u'(\sigma)\int_\sigma^T (t-\sigma)^{-\alpha}\varphi(t)\,\mathrm{d}t\,\mathrm{d}\sigma$$
$$= \int_0^T u(\sigma)\underbrace{\frac{-1}{\Gamma(1-\alpha)}\frac{\partial}{\partial\sigma}\left(\int_\sigma^T (t-\sigma)^{-\alpha}\varphi(t)\,\mathrm{d}t\right)}_{({}^C\partial_t^\alpha)^*[\varphi](\sigma)}\mathrm{d}\sigma$$
$$+\frac{1}{\Gamma(1-\alpha)}\left(u(T)\lim_{\sigma\to T^-}\int_\sigma^T (t-\sigma)^{-\alpha}\varphi(t)\,\mathrm{d}t - u(0)\int_0^T t^{-\alpha}\varphi(t)\,\mathrm{d}t\right).$$

Reverse time $\varphi(t)=\phi(T-t)$ so that, taking $\tau = T-\sigma$

$$({}^C\partial_t^\alpha)^*[\varphi](\sigma) = \left({}^R\partial_\tau^\alpha\right)[\phi](\tau) = \left({}^R\partial_\tau^\alpha\right)[\phi](T-\sigma).$$

Resolvent families

$$\begin{cases} {}^C\partial_t^\alpha u + \mathrm{L}u = f \\ u(0,x) = u_0 \end{cases} \implies u(t,x) = \mathcal{S}_\alpha(t)u_0 + \int_0^t \mathcal{P}_\alpha(t-\tau)f(\tau)\,\mathrm{d}\tau$$

$$\mathcal{S}_\alpha(t) = \int_0^\infty \Phi_\alpha(\tau)\mathcal{S}(\tau t^\alpha)\,\mathrm{d}\tau, \qquad \mathcal{P}_\alpha(t) = \alpha t^{\alpha-1}\int_0^\infty \tau\Phi_\alpha(\tau)\mathcal{S}(\tau t^\alpha)\,\mathrm{d}\tau$$

where the Wright-type of Mainardi function

$$\Phi_\alpha(t) = \sum_{k=0}^\infty \frac{(-t)^k}{k!\Gamma(1-\alpha(k+1))}$$

is bounded with power decay. For our class of L, the heat semigroup satisfies

$$\|\mathcal{S}(t)\|_{L^p\to L^q} \le Ct^{-\frac{n}{2s}(\frac{1}{p}-\frac{1}{q})}.$$

- $\mathcal{S}_\alpha, \mathcal{P}_\alpha$ nonlinear in t, not semigroup
- $\mathcal{S}_\alpha(t) : L^p \to L^q$ only when $\frac{n}{2s}(\frac{1}{p}-\frac{1}{q}) < 1$
- $\mathcal{P}_\alpha(t) : L^p \to L^q$ only when $\frac{n}{2s}(\frac{1}{p}-\frac{1}{q}) < 2$.

Work in progress

Theorem (C, Gómez-Castro, Vázquez, in progress)

For $\bullet \in \{C, R\}$,

$$\begin{cases} {}^{\bullet}\partial_t^{\alpha} u(t,x) + \mathrm{L}u(t,x) = f(t,x), & x \in \Omega, t \in (0,T) \\ u(t,x) = 0 & x \in \Omega^c, t > 0 \\ \displaystyle\lim_{x \to \zeta} \frac{u(t,x)}{u^{\star}(x)} = h(t,\zeta), & \zeta \in \partial\Omega, t > 0 \end{cases} \tag{P$_\bullet$}$$

has a unique boundary singular weak-dual solution for "nice data"

The end

Thank you very much!

Muchas gracias