Flatness results for stable solutions to some nonlocal problems

Eleonora Cinti

Università di Bologna

Madrid, June 14th, 2022.

(joint with X. Cabré, J. Serra, and E. Valdinoci)

A conjecture of De Giorgi (1978)

Let $u : \mathbb{R}^n \to (-1,1)$ be a solution of

$$-\Delta u = u - u^3 \quad \text{in} \quad \mathbb{R}^n$$

such that $\partial_{x_n} u > 0$.

Then, at least if $n \leq 8$, u is $1D$, that is all the level sets $\{u = t\}$ of u are hyperplanes.

A conjecture of De Giorgi (1978)

Let $u : \mathbb{R}^n \to (-1, 1)$ be a solution of

$$-\Delta u = u - u^3 \quad \text{in} \;\; \mathbb{R}^n$$

such that $\partial_{x_n} u > 0$.

Then, at least if $n \leq 8$, u is $1D$, that is all the level sets $\{u = t\}$ of u are hyperplanes.

Connection with area minimizing surfaces

The energy functional is

$$E_\Omega(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + W(u),$$

where $W(u) = \frac{1}{4}(1-u^2)^2$.

Theorem (Modica-Mortola)

$$"E_\varepsilon(u) = \frac{1}{2}\varepsilon\int |\nabla u|^2 + \frac{1}{\varepsilon}W(u) \quad \xrightarrow{\Gamma} \quad \mathrm{Per}"$$

Connection with area minimizing surfaces

The energy functional is

$$E_\Omega(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + W(u),$$

where $W(u) = \frac{1}{4}(1-u^2)^2$.

Theorem (Modica-Mortola)

$$''E_\varepsilon(u) = \frac{1}{2}\varepsilon \int |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \quad \xrightarrow{\Gamma} \quad \mathrm{Per}''$$

Classification for classical area minimizing surfaces

- Every minimal cone in $\mathbb{R}^n$ is an hyperplane, whenever $n \leq 7$
- In $\mathbb{R}^8$ the Simons cone defined as

$$\mathcal{C} := \{x \in \mathbb{R}^8 \,|\, x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2\}$$

  is a minimizer for the perimeter;
- If E is a minimizer of the perimeter functional in all $\mathbb{R}^n$, then E is a halfspace, whenever $n \leq 7$.
- If E is a minimizer of the perimeter functional and ∂E is a graph, then E is a halfspace, whenever $n \leq 8$.

Classification for classical area minimizing surfaces

- Every minimal cone in $\mathbb{R}^n$ is an hyperplane, whenever $n \leq 7$
- In $\mathbb{R}^8$ the Simons cone defined as

$$\mathcal{C} := \{x \in \mathbb{R}^8 \,|\, x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2\}$$

is a minimizer for the perimeter;

- If E is a minimizer of the perimeter functional in all $\mathbb{R}^n$, then E is a halfspace, whenever $n \leq 7$.
- If E is a minimizer of the perimeter functional and ∂E is a graph, then E is a halfspace, whenever $n \leq 8$.

Classification for classical area minimizing surfaces

- Every minimal cone in $\mathbb{R}^n$ is an hyperplane, whenever $n \leq 7$
- In $\mathbb{R}^8$ the Simons cone defined as

$$\mathcal{C} := \{x \in \mathbb{R}^8 \,|\, x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2\}$$

  is a minimizer for the perimeter;
- If E is a minimizer of the perimeter functional in all $\mathbb{R}^n$, then E is a halfspace, whenever $n \leq 7$.
- If E is a minimizer of the perimeter functional and ∂E is a graph, then E is a halfspace, whenever $n \leq 8$.

Classification for classical area minimizing surfaces

- Every minimal cone in $\mathbb{R}^n$ is an hyperplane, whenever $n \leq 7$
- In $\mathbb{R}^8$ the Simons cone defined as

$$\mathcal{C} := \{x \in \mathbb{R}^8 \,|\, x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2\}$$

  is a minimizer for the perimeter;
- If E is a minimizer of the perimeter functional in all $\mathbb{R}^n$, then E is a halfspace, whenever $n \leq 7$.
- If E is a minimizer of the perimeter functional and ∂E is a graph, then E is a halfspace, whenever $n \leq 8$.

Qs. What about stable sets for the Perimeter (i.e. $\frac{d^2}{dt^2}|_{\{t=0\}} Per(\Phi_t(E)) \geq 0$)?

- Embedded stable minimal surfaces in $\mathbb{R}^n$ are flat in dimension $n = 3$ [Fisher-Colbrie and Schoen, Do Carmo and Peng], and $n = 4$ [Chodosh-Li].
- **Conjecture** Embedded stable minimal surfaces in $\mathbb{R}^n$ are flat for $5 \leq n \leq 7$ **OPEN**.

Remark

The conjecture holds true for stable cones.

. One of the main obstruction is an **energy estimate** for **stable** sets:

$$\mathrm{Per}(E, B_R) \leq CR^{n-1}.$$

Qs. What about stable sets for the Perimeter (i.e. $\frac{d^2}{dt^2}|_{\{t=0\}} Per(\Phi_t(E)) \geq 0$)?

- Embedded stable minimal surfaces in $\mathbb{R}^n$ are flat in dimension $n = 3$ [Fisher-Colbrie and Schoen, Do Carmo and Peng], and $n = 4$ [Chodosh-Li].
- **Conjecture** Embedded stable minimal surfaces in $\mathbb{R}^n$ are flat for $5 \leq n \leq 7$ **OPEN**.

Remark

The conjecture holds true for stable cones.

. One of the main obstruction is an **energy estimate** for **stable** sets:

$$\mathrm{Per}(E, B_R) \leq CR^{n-1}.$$

Qs. What about stable sets for the Perimeter (i.e. $\frac{d^2}{dt^2}|_{\{t=0\}} Per(\Phi_t(E)) \geq 0$)?

- Embedded stable minimal surfaces in $\mathbb{R}^n$ are flat in dimension $n = 3$ [Fisher-Colbrie and Schoen, Do Carmo and Peng], and $n = 4$ [Chodosh-Li].
- **Conjecture** Embedded stable minimal surfaces in $\mathbb{R}^n$ are flat for $5 \leq n \leq 7$ **OPEN**.

Remark

The conjecture holds true for stable cones.

. One of the main obstruction is an **energy estimate** for **stable** sets:

$$\mathrm{Per}(E, B_R) \leq CR^{n-1}.$$

Qs. What about stable sets for the Perimeter (i.e. $\frac{d^2}{dt^2}|_{\{t=0\}} Per(\Phi_t(E)) \geq 0$)?

- Embedded stable minimal surfaces in $\mathbb{R}^n$ are flat in dimension $n = 3$ [Fisher-Colbrie and Schoen, Do Carmo and Peng], and $n = 4$ [Chodosh-Li].
- **Conjecture** Embedded stable minimal surfaces in $\mathbb{R}^n$ are flat for $5 \leq n \leq 7$ **OPEN**.

Remark

The conjecture holds true for stable cones.

. One of the main obstruction is an **energy estimate** for **stable** sets:

$$\mathrm{Per}(E, B_R) \leq CR^{n-1}.$$

Qs. What about stable sets for the Perimeter (i.e. $\frac{d^2}{dt^2}|_{\{t=0\}} Per(\Phi_t(E)) \geq 0$)?

- Embedded stable minimal surfaces in $\mathbb{R}^n$ are flat in dimension $n = 3$ [Fisher-Colbrie and Schoen, Do Carmo and Peng], and $n = 4$ [Chodosh-Li].
- **Conjecture** Embedded stable minimal surfaces in $\mathbb{R}^n$ are flat for $5 \leq n \leq 7$ **OPEN**.

Remark

The conjecture holds true for stable cones.

. One of the main obstruction is an **energy estimate** for **stable** sets:

$$\mathrm{Per}(E, B_R) \leq CR^{n-1}.$$

Qs. What about stable sets for the Perimeter (i.e. $\frac{d^2}{dt^2}|_{\{t=0\}} Per(\Phi_t(E)) \geq 0$)?

- Embedded stable minimal surfaces in $\mathbb{R}^n$ are flat in dimension $n = 3$ [Fisher-Colbrie and Schoen, Do Carmo and Peng], and $n = 4$ [Chodosh-Li].
- **Conjecture** Embedded stable minimal surfaces in $\mathbb{R}^n$ are flat for $5 \leq n \leq 7$ **OPEN**.

Remark

The conjecture holds true for stable cones.

. One of the main obstruction is an **energy estimate** for **stable** sets:

$$\mathrm{Per}(E, B_R) \leq CR^{n-1}.$$

Back to De Giorgi conjecture

For **Monotone solutions**

- $n = 2, 3$
- $4 \leq n \leq 8$ if, in addition, $u \to \pm 1$ for $x_n \to \pm\infty$
- counterexample for $n \geq 9$.

For **Minimizers**

- $2 \leq n \leq 7$
- counterexample for $n \geq 8$.

Back to De Giorgi conjecture

For **Monotone solutions**

- $n = 2, 3$
- $4 \leq n \leq 8$ if, in addition, $u \to \pm 1$ for $x_n \to \pm\infty$
- counterexample for $n \geq 9$.

For **Minimizers**

- $2 \leq n \leq 7$
- counterexample for $n \geq 8$.

For **stable solutions**

- for $n = 2$ TRUE;
- counterexample for $n \geq 8$;
- for $3 \leq n \leq 7$ OPEN.

The fractional Allen-Cahn equation

We consider now the same question for the nonlocal problem

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^n, \quad s \in (0,1).$$

The fractional Laplacian is defined as

$$(-\Delta)^s u(x) = C_{n,s}\, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy.$$

The fractional Allen-Cahn equation

We consider now the same question for the nonlocal problem

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^n, \quad s \in (0,1).$$

The fractional Laplacian is defined as

$$(-\Delta)^s u(x) = C_{n,s} \, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy.$$

The fractional Allen-Cahn equation

We consider now the same question for the nonlocal problem

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^n, \quad s \in (0,1).$$

The fractional Laplacian is defined as

$$(-\Delta)^s u(x) = C_{n,s} \, P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy.$$

Modica-Mortola type result

The energy functional is

$$E_{s,\Omega}(u) = \iint_{\mathbb{R}^n \times \mathbb{R}^n \setminus \Omega^c \times \Omega^c} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx\, dy + \int_\Omega W(u) dx.$$

After a suitable rescaling we have that

Theorem (Savin and Valdinoci)

$$E^{\varepsilon}_{s,\Omega}(u) \xrightarrow{\Gamma} \begin{cases} \mathrm{Per}_s & 0 < s < 1/2 \\ \mathrm{Per} & 1/2 \le s < 1. \end{cases}$$

Modica-Mortola type result

The energy functional is

$$E_{s,\Omega}(u) = \iint_{\mathbb{R}^n \times \mathbb{R}^n \setminus \Omega^c \times \Omega^c} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx\, dy + \int_\Omega W(u) dx.$$

After a suitable rescaling we have that

Theorem (Savin and Valdinoci)

$$E^\varepsilon_{s,\Omega}(u) \xrightarrow{\Gamma} \begin{cases} \mathrm{Per}_s & 0 < s < 1/2 \\ \mathrm{Per} & 1/2 \leq s < 1. \end{cases}$$

The case $0 < s < 1/2$

We recall that for $0 < s < 1/2$

$$E_s^{\varepsilon}(u) \xrightarrow{\Gamma} \mathrm{Per}_s.$$

What is the **fractional perimeter**?

$$\mathrm{Per}_s(E) = \frac{1}{2}[\chi_E]_{W^{2s,1}} = \int_E \int_{\mathbb{R}^n \setminus E} \frac{dx\, dy}{|x-y|^{n+2s}}.$$

The case $0 < s < 1/2$

We recall that for $0 < s < 1/2$

$$E_s^\varepsilon(u) \xrightarrow{\Gamma} \mathrm{Per}_s.$$

What is the **fractional perimeter**?

$$\mathrm{Per}_s(E) = \frac{1}{2}[\chi_E]_{W^{2s,1}} = \int_E \int_{\mathbb{R}^n \setminus E} \frac{dx\, dy}{|x-y|^{n+2s}}.$$

The case $0 < s < 1/2$

We recall that for $0 < s < 1/2$

$$E_s^{\varepsilon}(u) \xrightarrow{\Gamma} \mathrm{Per}_s.$$

What is the **fractional perimeter**?

$$\mathrm{Per}_s(E) = \frac{1}{2}[\chi_E]_{W^{2s,1}} = \int_E \int_{\mathbb{R}^n \setminus E} \frac{dx\, dy}{|x-y|^{n+2s}}.$$

Classification for nonlocal minimal surfaces

- Any nonlocal minimal cone in $\mathbb{R}^2$ is an halfplane (Savin-Valdinoci).
- Any nonlocal minimal set in $\mathbb{R}^2$ is an halfplane (Savin-Valdinoci).
- Any nonlocal minimal graph in $\mathbb{R}^3$ is an halfplane (Figalli-Valdinoci).
- For $s \sim 1/2$ and $n < 8$ any nonlocal minimal set in $\mathbb{R}^n$ is an halfspace (Caffarelli-Valdinoci).

Classification for nonlocal minimal surfaces

- Any nonlocal minimal cone in $\mathbb{R}^2$ is an halfplane (Savin-Valdinoci).
- Any nonlocal minimal set in $\mathbb{R}^2$ is an halfplane (Savin-Valdinoci).
- Any nonlocal minimal graph in $\mathbb{R}^3$ is an halfplane (Figalli-Valdinoci).
- For $s \sim 1/2$ and $n < 8$ any nonlocal minimal set in $\mathbb{R}^n$ is an halfspace (Caffarelli-Valdinoci).

Classification for nonlocal minimal surfaces

- Any nonlocal minimal cone in $\mathbb{R}^2$ is an halfplane (Savin-Valdinoci).
- Any nonlocal minimal set in $\mathbb{R}^2$ is an halfplane (Savin-Valdinoci).
- Any nonlocal minimal graph in $\mathbb{R}^3$ is an halfplane (Figalli-Valdinoci).
- For $s \sim 1/2$ and $n < 8$ any nonlocal minimal set in $\mathbb{R}^n$ is an halfspace (Caffarelli-Valdinoci).

Classification for nonlocal minimal surfaces

- Any nonlocal minimal cone in $\mathbb{R}^2$ is an halfplane (Savin-Valdinoci).
- Any nonlocal minimal set in $\mathbb{R}^2$ is an halfplane (Savin-Valdinoci).
- Any nonlocal minimal graph in $\mathbb{R}^3$ is an halfplane (Figalli-Valdinoci).
- For $s \sim 1/2$ and $n < 8$ any nonlocal minimal set in $\mathbb{R}^n$ is an halfspace (Caffarelli-Valdinoci).

Back to the fractional Allen-Cahn equation

For $1/2 \le s < 1$

- $n = 2, 3$ for monotone sol.s and minimizers (Cabré, C., Sire, Valdinoci)
- $4 \le n \le 7$ for minimizers (Savin);
- $4 \le n \le 8$ for monotone sol. + limits (Savin).
- $n = 4$, $s = 1/2$ for monotone solutions (Figalli, Serra)

For $0 < s < 1/2$

- $n = 2$ for minimizers (Cabré, Sire, Valdinoci) ;
- $n = 2, 3$ for monotone solutions (Dipierro, Farina, Valdinoci) ;
- $3 \le n \le 7$ and $s \sim 1/2$ for minimizers (Dipierro, Serra, Valdinoci)
- $3 < n \le 8$ and $s \sim 1/2$ for monotone sol.+ limits (Dipierro,Serra,Valdinoci).

Back to the fractional Allen-Cahn equation

For $1/2 \leq s < 1$

- $n = 2, 3$ for monotone sol.s and minimizers (Cabré, C., Sire, Valdinoci)
- $4 \leq n \leq 7$ for minimizers (Savin);
- $4 \leq n \leq 8$ for monotone sol. + limits (Savin).
- $n = 4$, $s = 1/2$ for monotone solutions (Figalli, Serra)

For $0 < s < 1/2$

- $n = 2$ for minimizers (Cabré, Sire, Valdinoci) ;
- $n = 2, 3$ for monotone solutions (Dipierro, Farina, Valdinoci) ;
- $3 \leq n \leq 7$ and $s \sim 1/2$ for minimizers (Dipierro, Serra, Valdinoci)
- $3 \leq n \leq 8$ and $s \sim 1/2$ for monotone sol.+ limits (Dipierro,Serra,Valdinoci).

(joint with X. C

Back to the fractional Allen-Cahn equation

For $1/2 \leq s < 1$

- $n = 2, 3$ for monotone sol.s and minimizers (Cabré, C., Sire, Valdinoci)
- $4 \leq n \leq 7$ for minimizers (Savin);
- $4 \leq n \leq 8$ for monotone sol. + limits (Savin).
- $n = 4$, $s = 1/2$ for monotone solutions (Figalli, Serra)

For $0 < s < 1/2$

- $n = 2$ for minimizers (Cabré, Sire, Valdinoci) ;
- $n = 2, 3$ for monotone solutions (Dipierro, Farina, Valdinoci) ;
- $3 \leq n \leq 7$ and $s \sim 1/2$ for minimizers (Dipierro, Serra, Valdinoci)
- $3 < n \leq 8$ and $s \sim 1/2$ for monotone sol.+ limits (Dipierro,Serra,Valdinoci). (joint with X. C

Back to the fractional Allen-Cahn equation

For $1/2 \leq s < 1$

- $n = 2, 3$ for monotone sol.s and minimizers (Cabré, C., Sire, Valdinoci)
- $4 \leq n \leq 7$ for minimizers (Savin);
- $4 \leq n \leq 8$ for monotone sol. + limits (Savin).
- $n = 4$, $s = 1/2$ for monotone solutions (Figalli, Serra)

For $0 < s < 1/2$

- $n = 2$ for minimizers (Cabré, Sire, Valdinoci) ;
- $n = 2, 3$ for monotone solutions (Dipierro, Farina, Valdinoci) ;
- $3 \leq n \leq 7$ and $s \sim 1/2$ for minimizers (Dipierro, Serra, Valdinoci)
- $3 < n \leq 8$ and $s \sim 1/2$ for monotone sol.+ limits (Dipierro,Serra,Valdinoci). (joint with X. C

Back to the fractional Allen-Cahn equation

For $1/2 \leq s < 1$

- $n = 2, 3$ for monotone sol.s and minimizers (Cabré, C., Sire, Valdinoci)
- $4 \leq n \leq 7$ for minimizers (Savin);
- $4 \leq n \leq 8$ for monotone sol. + limits (Savin).
- $n = 4$, $s = 1/2$ for monotone solutions (Figalli, Serra)

For $0 < s < 1/2$

- $n = 2$ for minimizers (Cabré, Sire, Valdinoci) ;
- $n = 2, 3$ for monotone solutions (Dipierro, Farina, Valdinoci) ;
- $3 \leq n \leq 7$ and $s \sim 1/2$ for minimizers (Dipierro, Serra, Valdinoci)
- $3 < n \leq 8$ and $s \sim 1/2$ for monotone sol.+ limits (Dipierro,Serra,Valdinoci).

Back to the fractional Allen-Cahn equation

For $1/2 \leq s < 1$

- $n = 2, 3$ for monotone sol.s and minimizers (Cabré, C., Sire, Valdinoci)
- $4 \leq n \leq 7$ for minimizers (Savin);
- $4 \leq n \leq 8$ for monotone sol. + limits (Savin).
- $n = 4$, $s = 1/2$ for monotone solutions (Figalli, Serra)

For $0 < s < 1/2$

- $n = 2$ for minimizers (Cabré, Sire, Valdinoci) ;
- $n = 2, 3$ for monotone solutions (Dipierro, Farina, Valdinoci) ;
- $3 \leq n \leq 7$ and $s \sim 1/2$ for minimizers (Dipierro, Serra, Valdinoci)
- $3 < n \leq 8$ and $s \sim 1/2$ for monotone sol.+ limits (Dipierro,Serra, Valdinoci). (joint with X. C

Back to the fractional Allen-Cahn equation

For $1/2 \leq s < 1$

- $n = 2, 3$ for monotone sol.s and minimizers (Cabré, C., Sire, Valdinoci)
- $4 \leq n \leq 7$ for minimizers (Savin);
- $4 \leq n \leq 8$ for monotone sol. + limits (Savin).
- $n = 4$, $s = 1/2$ for monotone solutions (Figalli, Serra)

For $0 < s < 1/2$

- $n = 2$ for minimizers (Cabré, Sire, Valdinoci) ;
- $n = 2, 3$ for monotone solutions (Dipierro, Farina, Valdinoci) ;
- $3 \leq n \leq 7$ and $s \sim 1/2$ for minimizers (Dipierro, Serra, Valdinoci)
- $3 < n \leq 8$ and $s \sim 1/2$ for monotone sol.+ limits (Dipierro,Serra,Valdinoci). (joint with X. C

Back to the fractional Allen-Cahn equation

For $1/2 \leq s < 1$

- $n = 2, 3$ for monotone sol.s and minimizers (Cabré, C., Sire, Valdinoci)
- $4 \leq n \leq 7$ for minimizers (Savin);
- $4 \leq n \leq 8$ for monotone sol. + limits (Savin).
- $n = 4$, $s = 1/2$ for monotone solutions (Figalli, Serra)

For $0 < s < 1/2$

- $n = 2$ for minimizers (Cabré, Sire, Valdinoci) ;
- $n = 2, 3$ for monotone solutions (Dipierro, Farina, Valdinoci) ;
- $3 \leq n \leq 7$ and $s \sim 1/2$ for minimizers (Dipierro, Serra, Valdinoci)
- $3 < n \leq 8$ and $s \sim 1/2$ for monotone sol.+ limits (Dipierro,Serra,Valdinoci). (Joint with X. C

The low-dimensional case

Principal ingredients in the proof of the De Giorgi conjecture in low dimensions:

- Stability of solutions;
- Energy estimate:

$$E_{s,B_R}(u) \leq CR^2 \log R;$$

- Liouville-type result.

The low-dimensional case

Principal ingredients in the proof of the De Giorgi conjecture in low dimensions:

- Stability of solutions;
- Energy estimate:

$$E_{s,B_R}(u) \leq CR^2 \log R;$$

- Liouville-type result.

The low-dimensional case

Principal ingredients in the proof of the De Giorgi conjecture in low dimensions:

- Stability of solutions;
- Energy estimate:

$$E_{s,B_R}(u) \leq CR^2 \log R;$$

- Liouville-type result.

The low-dimensional case

Principal ingredients in the proof of the De Giorgi conjecture in low dimensions:

- Stability of solutions;
- Energy estimate:

$$E_{s,B_R}(u) \leq CR^2 \log R;$$

- Liouville-type result.

The high-dimensional case

If $n > 3$ the proof is more involved and needs several ingredients:

- convergence (in the Hausdorff sense) of the blow-downs of solutions to area minimizing surfaces;
- classification of area minimizing surfaces;
- improvement of flatness.

The high-dimensional case

If $n > 3$ the proof is more involved and needs several ingredients:

- convergence (in the Hausdorff sense) of the blow-downs of solutions to area minimizing surfaces;
- classification of area minimizing surfaces;
- improvement of flatness.

The high-dimensional case

If $n > 3$ the proof is more involved and needs several ingredients:

- convergence (in the Hausdorff sense) of the blow-downs of solutions to area minimizing surfaces;
- classification of area minimizing surfaces;
- improvement of flatness.

The high-dimensional case

If $n > 3$ the proof is more involved and needs several ingredients:

- convergence (in the Hausdorff sense) of the blow-downs of solutions to area minimizing surfaces;
- classification of area minimizing surfaces;
- improvement of flatness.

Remark

The following implication holds for all $s \in (0,1]$:

$$\left.\begin{array}{c}\text{Stable sol'ns in } \mathbb{R}^{n-1} \text{ are 1D} \\ \text{and} \\ \text{Minimizers in } \mathbb{R}^{n} \text{ are 1D}\end{array}\right\} \Rightarrow \text{Monotone sol'ns in } \mathbb{R}^{n} \text{ are 1D.}$$

What about stable objects?

Theorem (C.,Serra, Valdinoci (2019))

Let E be a stable set for the s-perimeter in B_{4R}.

Then,

$$\mathrm{Per}_{s,B_R}(E) \le C(n,s) R^{n-2s};$$

$$\mathrm{Per}_{B_R}(E) \le C(n,s) R^{n-1}.$$

Corollary (C.,Serra, Valdinoci (2019))

Let E be stable set for the s-perimeter in $\mathbb{R}^2$. Then E is an half-plane.

What about stable objects?

Theorem (C.,Serra, Valdinoci (2019))

Let E be a stable set for the s-perimeter in B_{4R}.

Then,

$$\mathrm{Per}_{s,B_R}(E) \leq C(n,s)R^{n-2s};$$

$$\mathrm{Per}_{B_R}(E) \leq C(n,s)R^{n-1}.$$

Corollary (C.,Serra, Valdinoci (2019))

Let E be stable set for the s-perimeter in $\mathbb{R}^2$. Then E is an half-plane.

What about stable objects?

Theorem (C.,Serra, Valdinoci (2019))

Let E be a stable set for the s-perimeter in B_{4R}.

Then,

$$\mathrm{Per}_{s,B_R}(E) \leq C(n,s)R^{n-2s};$$

$$\mathrm{Per}_{B_R}(E) \leq C(n,s)R^{n-1}.$$

Corollary (C.,Serra, Valdinoci (2019))

Let E be stable set for the s-perimeter in $\mathbb{R}^2$. Then E is an half-plane.

What about stable objects?

Theorem (C.,Serra, Valdinoci (2019))

Let E be a stable set for the s-perimeter in B_{4R}.

Then,

$$\mathrm{Per}_{s,B_R}(E) \le C(n,s)R^{n-2s};$$

$$\mathrm{Per}_{B_R}(E) \le C(n,s)R^{n-1}.$$

Corollary (C.,Serra, Valdinoci (2019))

Let E be stable set for the s-perimeter in $\mathbb{R}^2$. Then E is an half-plane.

What about stable objects?

Theorem (C.,Serra, Valdinoci (2019))

Let E be a stable set for the s-perimeter in B_{4R}.

Then,

$$\mathrm{Per}_{s,B_R}(E) \leq C(n,s)R^{n-2s};$$

$$\mathrm{Per}_{B_R}(E) \leq C(n,s)R^{n-1}.$$

Corollary (C.,Serra, Valdinoci (2019))

Let E be stable set for the s-perimeter in $\mathbb{R}^2$. Then E is an half-plane.

Theorem (Cabré, C., Serra (2020))

There exists $s_* \in (0, 1/2)$ *such that for every* $s \in (s_*, 1/2)$ *the following statement holds. Let* $\Sigma \subset \mathbb{R}^3$ *be a cone with nonempty boundary and* C^2 *away from* 0*.*

Assume that Σ *is stable.*

Then, Σ *is a half space.*

Idea of the proof. The proof combines the following three main ingredients:

- uniform perimeter estimates for stable sets of [C., Serra, Valdinoci];
- the second variation formula for the s-perimeter;
- the fractional Hardy inequality with optimal constant (with the precise dependence as $s \uparrow 1/2$).

Theorem (Cabré, C., Serra (2020))

There exists $s_* \in (0,1/2)$ *such that for every* $s \in (s_*, 1/2)$ *the following statement holds. Let* $\Sigma \subset \mathbb{R}^3$ *be a cone with nonempty boundary and* C^2 *away from* 0.

Assume that Σ *is stable.*

Then, Σ *is a half space.*

Idea of the proof. The proof combines the following three main ingredients:

- uniform perimeter estimates for stable sets of [C., Serra, Valdinoci];
- the second variation formula for the s-perimeter;
- the fractional Hardy inequality with optimal constant (with the precise dependence as $s \uparrow 1/2$).

Theorem (Cabré, C., Serra (2020))

There exists $s_* \in (0, 1/2)$ *such that for every* $s \in (s_*, 1/2)$ *the following statement holds. Let* $\Sigma \subset \mathbb{R}^3$ *be a cone with nonempty boundary and* C^2 *away from* 0.

Assume that Σ *is stable.*

Then, Σ *is a half space.*

Idea of the proof. The proof combines the following three main ingredients:

- uniform perimeter estimates for stable sets of [C., Serra, Valdinoci];
- the second variation formula for the s-perimeter;
- the fractional Hardy inequality with optimal constant (with the precise dependence as $s \uparrow 1/2$).

Theorem (Cabré, C., Serra (2020))

There exists $s_* \in (0, 1/2)$ *such that for every* $s \in (s_*, 1/2)$ *the following statement holds. Let* $\Sigma \subset \mathbb{R}^3$ *be a cone with nonempty boundary and* C^2 *away from* 0.

Assume that Σ *is stable.*

Then, Σ *is a half space.*

Idea of the proof. The proof combines the following three main ingredients:

- uniform perimeter estimates for stable sets of [C., Serra, Valdinoci];
- the second variation formula for the s-perimeter;
- the fractional Hardy inequality with optimal constant (with the precise dependence as $s \uparrow 1/2$).

Theorem (Cabré, C., Serra (2020))

There exists $s_* \in (0, 1/2)$ *such that for every* $s \in (s_*, 1/2)$ *the following statement holds. Let* $\Sigma \subset \mathbb{R}^3$ *be a cone with nonempty boundary and* C^2 *away from* 0*.*

Assume that Σ *is stable.*

Then, Σ *is a half space.*

Idea of the proof. The proof combines the following three main ingredients:

- uniform perimeter estimates for stable sets of [C., Serra, Valdinoci];
- the second variation formula for the s-perimeter;
- the fractional Hardy inequality with optimal constant (with the precise dependence as $s \uparrow 1/2$).

Fractional Allen-Cahn for $0 < s < 1/2$

Theorem 1 (Cabré, C., Serra (2021))

Let $0 < s < 1/2$. *Let* u *be a bounded stable solution of* $(-\Delta)^s u = u - u^3$ *in* $\mathbb{R}^n$.

Then,

$$\int_{B_R} |\nabla u|\, dx \leq C_{n,s} R^{n-1} \quad \text{for all } R \geq 1, \tag{1}$$

and

$$E_{s,B_R}(u) \leq C_{n,s} R^{n-2s} \quad \text{for all } R \geq 1. \tag{2}$$

Sketch of the Proof of Theorem 1.

- To prove the BV-estimate we use the ideas developed in [C., Serra, Valdinoci] for analogue estimate for stable s-minimal sets;
- After interpolation, the BV-estimate $\Rightarrow$ estimate for the Dirichlet energy.
- We prove that for stable solutions

$$E_s^{Pot}(u) \leq C E_s^{Dir}(u).$$

Sketch of the Proof of Theorem 1.

- To prove the BV-estimate we use the ideas developed in [C., Serra, Valdinoci] for analogue estimate for stable s-minimal sets;
- After interpolation, the BV-estimate $\Rightarrow$ estimate for the Dirichlet energy.
- We prove that for stable solutions

$$E_s^{Pot}(u) \leq C E_s^{Dir}(u).$$

Sketch of the Proof of Theorem 1.

- To prove the BV-estimate we use the ideas developed in [C., Serra, Valdinoci] for analogue estimate for stable s-minimal sets;
- After interpolation, the BV-estimate $\Rightarrow$ estimate for the Dirichlet energy.
- We prove that for stable solutions

$$E_s^{Pot}(u) \leq C E_s^{Dir}(u).$$

Sketch of the Proof of Theorem 1.

- To prove the BV-estimate we use the ideas developed in [C., Serra, Valdinoci] for analogue estimate for stable s-minimal sets;
- After interpolation, the BV-estimate $\Rightarrow$ estimate for the Dirichlet energy.
- We prove that for stable solutions

$$E_s^{Pot}(u) \leq C E_s^{Dir}(u).$$

Sketch of the Proof of Theorem 1.

- To prove the BV-estimate we use the ideas developed in [C., Serra, Valdinoci] for analogue estimate for stable s-minimal sets;
- After interpolation, the BV-estimate $\Rightarrow$ estimate for the Dirichlet energy.
- We prove that for stable solutions

$$E_s^{Pot}(u) \leq C E_s^{Dir}(u).$$

Theorem 2 (Cabré, C., Serra)

Let u be a bounded stable solution of $(-\Delta)^s u = u - u^3$ in $\mathbb{R}^n$.

Then, for any given blow-down sequence $u_{R_j}(x) = u(Rx)$ with $R_j \uparrow \infty$, there is a subsequence R_{j_m}

$$u_{R_{j_m}} \to \chi_\Sigma - \chi_{\Sigma^c} \quad \text{in } L^1(B_1),$$

for some cone Σ that is a stable set for the fractional perimeter Per_s under smooth, compactly supported deformations.

Moreover, for all given $t \in (-1,1)$, we have

$$d_{\mathrm{Hausdorff}}\big(\,\{u_R \le t\} \cap B_1\,,\, \Sigma \cap B_1\,\big) \to 0.$$

Sketch of the proof. The proof combines Theorem 1, a monotonicity formula and density estimates for stable solutions.

Theorem 2 (Cabré, C., Serra)

Let u be a bounded stable solution of $(-\Delta)^s u = u - u^3$ in $\mathbb{R}^n$.

Then, for any given blow-down sequence $u_{R_j}(x) = u(Rx)$ with $R_j \uparrow \infty$, there is a subsequence R_{j_m}

$$u_{R_{j_m}} \to \chi_\Sigma - \chi_{\Sigma^c} \quad \text{in } L^1(B_1),$$

for some cone Σ that is a stable set for the fractional perimeter Per_s under smooth, compactly supported deformations.

Moreover, for all given $t \in (-1, 1)$, we have

$$d_{\mathrm{Hausdorff}}\left(\{u_R \le t\} \cap B_1 \,,\, \Sigma \cap B_1 \right) \to 0.$$

Sketch of the proof. The proof combines Theorem 1, a monotonicity formula and density estimates for stable solutions.

Abstract classification result

Theorem 3 (Cabré, C., Serra)

Assume that for some pair (n,s) with $s \in (0, 1/2)$ *half spaces are the only stable cones for the s-perimeter.*

Then, every stable solution of $(-\Delta)^s u = u - u^3$ *in* $\mathbb{R}^n$ *is* $1D$.

Ingredients of the proof. The proof combines Theorem 2 and the *improvement of flatness* result by Dipierro, Serra, Valdinoci.

Abstract classification result

Theorem 3 (Cabré, C., Serra)

Assume that for some pair (n,s) with $s \in (0, 1/2)$ *half spaces are the only stable cones for the s-perimeter.*

Then, every stable solution of $(-\Delta)^s u = u - u^3$ *in* $\mathbb{R}^n$ *is* $1D$*.*

Ingredients of the proof. The proof combines Theorem 2 and the *improvement of flatness* result by Dipierro, Serra, Valdinoci.

Corollary

Let $0 < s < 1/2$. Let u be a bounded solution of

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^n.$$

Then, u is $1D$ in the following cases:

- $n = 3$, s close to $1/2$, u is stable;
- $n = 4$, s close to $1/2$, u is monotone.

Theorem (Figalli, Serra)

Let u be a bounded stable solution of $(-\Delta)^{1/2} u = u - u^3 \quad \text{in } \mathbb{R}^n$.

Then u is $1D$ in the following cases:

- $n = 3$, u is stable;
- $n = 4$, u is monotone.

Corollary

Let $0 < s < 1/2$. Let u be a bounded solution of

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^n.$$

Then, u is $1D$ in the following cases:

- *$n = 3$, s close to $1/2$, u is stable;*
- *$n = 4$, s close to $1/2$, u is monotone.*

Theorem (Figalli, Serra)

Let u be a bounded stable solution of $(-\Delta)^{1/2} u = u - u^3 \quad \text{in } \mathbb{R}^n$.

Then u is $1D$ in the following cases:

- *$n = 3$, u is stable;*
- *$n = 4$, u is monotone.*

Corollary

Let $0 < s < 1/2$. Let u be a bounded solution of

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^n.$$

Then, u is $1D$ in the following cases:

- *$n = 3$, s close to $1/2$, u is stable;*
- *$n = 4$, s close to $1/2$, u is monotone.*

Theorem (Figalli, Serra)

Let u be a bounded stable solution of $(-\Delta)^{1/2} u = u - u^3 \quad \text{in } \mathbb{R}^n$.

Then u is $1D$ in the following cases:

- *$n = 3$, u is stable;*
- *$n = 4$, u is monotone.*

Corollary

Let $0 < s < 1/2$. *Let* u *be a bounded solution of*

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^n.$$

Then, u *is* $1D$ *in the following cases:*

- $n = 3$, *s close to* $1/2$, u *is stable;*
- $n = 4$, *s close to* $1/2$, u *is monotone.*

Theorem (Figalli, Serra)

Let u *be a bounded stable solution of* $(-\Delta)^{1/2} u = u - u^3 \quad \text{in } \mathbb{R}^n$.

Then u *is* $1D$ *in the following cases:*

- $n = 3$, u *is stable;*
- $n = 4$, u *is monotone.*

Stable minimal surfaces

	Per_s	Per
perimeter estimates	$\forall\, n$	$n = 2,\ 3,\ 4$
flatness of stable cones	$n = 2,\ n = 3$ and $s \sim 1/2$	$2 \leq n \leq 7$

Stable solutions to the fractional Allen-Cahn

	Case $0 < s < 1/2$	Case $1/2 \leq s \leq 1$
energy estimates	$\forall\, n$	$n = 2$, $n = 3$ and $s = 1/2$
1D symmetry	$n = 2$, $n = 3$ and $s \sim 1/2$	$n = 2$, $n = 3$ and $s = 1/2$

Muchas gracias!!