Stable cones in the fractional Alt-Caffarelli problem

Xavier Fernández-Real

Joint work with X. Ros-Oton

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Regularity for nonlinear diffusion equations.

Green functions and functional inequalities

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The classical one-phase problem

- The one-phase problem is a classical (variational) free boundary problem, also known as the Bernoulli problem or the Alt-Caffarelli problem,
- It is a typical model for flame propagation or jet-flows,
- From a mathematical point of view, it was originally studied by Alt and Caffarelli in 1980s, with many contributions since then that keep it an active research topic.

The variational problem consists in minimizing an energy that presents a discontinuity at $u = 0$. That is,

$$\boxed{\mathcal{J}[u] := [u]^2_{H^1(B_1)} + |\{u > 0\} \cap B_1|.}$$

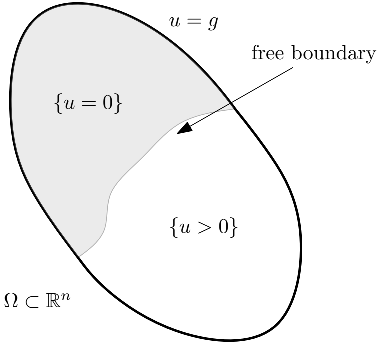
The classical one-phase problem

Given

- a domain $\Omega \subset \mathbb{R}^n$
- a boundary datum $g : \partial\Omega \to \mathbb{R}$,

we minimize

$$\mathcal{J}[u] := \int_\Omega |\nabla u|^2 + |\{u > 0\} \cap \Omega| = \int_\Omega \left(|\nabla u|^2 + \chi_{\{u>0\}}\right).$$

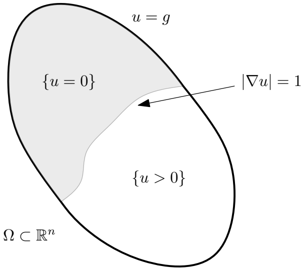


The PDE satisfied by u

The Euler-Lagrange equations characterising critical points of the previous functional are

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \\ |\nabla u| = 1 & \text{on } \partial\{u > 0\}. \end{cases}$$

Not all critical points are minimizers (or even, stable)!



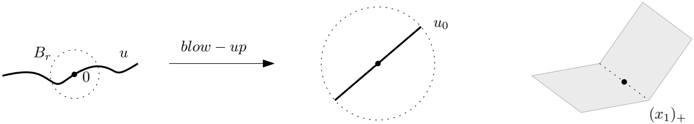
- There exists a minimizer. However, it is not necessarily unique!
- The minimizer satisfies $u \in \mathrm{Lip}(\Omega)$.
- In dimensions $n \leq 4$, free boundaries are C^∞.
- In dimensions $n \geq 7$, there may be singular points.

Main open problem:

What about dimensions $n = 5, 6$?

Strategy of the proofs

The strategy is heavily inspired by that in the regularity of minimizing minimal surfaces.



- The problem scale invariance allows us to do Lipschitz blow-ups, $u_r(x) = \frac{u(rx)}{r}$, around free boundary points.
- A monotonicity formula implies that blow-ups are 1-homogeneous.
- We need to classify blow-ups. In dimensions $n \leq 4$, blow-ups are 1-dimensional.
- In particular, for $n \leq 4$, the free boundary is contained in an increasingly flatter strip.
- Then we need an improvement of flatness result. If the free boundary is flat enough, then it is $C^{1,\alpha}$.
- We finish by upgrading the regularity from $C^{1,\alpha}$ to C^∞.

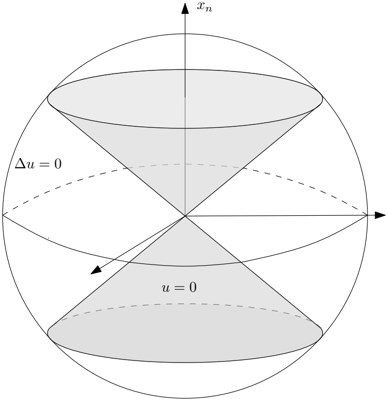
Main obstruction: Classification of blow-ups

Blow-ups are global 1-homogeneous solutions to the one-phase problem. In particular, if u_0 is a blow-up, then it must satisfy

$$(*) \quad \begin{cases} \Delta u_0 = 0 & \text{in } \{u_0 > 0\} \\ |\nabla u_0| = 1 & \text{on } \partial\{u_0 > 0\} \\ u_0 \text{ is 1-homogeneous} \end{cases}$$

We know:

- Solutions to $(*)$ in $\mathbb{R}^2$ are necessarily 1-dimensional.
- For any $n \geq 3$, there always exist an axially symmetric solution to $(*)$.
- Such solution is not a minimizer for $n \leq 6$.
- For $n \geq 7$, it is a minimizer.



Classification of blow-ups

Theorem (Caffarelli-Jerison-Kenig 2000, Jerison-Savin 2015)

For $n = 3, 4$, all blow-ups are 1-dimensional.

- This is a difficult theorem. It remains an open question for $n = 5, 6$.
- Minimality must be used. That is, the PDE is not enough.

Theorem (Caffarelli-Jerison-Kenig 2000)

If u is a minimizer, then

$$\int_{\{u>0\}} |\nabla \eta|^2 \geq \int_{\partial\{u>0\}} H\eta^2 \qquad \textit{for all} \quad \eta \in C_c^\infty(\Omega),$$

where H denotes the mean curvature of the free boundary.

The previous theorem corresponds to the **stability condition**.

Classification of blow-ups

The stability condition

$$\int_{\{u>0\}} |\nabla \eta|^2 \geq \int_{\partial\{u>0\}} H\eta^2 \qquad \text{for all} \quad \eta \in C_c^\infty(\Omega),$$

is used to

- Classify blow-ups in dimensions $n \leq 4$,
- Understand axially symmetric solutions for $n \leq 6$.

Theorem

Let k_1^* be the lowest dimension in which non-trivial blow-ups appear (currently, we know $5 \leq k_1^* \leq 7$). Then, free boundaries are smooth up to dimension $k_1^* - 1$, and in general the singular set has dimension $n - k_1^*$.

The thin (or fractional) one-phase free boundary problem

As before, we minimize an energy that presents a discontinuity at $u = 0$. That is, for $s \in (0, 1)$,

$$\boxed{\mathcal{J}_s[u] := [u]^2_{H^s(\mathbb{R}^n)} + |\{u > 0\} \cap B_1|.}$$

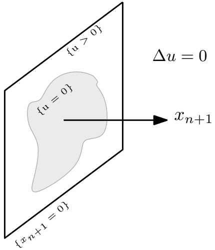
We consider now fractional energies. In particular, we say that u is a minimizer of the previous functional if $\tilde{\mathcal{J}}[u] \le \tilde{\mathcal{J}}[v]$ for all v such that $v - u \in H^s(\mathbb{R}^n)$ and $v \equiv u$ in $\mathbb{R}^n \setminus B_1$, where

$$\tilde{\mathcal{J}}_s[v] := \frac{c_{n,s}}{2} \iint_{\mathbb{R}^{2n} \setminus (B_1^c)^2} \frac{(v(x) - v(y))^2}{|x - y|^{n+2s}}\, dx\, dy + |\{v > 0\} \cap B_1|.$$

First studied by Caffarelli-Roquejoffre-Sire in 2010, motivated by flame propagation models.

By the Caffarelli-Silvestre extension, we can consider the thin one-phase free boundary problem. Namely, minimization of

$$\mathcal{I}[w] := [w]^2_{H^1(B_1, |y|^{1-2s})} + \mathcal{H}^n(\{x_{n+1} = 0, w > 0\}).$$



The thin (or fractional) one-phase free boundary problem

The steps in the study of the free boundary follow the same structure as in the local case.

However, one needs new techniques an approaches for each result.

Theorem (De Silva, Savin 2015; Engelstein, Kauranen, Prats, Sakellaris, Sire 2021)

Free boundaries are smooth outside of a singular set Σ with

$$\dim(\Sigma) \leq n - k_s^*$$

where the value k_s^* is the lowest dimension in which nontrivial blow-ups appear.

This is the motivation of our work.

What is the first dimension in which non-trivial blow-ups appear?

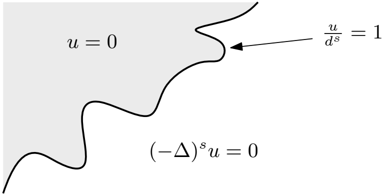
In $\mathbb{R}^2$, all blow-ups are 1D.

In particular, there are no singular points in $\mathbb{R}^2$.

Criticality condition

The PDE satisfied by minimizers (and more generally, by critical points) is the following

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \{u > 0\} \\ u/d^s = 1 & \text{on } \partial\{u > 0\}. \end{cases}$$

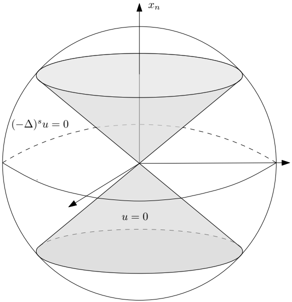


- Solutions are C^s, and blow-ups are homogeneous with homogeneity s.
- Blow-ups, therefore, satisfy.

$$\boxed{\begin{cases} (-\Delta)^s u_0 = 0 & \text{in } \{u_0 > 0\} \\ u_0/d^s = 1 & \text{on } \partial\{u_0 > 0\} \\ u_0 \text{ is } s\text{-homogeneous} \end{cases}}$$

Axially symmetric solutions

$$(*) \quad \begin{cases} (-\Delta)^s u_0 = 0 & \text{in } \{u_0 > 0\} \\ u_0/d^s = 1 & \text{on } \partial\{u_0 > 0\} \\ u_0 \text{ is } s\text{-homogeneous} \end{cases}$$



- One can show that there is an axially symmetric solution to $(*)$ in any dimension $n \geq 2$.

In which dimensions are these axially symmetric solutions minimizers?

For this, we need the following:

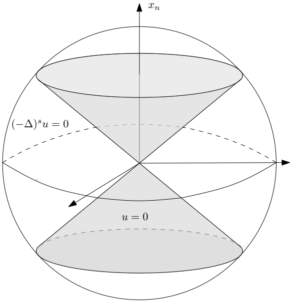
What is the stability condition for the thin one-phase problem?

Main results

Theorem (F., Ros-Oton, 2022)

Let u_0 be any global s-homogeneous stable solution, with $s \in (0,1)$. Assume that u_0 is axially symmetric. Then, if $n \leq 5$, u_0 is one-dimensional.

- Nontrivial axially symmetric solutions must be unstable if $n \leq 5$, in particular they cannot be blow-ups.
- At least for $s \sim 1$, we expect them to be minimizers in dimensions $n \geq 7$.
- Our approach is completely different from the one by Caffarelli-Jerison-Kenig for $s = 1$. (In particular, they do it by some delicate numerical computations.)



Main results

Theorem (Stability condition; F., Ros-Oton, 2022)

Let u be a global, stable, s-homogeneous solution, and let Γ be the free boundary. Then

$$\boxed{\iint_{\Gamma\times\Gamma} \big(f(x)-f(y)\big)^2 \mathcal{K}_{\Gamma,s}(x,y) \geq \int_\Gamma H_{\Gamma,s} f^2} \qquad \textit{for all} \quad f \in C_c^\infty(\Gamma),$$

where $\mathcal{K}_{\Gamma,s}$ is $(-n)$-homogeneous, $H_{\Gamma,s}$ is (-1)-homogeneous, with

$$\mathcal{K}_{\Gamma,s}(x,y) \asymp \frac{1}{|x-y|^n} \quad \textit{for all} \quad x,y\in\Gamma, \qquad H_{\Gamma,s}(x) \asymp \frac{1}{|x|} \quad \textit{for all} \quad x\in\Gamma.$$

More precisely, $\mathcal{K}_{\Gamma,s}$ and $\mathcal{H}_{\Gamma,s}$ are given by

$$\boxed{\mathcal{K}_{\Gamma,s}(x,y) := \lim_{\substack{\bar x\to x\\ \bar y\to y}} \frac{G_{\{u>0\},s}(\bar x,\bar y)}{d^s(\bar x)d^s(\bar y)}}, \qquad \text{and} \qquad \boxed{H_{\Gamma,s}(x) := \int_\Gamma |\nu(x)-\nu(y)|^2 \mathcal{K}_{\Gamma,s}(x,y)},$$

where $G_{\{u>0\},s}$ denotes the Green function for $(-\Delta)^s$ in $\{u>0\}$.

How to find the stability condition

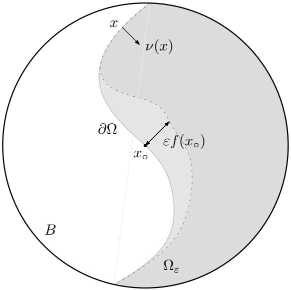
- How do we find the stability condition?
- If u is a minimizer, then $\mathcal{J}[u_\varepsilon] \geq \mathcal{J}[u]$ for any u_ε.
- Roughly, we consider domain variations; for $\phi \in C^\infty(B_1; B_1)$ we take

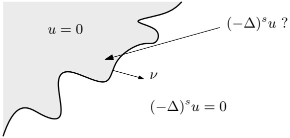
$$u_\varepsilon(x) = u(x + \varepsilon\phi(x)).$$

- Since u is a minimizer, we have

$$\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} \mathcal{J}[u_\varepsilon] \geq 0.$$

- We then need very fine estimates for s-harmonic functions up to the boundary.
- We finally combine all this with integration by parts formulas and some nice cancellations, to get the stability condition result above.





Main results

Recall the stability condition:

$$\iint_{\Gamma\times\Gamma} \big(f(x)-f(y)\big)^2 \mathcal{K}_{\Gamma,s}(x,y) \geq \int_\Gamma H_{\Gamma,s} f^2 \qquad \text{for all} \quad f\in C_c^\infty(\Gamma),$$

How can we use this condition? It is equivalent to:

$$\boxed{\int_\Gamma f\, T_{\Gamma,s} f \geq \kappa_s \int_\Gamma U_1 f^2} \qquad \text{for all} \quad f\in C_c^\infty(\Gamma),$$

where

$$\begin{aligned} U_1 &:= \partial_\nu\left(\frac{u}{d^s}\right), \quad u \approx d^s + U_1 d^{1+s} + \dots \\ T_{\Gamma,s} f &:= \partial_\nu\left(\frac{F}{d^{s-1}}\right), \quad F \approx f d^{s-1} + T_{\Gamma,s} f d^s + \dots \end{aligned}$$

$$\boxed{\begin{cases} (-\Delta)^s F = 0 & \text{in} \quad \{u>0\} \\ F = 0 & \text{in} \quad \{u=0\} \\ \frac{F}{d^{s-1}} = f & \text{on} \quad \Gamma \end{cases}}$$

The equivalence is non-trivial!

Proof of theorem

Theorem

Let u_0 be any global s-homogeneous stable solution, with $s \in (0,1)$. Assume that u_0 is axially symmetric. Then, if $n \leq 5$, u_0 is one-dimensional.

Sketch of the proof: Our stability condition can be rewritten as

$$\int_\Gamma \frac{F}{d^{s-1}} \partial_\nu \left(\frac{F}{d^{s-1}} \right) \leq \kappa_s \int_\Gamma \left(\frac{F}{d^{s-1}} \right)^2 \partial_\nu \left(\frac{u}{d^s} \right) \qquad \text{for all} \quad f \in C_c^\infty(\Gamma),$$

for any large solution $F \approx d^{s-1}$ such that $(-\Delta)^s F = 0$ in $\{u > 0\}$.

- If $(-\Delta)^s F \neq 0$, we still have an inequality with extra terms.
- Suppose now $u(x) = u(|x'|, x_n) = u(\tau, x_n)$, and take $F = \eta\, \partial_\tau u$ for $\eta \in C_c^\infty(\mathbb{R}^n)$. Notice that $u \approx d^s$, so $\nabla u \approx d^{s-1}$.
- In the extension variables we have "magic cancellations" that give

$$\int_{\mathbb{R}^{n+1}_+} (\partial_\tau u)^2 |\nabla \eta|^2 y^{1-2s} \geq (n-2) \int_{\mathbb{R}^{n+1}_+} (\partial_\tau u)^2 \frac{\eta^2}{\tau^2} y^{1-2s} \qquad \text{for all} \quad f \in C_c^\infty(\Gamma).$$

- $\eta(x) = |x'|^{-\alpha}$ and optimize in α.

Open problems

- We then use the right test function to prove that axially symmetric solutions are unstable in dimensions $n \leq 5$, independently of $s \in (0, 1)$!
- Is the same true for $n = 6$?

Conjecture

Let u be any stable, s-homogeneous solution to the fractional one-phase problem. If $n \leq 6$, then u is one-dimensional.

- Observe that this is open even for $s = 1$.
- Currently, the only known cases are for $n \leq 4$ if $s = 1$, and $n = 2$ if $s \in (0, 1)$.

Thank you!