On the stability of critical points of the Sobolev inequality

"Nonlinear diffusion equations, Green functions, functional inequalities" - UAM - Madrid

Federico Glaudo

ETH Zürich

June 14, 2022

Sobolev inequality

The Sobolev inequality states that, for any $u \in \dot{H}^1(\mathbb{R}^n)$, we have

$$\Big(\int_{\mathbb{R}^n} |\nabla u|^2\Big)^{\frac{1}{2}} \geq S\Big(\int_{\mathbb{R}^n} u^{2^*}\Big)^{\frac{1}{2^*}},$$

where $2^* = \frac{2n}{n-2}$ and $S = S(n)$ is a dimensional constant.

Sobolev inequality

The Sobolev inequality states that, for any $u \in \dot{H}^1(\mathbb{R}^n)$, we have

$$\Big(\int_{\mathbb{R}^n} |\nabla u|^2 \Big)^{\frac{1}{2}} \geq S \Big(\int_{\mathbb{R}^n} u^{2^*} \Big)^{\frac{1}{2^*}},$$

where $2^* = \frac{2n}{n-2}$ and $S = S(n)$ is a dimensional constant.
Equality is achieved by Talenti bubbles (Aubin 1976, Talenti 1976) Given $z \in \mathbb{R}^n$ and $\lambda > 0$, the Talenti bubble with center z and concentration λ is given by

$$U[z, \lambda] := \big(n(n-2) \big)^{\frac{n-2}{4}} \lambda^{\frac{n-2}{2}} \frac{1}{(1 + \lambda^2 |x - z|^2)^{\frac{n-2}{2}}}.$$

Sobolev inequality

The Sobolev inequality states that, for any $u \in \dot{H}^1(\mathbb{R}^n)$, we have

$$\Big(\int_{\mathbb{R}^n} |\nabla u|^2\Big)^{\frac{1}{2}} \geq S\Big(\int_{\mathbb{R}^n} u^{2^*}\Big)^{\frac{1}{2^*}},$$

where $2^* = \frac{2n}{n-2}$ and $S = S(n)$ is a dimensional constant.
Equality is achieved by Talenti bubbles (Aubin 1976, Talenti 1976) Given $z \in \mathbb{R}^n$ and $\lambda > 0$, the Talenti bubble with center z and concentration λ is given by

$$U[z,\lambda] := \big(n(n-2)\big)^{\frac{n-2}{4}} \lambda^{\frac{n-2}{2}} \frac{1}{(1+\lambda^2|x-z|^2)^{\frac{n-2}{2}}}.$$

Remark: The scalar coefficient in the definition of U guarantees that $\|\nabla U[z,\lambda]\|_{L^2}$ $(= S^{n/2})$ and $\|U[z,\lambda]\|_{L^{2^*}}$ do not depend on z and λ.

Sharp quantitative stability

The stability, in a sharp quantitative form, of the Sobolev inequality was established in Bianchi-Egnell 1991.

Sharp quantitative stability

The stability, in a sharp quantitative form, of the Sobolev inequality was established in Bianchi-Egnell 1991.

Theorem

For any $u \in \dot{H}^1(\mathbb{R}^n)$, it holds

$$\int_{\mathbb{R}^n} |\nabla u|^2 - S^2 \Big(\int_{\mathbb{R}^n} u^{2^*} \Big)^{\frac{2}{2^*}} \gtrsim \mathrm{dist}_{\dot{H}^1}(u, \mathcal{M}_{TB})^2,$$

Sharp quantitative stability

The stability, in a sharp quantitative form, of the Sobolev inequality was established in Bianchi-Egnell 1991.

Theorem

For any $u \in \dot{H}^1(\mathbb{R}^n)$, it holds

$$\int_{\mathbb{R}^n} |\nabla u|^2 - S^2\Big(\int_{\mathbb{R}^n} u^{2^*}\Big)^{\frac{2}{2^*}} \gtrsim \mathrm{dist}_{\dot{H}^1}(u, \mathcal{M}_{TB})^2,$$

where $\mathcal{M}_{TB}$ denotes the manifold of Talenti bubbles, i.e.,

$$\mathrm{dist}_{\dot{H}^1}(u, \mathcal{M}_{TB}) := \inf_{c\in\mathbb{R}, z\in\mathbb{R}^n, \lambda>0} \Big\| \nabla\Big(u - cU[z,\lambda]\Big)\Big\|_{L^2}.$$

Sharp quantitative stability

The stability, in a sharp quantitative form, of the Sobolev inequality was established in Bianchi-Egnell 1991.

Theorem

For any $u \in \dot{H}^1(\mathbb{R}^n)$, it holds

$$\int_{\mathbb{R}^n} |\nabla u|^2 - S^2\Big(\int_{\mathbb{R}^n} u^{2^*}\Big)^{\frac{2}{2^*}} \gtrsim \mathrm{dist}_{\dot{H}^1}(u, \mathcal{M}_{TB})^2,$$

where $\mathcal{M}_{TB}$ denotes the manifold of Talenti bubbles, i.e.,

$$\mathrm{dist}_{\dot{H}^1}(u, \mathcal{M}_{TB}) \coloneqq \inf_{c\in\mathbb{R}, z\in\mathbb{R}^n, \lambda>0} \Big\|\nabla\Big(u - cU[z,\lambda]\Big)\Big\|_{L^2}.$$

Remark: This result is sharp both with respect to the exponent and with respect to the choice of the distance.

Critical points

Consider the Sobolev-ratio functional

$$F(u) := \frac{\left(\int |\nabla u|^2\right)^{\frac{1}{2}}}{\left(\int |u|^{2^*}\right)^{\frac{1}{2^*}}},$$

Critical points

Consider the Sobolev-ratio functional

$$F(u) := \frac{\left(\int |\nabla u|^2\right)^{\frac{1}{2}}}{\left(\int |u|^{2^*}\right)^{\frac{1}{2^*}}},$$

its first variation is

$$F(u+\varphi) = \frac{\left(\int |\nabla u|^2 + 2\nabla u \cdot \nabla \varphi\right)^{\frac{1}{2}}}{\left(\int |u|^{2^*} + 2^* |u|^{2^*-2} u\varphi\right)^{\frac{1}{2^*}}} + \mathcal{O}(\|\nabla \varphi\|_{L^2}^2).$$

Critical points

Consider the Sobolev-ratio functional

$$F(u) := \frac{\left(\int |\nabla u|^2\right)^{\frac{1}{2}}}{\left(\int |u|^{2^*}\right)^{\frac{1}{2^*}}},$$

its first variation is

$$F(u+\varphi) = \frac{\left(\int |\nabla u|^2 + 2\nabla u \cdot \nabla\varphi\right)^{\frac{1}{2}}}{\left(\int |u|^{2^*} + 2^*|u|^{2^*-2}u\varphi\right)^{\frac{1}{2^*}}} + \mathcal{O}(\|\nabla\varphi\|_{L^2}^2).$$

Hence $\mathrm{d}F(u)[\varphi] = 0$ if and only if $\int_{\mathbb{R}^n} \nabla u \cdot \nabla\varphi = \lambda \int_{\mathbb{R}^n} |u|^{2^*-2}u\varphi$ (where λ depends on u but not on φ),

Critical points

Consider the Sobolev-ratio functional

$$F(u) := \frac{\left(\int |\nabla u|^2\right)^{\frac{1}{2}}}{\left(\int |u|^{2^*}\right)^{\frac{1}{2^*}}},$$

its first variation is

$$F(u+\varphi) = \frac{\left(\int |\nabla u|^2 + 2\nabla u \cdot \nabla \varphi\right)^{\frac{1}{2}}}{\left(\int |u|^{2^*} + 2^*|u|^{2^*-2}u\varphi\right)^{\frac{1}{2^*}}} + \mathcal{O}(\|\nabla\varphi\|_{L^2}^2).$$

Hence $\mathrm{d}F(u)[\varphi] = 0$ if and only if $\int_{\mathbb{R}^n} \nabla u \cdot \nabla \varphi = \lambda \int_{\mathbb{R}^n} |u|^{2^*-2} u\varphi$ (where λ depends on u but not on φ), and therefore

$$\mathrm{d}F(u) = 0 \iff -\Delta u = \lambda |u|^{2^*-2} u.$$

Solutions of the Euler-Lagrange equation

Up to scaling, the Euler-Lagrange equation associated to the Sobolev inequality is

$$-\Delta u = |u|^{2^*-2}u. \tag{EL}$$

Solutions of the Euler-Lagrange equation

Up to scaling, the Euler-Lagrange equation associated to the Sobolev inequality is

$$-\Delta u = |u|^{2^*-2}u. \tag{EL}$$

Remark: The Talenti bubbles (with our normalization) satisfy (EL).

Solutions of the Euler-Lagrange equation

Up to scaling, the Euler-Lagrange equation associated to the Sobolev inequality is

$$-\Delta u = |u|^{2^*-2}u. \tag{EL}$$

Remark: The Talenti bubbles (with our normalization) satisfy (EL).
In Ding 1986 it is shown that (EL) has infinitely many, non conformally equivalent, sign-changing solutions (different from Talenti bubbles).

Solutions of the Euler-Lagrange equation

Up to scaling, the Euler-Lagrange equation associated to the Sobolev inequality is

$$-\Delta u = |u|^{2^*-2}u. \tag{EL}$$

Remark: The Talenti bubbles (with our normalization) satisfy (EL).
In Ding 1986 it is shown that (EL) has infinitely many, non conformally equivalent, sign-changing solutions (different from Talenti bubbles).
If we restrict to nonnegative solutions, Gidas-Ni-Nirenberg 1979 shows:

Theorem

If $u \in \dot{H}^1(\mathbb{R}^n)$ solves (EL) and $u \geq 0$ in $\mathbb{R}^n$, then u is a Talenti bubble.

Solutions of the Euler-Lagrange equation

Up to scaling, the Euler-Lagrange equation associated to the Sobolev inequality is

$$-\Delta u = |u|^{2^*-2}u. \tag{EL}$$

Remark: The Talenti bubbles (with our normalization) satisfy (EL).
In Ding 1986 it is shown that (EL) has infinitely many, non conformally equivalent, sign-changing solutions (different from Talenti bubbles).
If we restrict to nonnegative solutions, Gidas-Ni-Nirenberg 1979 shows:

Theorem

If $u \in \dot{H}^1(\mathbb{R}^n)$ solves (EL) and $u \geq 0$ in $\mathbb{R}^n$, then u is a Talenti bubble.

Thus, the positive critical points of the Sobolev inequality are exactly the Talenti bubbles, so critical points and minimizers coincide.

Almost critical points

Question: If a function $u \geq 0$ *almost solves* $-\Delta u = u^{2^*-1}$ (i.e., $\Delta u + u^{2^*-1}$ is small in a suitable norm), is it close to a Talenti bubble?

Almost critical points

Question: If a function $u \geq 0$ *almost solves* $-\Delta u = u^{2^*-1}$ (i.e., $\Delta u + u^{2^*-1}$ is small in a suitable norm), is it close to a Talenti bubble?
No, consider $u = U_- + U_+ = U[-Re_1, 1] + U[Re_1, 1]$ for $R \gg 1$.

Almost critical points

Question: If a function $u \geq 0$ *almost solves* $-\Delta u = u^{2^*-1}$ (i.e., $\Delta u + u^{2^*-1}$ is small in a suitable norm), is it close to a Talenti bubble?

No, consider $u = U_- + U_+ = U[-Re_1, 1] + U[Re_1, 1]$ for $R \gg 1$. We have

$$\begin{aligned}\Delta u + u^{2^*-1} =& \left(\Delta U_- + U_-^{2^*-1}\right) + \left(\Delta U_+ + U_+^{2^*-1}\right) \\ &+ (U_- + U_+)^{2^*-1} - U_-^{2^*-1} - U_+^{2^*-1} \\ =& (U_- + U_+)^{2^*-1} - U_-^{2^*-1} - U_+^{2^*-1},\end{aligned}$$

which is small because U_- and U_+ are *weakly interacting* (because they are concentrated in different regions of the space).

Almost critical points

Question: If a function $u \geq 0$ *almost solves* $-\Delta u = u^{2^*-1}$ (i.e., $\Delta u + u^{2^*-1}$ is small in a suitable norm), is it close to a Talenti bubble?
No, consider $u = U_- + U_+ = U[-Re_1, 1] + U[Re_1, 1]$ for $R \gg 1$. We have

$$\begin{aligned}\Delta u + u^{2^*-1} =& \left(\Delta U_- + U_-^{2^*-1}\right) + \left(\Delta U_+ + U_+^{2^*-1}\right) \\ &+ (U_- + U_+)^{2^*-1} - U_-^{2^*-1} - U_+^{2^*-1} \\ =& (U_- + U_+)^{2^*-1} - U_-^{2^*-1} - U_+^{2^*-1},\end{aligned}$$

which is small because U_- and U_+ are *weakly interacting* (because they are concentrated in different regions of the space).
New question: If a function $u \geq 0$ *almost solves* $-\Delta u = u^{2^*-1}$ (i.e., $\Delta u + u^{2^*-1}$ is small in a suitable norm), is it close to a sum of weakly interacting Talenti bubbles?

Almost critical points

Question: If a function $u \geq 0$ *almost solves* $-\Delta u = u^{2^*-1}$ (i.e., $\Delta u + u^{2^*-1}$ is small in a suitable norm), is it close to a Talenti bubble?
No, consider $u = U_- + U_+ = U[-Re_1, 1] + U[Re_1, 1]$ for $R \gg 1$. We have

$$\begin{aligned}\Delta u + u^{2^*-1} =& \left(\Delta U_- + U_-^{2^*-1}\right) + \left(\Delta U_+ + U_+^{2^*-1}\right) \\ &+ (U_- + U_+)^{2^*-1} - U_-^{2^*-1} - U_+^{2^*-1} \\ =& (U_- + U_+)^{2^*-1} - U_-^{2^*-1} - U_+^{2^*-1},\end{aligned}$$

which is small because U_- and U_+ are *weakly interacting* (because they are concentrated in different regions of the space).
New question: If a function $u \geq 0$ *almost solves* $-\Delta u = u^{2^*-1}$ (i.e., $\Delta u + u^{2^*-1}$ is small in a suitable norm), is it close to a sum of weakly interacting Talenti bubbles?
Yes, as was shown in Struwe 1984.

Qualitative stability of critical points

Theorem

Let $u \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative function $u \geq 0$ such that

$$(k - \frac{1}{2})S^n \leq \int |\nabla u|^2 \leq (k + \frac{1}{2})S^n,$$

for some $k \in \mathbb{N}$ (which represents the number of bubbles).

Qualitative stability of critical points

Theorem

Let $u \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative function $u \geq 0$ such that

$$(k - \frac{1}{2})S^n \leq \int |\nabla u|^2 \leq (k + \frac{1}{2})S^n,$$

for some $k \in \mathbb{N}$ (which represents the number of bubbles). Then,

$$\inf_{(z_i,\lambda_i)_{1\leq i\leq k}} \left\| \nabla \Big(u - \sum_{i=1}^{k} U[z_i, \lambda_i] \Big) \right\|_{L^2} = \omega\big(\| \Delta u + u^{2^*-1} \|_{H^{-1}} \big),$$

for a suitable modulus of continuity ω (which does not depend on u).

Qualitative stability of critical points

Theorem

Let $u \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative function $u \geq 0$ such that

$$(k - \frac{1}{2})S^n \leq \int |\nabla u|^2 \leq (k + \frac{1}{2})S^n,$$

for some $k \in \mathbb{N}$ (which represents the number of bubbles). Then,

$$\inf_{(z_i,\lambda_i)_{1\leq i\leq k}} \left\| \nabla\Big(u - \sum_{i=1}^{k} U[z_i, \lambda_i]\Big) \right\|_{L^2} = \omega\big(\|\Delta u + u^{2^*-1}\|_{H^{-1}}\big),$$

for a suitable modulus of continuity ω (which does not depend on u). Moreover the Talenti bubbles $U[z_1, \lambda_1], U[z_2, \lambda_2], \dots, U[z_k, \lambda_k]$ are weakly interacting.

Quantitative stability – Single bubble case

Motivated by an application to the *fast diffusion equation*, one looks for the sharp quantitative version of the previous result, i.e., finding the optimal modulus of continuity ω.

Quantitative stability – Single bubble case

Motivated by an application to the *fast diffusion equation*, one looks for the sharp quantitative version of the previous result, i.e., finding the optimal modulus of continuity ω.
In Ciraolo-Figalli-Maggi 2018 the single bubble case is settled.

Quantitative stability – Single bubble case

Motivated by an application to the *fast diffusion equation*, one looks for the sharp quantitative version of the previous result, i.e., finding the optimal modulus of continuity ω.
In Ciraolo-Figalli-Maggi 2018 the single bubble case is settled.

Theorem

Let $u \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative function $u \geq 0$ such that

$$\frac{1}{2}S^n \leq \int |\nabla u|^2 \leq \frac{3}{2}S^n.$$

Quantitative stability – Single bubble case

Motivated by an application to the *fast diffusion equation*, one looks for the sharp quantitative version of the previous result, i.e., finding the optimal modulus of continuity ω.
In Ciraolo-Figalli-Maggi 2018 the single bubble case is settled.

Theorem

Let $u \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative function $u \geq 0$ such that

$$\frac{1}{2}S^n \leq \int |\nabla u|^2 \leq \frac{3}{2}S^n.$$

Then,

$$\inf_{z\in\mathbb{R}^n,\lambda>0} \left\|\nabla\Big(u - U[z,\lambda]\Big)\right\|_{L^2} \lesssim \|\Delta u + u^{2^*-1}\|_{H^{-1}}.$$

Quantitative stability – Single bubble case

Motivated by an application to the *fast diffusion equation*, one looks for the sharp quantitative version of the previous result, i.e., finding the optimal modulus of continuity ω.
In Ciraolo-Figalli-Maggi 2018 the single bubble case is settled.

Theorem

Let $u \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative function $u \geq 0$ such that

$$\frac{1}{2} S^n \leq \int |\nabla u|^2 \leq \frac{3}{2} S^n.$$

Then,

$$\inf_{z \in \mathbb{R}^n, \lambda > 0} \left\| \nabla \Big(u - U[z, \lambda] \Big) \right\|_{L^2} \lesssim \| \Delta u + u^{2^*-1} \|_{H^{-1}}.$$

Remark: While this result paved the way for the multibubble case, the latter is significantly more delicate and features unexpectedly different behaviors depending on the dimension.

Quantitative stability – Multibubble case (low dimension)

For the dimensions $3 \leq n \leq 5$, the sharp quantitative stability was established in Figalli-G. 2020.

Quantitative stability – Multibubble case (low dimension)

For the dimensions $3 \le n \le 5$, the sharp quantitative stability was established in Figalli-G. 2020.

Theorem

Given $3 \le n \le 5$, let $u \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative function $u \ge 0$ such that

$$(k - \frac{1}{2})S^n \le \int |\nabla u|^2 \le (k + \frac{1}{2})S^n,$$

for some $k \in \mathbb{N}$ (which represents the number of bubbles).

Quantitative stability – Multibubble case (low dimension)

For the dimensions $3 \leq n \leq 5$, the sharp quantitative stability was established in Figalli-G. 2020.

Theorem

Given $3 \leq n \leq 5$, let $u \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative function $u \geq 0$ such that

$$(k - \frac{1}{2})S^n \leq \int |\nabla u|^2 \leq (k + \frac{1}{2})S^n,$$

for some $k \in \mathbb{N}$ (which represents the number of bubbles). Then, there exist k Talenti bubbles $U_1, U_2, \dots, U_k$ such that

$$\left\| \nabla\Big(u - \sum_{i=1}^{k} U_i \Big) \right\|_{L^2} \lesssim \|\Delta u + u^{2^*-1}\|_{H^{-1}}.$$

Moreover, the interaction between the bubbles can be estimated by

$$\int_{\mathbb{R}^n} U_i^{2^*-1} U_j \lesssim \|\Delta u + u^{2^*-1}\|_{H^{-1}}.$$

Quantitative stability – Multibubble case (high dimension)

In Figalli-G. 2020 it was shown, by constructing a counterexample, that the last result is false in dimension $n \geq 6$.

Quantitative stability – Multibubble case (high dimension)

In Figalli-G. 2020 it was shown, by constructing a counterexample, that the last result is false in dimension $n \geq 6$.

It was also conjectured that the sharp stability is valid if $\|\Delta u + u^{2^*-1}\|_{H^{-1}}$ is replaced by $\|\Delta u + u^{2^*-1}\|_{H^{-1}}^{\alpha}$ for an appropriate exponent $\alpha = \alpha(n)$.

Quantitative stability – Multibubble case (high dimension)

In Figalli-G. 2020 it was shown, by constructing a counterexample, that the last result is false in dimension $n \geq 6$.

It was also conjectured that the sharp stability is valid if $\|\Delta u + u^{2^*-1}\|_{H^{-1}}$ is replaced by $\|\Delta u + u^{2^*-1}\|_{H^{-1}}^{\alpha}$ for an appropriate exponent $\alpha = \alpha(n)$.

This was shown in Deng-Sun-Wei 2021.

Quantitative stability – Multibubble case (high dimension)

In Figalli-G. 2020 it was shown, by constructing a counterexample, that the last result is false in dimension $n \geq 6$.
It was also conjectured that the sharp stability is valid if $\|\Delta u + u^{2^*-1}\|_{H^{-1}}$ is replaced by $\|\Delta u + u^{2^*-1}\|_{H^{-1}}^{\alpha}$ for an appropriate exponent $\alpha = \alpha(n)$.
This was shown in Deng-Sun-Wei 2021.

Theorem

Given $n \geq 6$, let $u \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative function $u \geq 0$ such that

$$(k - \frac{1}{2})S^n \leq \int |\nabla u|^2 \leq (k + \frac{1}{2})S^n,$$

for some $k \in \mathbb{N}$ (which represents the number of bubbles).

Quantitative stability – Multibubble case (high dimension)

In Figalli-G. 2020 it was shown, by constructing a counterexample, that the last result is false in dimension $n \geq 6$.
It was also conjectured that the sharp stability is valid if $\|\Delta u + u^{2^*-1}\|_{H^{-1}}$ is replaced by $\|\Delta u + u^{2^*-1}\|_{H^{-1}}^{\alpha}$ for an appropriate exponent $\alpha = \alpha(n)$.
This was shown in Deng-Sun-Wei 2021.

Theorem

Given $n \geq 6$, let $u \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative function $u \geq 0$ such that

$$(k - \frac{1}{2})S^n \leq \int |\nabla u|^2 \leq (k + \frac{1}{2})S^n,$$

for some $k \in \mathbb{N}$ (which represents the number of bubbles). Then, there exist k Talenti bubbles $U_1, U_2, \dots, U_k$ such that

$$\left\|\nabla\Big(u - \sum_{i=1}^{k} U_i\Big)\right\|_{L^2} \lesssim \Phi(\|\Delta u + u^{2^*-1}\|_{H^{-1}}),$$

where $\Phi(t) := t|\log(t)|^{\frac{1}{2}}$ if $n = 6$ and $\Phi(t) = t^{(n+2)/(2(n-2))}$ for $n \geq 7$.

Quantitative stability – Sketch of proof

Let us assume that $3 \leq n \leq 5$. We want to prove the sharp stability.

Quantitative stability – Sketch of proof

Let us assume that $3 \leq n \leq 5$. We want to prove the sharp stability. Given $u \geq 0$ so that $\|\Delta u + u^{2^*-2}\|_{H^{-1}} \ll 1$,

Quantitative stability – Sketch of proof

Let us assume that $3 \le n \le 5$. We want to prove the sharp stability. Given $u \ge 0$ so that $\|\Delta u + u^{2^*-2}\|_{H^{-1}} \ll 1$, let

- $\alpha_1, \alpha_2, \dots, \alpha_k \in [1-\varepsilon, 1+\varepsilon]$ be the coefficients
- $U_1, U_2, \dots, U_k$ be the Talenti bubbles

which minimize

$$\|\nabla \rho\|_{L^2} := \left\| \nabla \Big(u - \sum_{i=1}^{k} \alpha_i U_i \Big) \right\|_{L^2}.$$

Quantitative stability – Sketch of proof

Let us assume that $3 \leq n \leq 5$. We want to prove the sharp stability. Given $u \geq 0$ so that $\|\Delta u + u^{2^*-2}\|_{H^{-1}} \ll 1$, let

- $\alpha_1, \alpha_2, \dots, \alpha_k \in [1-\varepsilon, 1+\varepsilon]$ be the coefficients
- $U_1, U_2, \dots, U_k$ be the Talenti bubbles

which minimize

$$\|\nabla \rho\|_{L^2} := \left\| \nabla \Big(u - \sum_{i=1}^{k} \alpha_i U_i \Big) \right\|_{L^2}.$$

Remark: Notice that in the statement of the theorem we have $\alpha_i = 1$, so we will have to take care of it eventually.

Quantitative stability – Sketch of proof

Let us assume that $3 \le n \le 5$. We want to prove the sharp stability. Given $u \ge 0$ so that $\|\Delta u + u^{2^*-2}\|_{H^{-1}} \ll 1$, let

- $\alpha_1, \alpha_2, \dots, \alpha_k \in [1-\varepsilon, 1+\varepsilon]$ be the coefficients
- $U_1, U_2, \dots, U_k$ be the Talenti bubbles

which minimize

$$\|\nabla \rho\|_{L^2} := \left\| \nabla \Big(u - \sum_{i=1}^{k} \alpha_i U_i \Big) \right\|_{L^2}.$$

Remark: Notice that in the statement of the theorem we have $\alpha_i = 1$, so we will have to take care of it eventually. We allow ourselves to find a minimizer in a larger class, then we show that the minimizer is *almost* in the initial class. A bit like when you try to solve a PDE you look for a very weak solution and then you prove its regularity.

Quantitative stability – Sketch of proof

Let us assume that $3 \leq n \leq 5$. We want to prove the sharp stability. Given $u \geq 0$ so that $\|\Delta u + u^{2^*-2}\|_{H^{-1}} \ll 1$, let

- $\alpha_1, \alpha_2, \ldots, \alpha_k \in [1-\varepsilon, 1+\varepsilon]$ be the coefficients
- $U_1, U_2, \ldots, U_k$ be the Talenti bubbles

which minimize

$$\|\nabla\rho\|_{L^2} := \left\|\nabla\Big(u - \sum_{i=1}^{k} \alpha_i U_i\Big)\right\|_{L^2}.$$

Remark: Notice that in the statement of the theorem we have $\alpha_i = 1$, so we will have to take care of it eventually. We allow ourselves to find a minimizer in a larger class, then we show that the minimizer is *almost* in the initial class. A bit like when you try to solve a PDE you look for a very weak solution and then you prove its regularity.
By computing the first variation, we deduce that $\rho \perp_{\dot{H}^1} U_i, \partial_z U_i, \partial_\lambda U_i$.

Quantitative stability – Main estimate

By testing ρ against suitable test functions and using abundantly the orthogonality conditions, one obtains

Quantitative stability – Main estimate

By testing ρ against suitable test functions and using abundantly the orthogonality conditions, one obtains

$$
\begin{aligned}
\|\nabla\rho\|_{L^2}^2 \leq\ & \|\nabla\rho\|_{L^2}\|\Delta u + u^{2^*-1}\|_{H^{-1}} \\
&+ (2^*-1)\int\Big(\sum\alpha_i U_i\Big)^{2^*-2}\rho^2 \\
&+ \|\nabla\rho\|_{L^2}\sum_{i\neq j}\int U_i^{2^*-1}U_j
\end{aligned}
$$

Quantitative stability – Main estimate

By testing ρ against suitable test functions and using abundantly the orthogonality conditions, one obtains

$$\|\nabla\rho\|_{L^2}^2 \le \|\nabla\rho\|_{L^2}\|\Delta u + u^{2^*-1}\|_{H^{-1}}$$
$$+ (2^*-1)\int\Big(\sum \alpha_i U_i\Big)^{2^*-2}\rho^2$$
$$+ \|\nabla\rho\|_{L^2}\sum_{i\neq j}\int U_i^{2^*-1}U_j$$

1. The first term of the RHS is good, so we do not have to handle it.

Quantitative stability – Main estimate

By testing ρ against suitable test functions and using abundantly the orthogonality conditions, one obtains

$$\begin{aligned}\|\nabla\rho\|_{L^2}^2 &\leq \|\nabla\rho\|_{L^2}\|\Delta u + u^{2^*-1}\|_{H^{-1}} \\ &+ (2^*-1)\int\Big(\sum \alpha_i U_i\Big)^{2^*-2}\rho^2 \leq (1-\delta)\|\nabla\rho\|_{L^2}^2 \\ &+ \|\nabla\rho\|_{L^2}\sum_{i\neq j}\int U_i^{2^*-1}U_j\end{aligned}$$

1. The first term of the RHS is good, so we do not have to handle it.
2. The second term will be controlled through a *nondegeneracy condition* (using that ρ is orthogonal to the first eigenfunctions of a spectral operator).

Quantitative stability – Main estimate

By testing ρ against suitable test functions and using abundantly the orthogonality conditions, one obtains

$$\begin{aligned}\|\nabla\rho\|_{L^2}^2 &\le \|\nabla\rho\|_{L^2}\|\Delta u + u^{2^*-1}\|_{H^{-1}} \\ &+ (2^*-1)\int\Big(\sum \alpha_i U_i\Big)^{2^*-2}\rho^2 \le (1-\delta)\|\nabla\rho\|_{L^2}^2 \\ &+ \|\nabla\rho\|_{L^2}\sum_{i\neq j}\int U_i^{2^*-1}U_j \le \|\nabla\rho\|_{L^2}\cdot\Big(\varepsilon\|\nabla\rho\|_{L^2} + C\|\Delta u + u^{2^*-1}\|_{H^{-1}}\Big)\end{aligned}$$

1. The first term of the RHS is good, so we do not have to handle it.
2. The second term will be controlled through a *nondegeneracy condition* (using that ρ is orthogonal to the first eigenfunctions of a spectral operator).
3. The third term is expected to be small because it involves interactions between different bubbles, which should be weakly interacting. It is the hard term to control.

Quantitative stability – Nondegeneracy

We want to show (for some $\delta > 0$)

$$(2^* - 1) \int \Big(\sum \alpha_i U_i \Big)^{2^*-2} \rho^2 \leq (1 - \delta) \int |\nabla \rho|^2.$$

Quantitative stability – Nondegeneracy

We want to show (for some $\delta > 0$)

$$(2^* - 1) \int \Big(\sum \alpha_i U_i \Big)^{2^*-2} \rho^2 \leq (1 - \delta) \int |\nabla \rho|^2.$$

Since bubble are (nonquantitatively) weakly interacting, this follow from the equivalent inequality for a single bubble

$$\frac{\int |\nabla \rho|^2}{\int U^{2^*-2} \rho^2} > 2^* - 1. \tag{ND}$$

Quantitative stability – Nondegeneracy

We want to show (for some $\delta > 0$)

$$(2^* - 1) \int \Big(\sum \alpha_i U_i \Big)^{2^*-2} \rho^2 \leq (1 - \delta) \int |\nabla \rho|^2.$$

Since bubble are (nonquantitatively) weakly interacting, this follow from the equivalent inequality for a single bubble

$$\frac{\int |\nabla \rho|^2}{\int U^{2^*-2} \rho^2} > 2^* - 1. \qquad \text{(ND)}$$

Consider the associated elliptic operator $T[v] := \frac{-\Delta v}{U^{2^*-2}}$;

Quantitative stability – Nondegeneracy

We want to show (for some $\delta > 0$)

$$(2^* - 1) \int \Big(\sum \alpha_i U_i \Big)^{2^*-2} \rho^2 \leq (1-\delta) \int |\nabla \rho|^2.$$

Since bubble are (nonquantitatively) weakly interacting, this follow from the equivalent inequality for a single bubble

$$\frac{\int |\nabla \rho|^2}{\int U^{2^*-2} \rho^2} > 2^* - 1. \tag{ND}$$

Consider the associated elliptic operator $T[v] := \frac{-\Delta v}{U^{2^*-2}}$; we have

$$TU = U \ , \quad T\partial_\lambda U = (2^*-1)\partial_\lambda U \ , \quad T\partial_z U = (2^*-1)\partial_z U.$$

Quantitative stability – Nondegeneracy

We want to show (for some $\delta > 0$)

$$(2^* - 1) \int \Big(\sum \alpha_i U_i \Big)^{2^*-2} \rho^2 \leq (1-\delta) \int |\nabla \rho|^2.$$

Since bubble are (nonquantitatively) weakly interacting, this follow from the equivalent inequality for a single bubble

$$\frac{\int |\nabla \rho|^2}{\int U^{2^*-2} \rho^2} > 2^* - 1. \qquad \text{(ND)}$$

Consider the associated elliptic operator $T[v] := \frac{-\Delta v}{U^{2^*-2}}$; we have

$$TU = U \ , \quad T\partial_\lambda U = (2^*-1)\partial_\lambda U \ , \quad T\partial_z U = (2^*-1)\partial_z U.$$

Bianchi-Egnell 1991 show that 1 and $2^* - 1$ are the first two eigenvalues of the operator and the corresponding eigenfunctions are $U, \partial_\lambda U, \partial_z U$.

Quantitative stability – Nondegeneracy

We want to show (for some $\delta > 0$)

$$(2^* - 1) \int \Big(\sum \alpha_i U_i \Big)^{2^*-2} \rho^2 \leq (1 - \delta) \int |\nabla \rho|^2.$$

Since bubble are (nonquantitatively) weakly interacting, this follow from the equivalent inequality for a single bubble

$$\frac{\int |\nabla \rho|^2}{\int U^{2^*-2} \rho^2} > 2^* - 1. \qquad \text{(ND)}$$

Consider the associated elliptic operator $T[v] := \frac{-\Delta v}{U^{2^*-2}}$; we have

$$TU = U \ , \quad T\partial_\lambda U = (2^* - 1)\partial_\lambda U \ , \quad T\partial_z U = (2^* - 1)\partial_z U.$$

Bianchi-Egnell 1991 show that 1 and $2^* - 1$ are the first two eigenvalues of the operator and the corresponding eigenfunctions are $U, \partial_\lambda U, \partial_z U$. Since ρ is orthogonal to all of them, (ND) follows.

Quantitative stability – Hard term

It remains to prove, for $i \neq j$,

$$\int U_i^{2^*-1} U_j \leq \varepsilon \|\nabla \rho\|_{L^2} + C\|\Delta u + u^{2^*-1}\|_{H^{-1}}$$

Quantitative stability – Hard term

It remains to prove, for $i \neq j$,

$$\int U_i^{2^*-1} U_j \leq \varepsilon \|\nabla \rho\|_{L^2} + C\|\Delta u + u^{2^*-1}\|_{H^{-1}}$$

and (remember that $\alpha_i = 1$ in the statement we want to prove)

$$|\alpha_i - 1| \leq \varepsilon \|\nabla \rho\|_{L^2} + C\|\Delta u + u^{2^*-1}\|_{H^{-1}}.$$

Quantitative stability – Hard term

It remains to prove, for $i \neq j$,

$$\int U_i^{2^*-1} U_j \leq \varepsilon \|\nabla \rho\|_{L^2} + C\|\Delta u + u^{2^*-1}\|_{H^{-1}}$$

and (remember that $\alpha_i = 1$ in the statement we want to prove)

$$|\alpha_i - 1| \leq \varepsilon \|\nabla \rho\|_{L^2} + C\|\Delta u + u^{2^*-1}\|_{H^{-1}}.$$

This is the core of the proof and is done by induction over i, starting from the most concentrated bubble U_i and through a delicate localization procedure.

Application to a fast diffusion equation

Consider the fast diffusion equation with critical exponent

$$\begin{cases} u(0, \cdot) = u_0 \geq 0, \\ \partial_t u = \Delta(u^{1/(2^*-1)}). \end{cases} \tag{FD}$$

Application to a fast diffusion equation

Consider the fast diffusion equation with critical exponent

$$\begin{cases} u(0,\,\cdot\,) = u_0 \geq 0, \\ \partial_t u = \Delta\big(u^{1/(2^*-1)}\big). \end{cases} \tag{FD}$$

For any $T > 0, z \in \mathbb{R}^n, \lambda > 0$; the function

$$u_{T,z,\lambda}(t,x) = C(n)(T-t)^{\frac{2^*-1}{2^*-2}} U[z,\lambda]$$

solves (FD).

Application to a fast diffusion equation

Consider the fast diffusion equation with critical exponent

$$\begin{cases} u(0,\,\cdot\,) = u_0 \geq 0, \\ \partial_t u = \Delta\big(u^{1/(2^*-1)}\big). \end{cases} \tag{FD}$$

For any $T > 0, z \in \mathbb{R}^n, \lambda > 0$; the function

$$u_{T,z,\lambda}(t,x) = C(n)(T-t)^{\frac{2^*-1}{2^*-2}} U[z,\lambda]$$

solves (FD).

For any $u_0 \in L^{(2^*)'}$ there exists an extinction time $T > 0$ and $z \in \mathbb{R}^n, \lambda > 0$ such that (del Pino-Sáez 2001)

$$\left\| \frac{u(t)}{u_{T,z,\lambda}(t)} - 1 \right\|_\infty \to 0 \quad \text{as } t \to T^-.$$

Application to a fast diffusion equation

Consider the fast diffusion equation with critical exponent

$$\begin{cases} u(0,\,\cdot\,) = u_0 \geq 0, \\ \partial_t u = \Delta(u^{1/(2^*-1)}). \end{cases} \tag{FD}$$

For any $T > 0, z \in \mathbb{R}^n, \lambda > 0$; the function

$$u_{T,z,\lambda}(t,x) = C(n)(T-t)^{\frac{2^*-1}{2^*-2}} U[z,\lambda]$$

solves (FD).

For any $u_0 \in L^{(2^*)'}$ there exists an extinction time $T > 0$ and $z \in \mathbb{R}^n, \lambda > 0$ such that (del Pino-Sáez 2001)

$$\left\| \frac{u(t)}{u_{T,z,\lambda}(t)} - 1 \right\|_\infty \to 0 \quad \text{as } t \to T^-.$$

Applying the quantitative stability for a single bubble, one obtains

$$\left\| \frac{u(t)}{u_{T,z,\lambda}(t)} - 1 \right\|_\infty \leq C(n,u_0)(T-t)^{\kappa(n)} \quad \text{for } 0 < t < T^-.$$

Ongoing work and future developments

- The analogous result for the **Caffarelli-Kohn-Nirenberg inequality** (Sobolev inequality with power weights) is shown in Wei-Wu 2022.

Ongoing work and future developments

- The analogous result for the **Caffarelli-Kohn-Nirenberg inequality** (Sobolev inequality with power weights) is shown in Wei-Wu 2022.
- The analogous result for the **Hardy-Littlewood-Sobolev inequality** (Sobolev inequality but $\|u\|_{H^1}$ is replaced by $\|u\|_{H^s}$) is being investigated by Aryan.

Ongoing work and future developments

- The analogous result for the **Caffarelli-Kohn-Nirenberg inequality** (Sobolev inequality with power weights) is shown in Wei-Wu 2022.
- The analogous result for the **Hardy-Littlewood-Sobolev inequality** (Sobolev inequality but $\|u\|_{H^1}$ is replaced by $\|u\|_{H^s}$) is being investigated by Aryan.
- The analogous result for the **fractional Caffarelli-Kohn-Nirenberg inequality** (a mix of the two inequalities mentioned above) is being investigated by De Nitti-Glaudo-König.

Ongoing work and future developments

- The analogous result for the **Caffarelli-Kohn-Nirenberg inequality** (Sobolev inequality with power weights) is shown in Wei-Wu 2022.
- The analogous result for the **Hardy-Littlewood-Sobolev inequality** (Sobolev inequality but $\|u\|_{H^1}$ is replaced by $\|u\|_{H^s}$) is being investigated by Aryan.
- The analogous result for the **fractional Caffarelli-Kohn-Nirenberg inequality** (a mix of the two inequalities mentioned above) is being investigated by De Nitti-Glaudo-König.
- The analogous result for the **isoperimetric inequality** is an interesting open problem. Currently it is known that if a domain has *almost constant mean curvature* then it is close to a *necklace of spheres*. The sharp quantitative version of this result with the natural norms is not known.

Thank you for your attention

Federico Glaudo

ETH Zürich