Levy Fokker-Planck equations

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Summary

$$\boxed{Lu = (-\Delta)^s u - \frac{x}{2s} \cdot \nabla u - \frac{n}{2} u \quad \text{in } \mathbb{R}^n}$$

Comments:

- $s \in (0, 1)$.
- May reduce to radial solutions $u(r)$, $r = |x|$

Objectives:

- Eigenvalue problem $Lu = \nu u$
- Regularity estimates for $Lu = f$

Joint work with H. Chan, M. Fontelos, J. Wei

The fractional Laplacian $(-\Delta)^s$

- Infinitesimal generator of a Levy process
- Pseudo-differential operator, principal symbol $|\xi|^{2s}$.

$$\widehat{(-\Delta)^s} f = |\xi|^{2s} \hat{f}$$

- Singular integral formulation

$$(-\Delta)^s f(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{f(x) - f(\xi)}{|x-\xi|^{n+2s}} \, d\xi$$

- Nonlocal operator
- Extension viewpoint

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Fractional heat equation

$$\partial_\tau \varphi + (-\Delta_y)^s \varphi = 0, \quad \tau > 0, y \in \mathbb{R}^n$$

- **Self-similar variables**: $\tau = \log t, \quad x = \frac{y}{t^{\frac{1}{2s}}}$

$$\partial_t u + (-\Delta_x)^s - \tfrac{1}{2s} x \cdot \nabla u = 0, \quad x \in \mathbb{R}^n, t > 0$$

- **Finite time blow-up**: $t = -\log(T - \tau)$, $x = \frac{y}{(T-t)^{\frac{1}{2s}}}$

$$\partial_t u + (-\Delta_x)^s + \tfrac{1}{2s} x \cdot \nabla u = 0, \quad x \in \mathbb{R}^n, t > 0$$

- Note: L^* is the adjoint of L.

Theorem 1

Theorem*

L has eigenvalues $\nu = k$, $k \in \mathbb{N}$, with eigenfunctions

$$w_k(r) = e^{-\frac{r^{2s}}{4}} L_k^{\left(\frac{n-2s}{2s}\right)}\left(\frac{r^{2s}}{4}\right)$$

The eigenfunctions form a complete orthonormal basis.

* Up to a change of basis

Notes:

- Generalized Laguerre polynomials

$$L_k^{(\alpha)}(\rho) = \frac{1}{k!}\rho^{-\alpha}e^{\rho}\frac{d^k}{d\rho^k}(\rho^{\alpha+k}e^{-\rho}),$$

- Function spaces:

$$\|w\|_{L_w^2}^2 = \int_0^\infty w^2(r)e^{\frac{r^{2s}}{4}}r^{n-1}\,dr$$

Local case $s = 1 \to$ Fourier transform

$$\boxed{-\Delta u - \frac{x}{2} \cdot \nabla u - \frac{n}{2} = \frac{\nu}{2} u}$$

- Fourier transform in $\mathbb{R}^n$: $\zeta = |\xi|$, $\hat{u} = \phi(\zeta)$

$$\zeta^2 \phi + \frac{1}{2} \zeta \phi_\zeta = \frac{\nu}{2} \phi$$

- Solution of the ODE: $\phi(\zeta) = \zeta^\nu e^{-\zeta^2}$
- Exponential decay for u as $r \to \infty$ $\Leftrightarrow$ smoothness at the origin $\zeta = 0$ for ϕ
- $\nu = 2k$, $k \in \mathbb{Z}$
- Inverse Fourier (dim=1)

$$u(r) = D_k(e^{-\frac{r^2}{4}}) = e^{-\frac{r^2}{4}} H_k(r)$$

- Rodrigues formula

Mellin transform of $u = u(r)$

- **Definition**
$$\mathcal{M}u(z) := \int_0^\infty r^z u(r)\,\frac{dr}{r}.$$
- Usually, $z = \lambda i$, $\lambda \in \mathbb{R}$
- Equivalent to **Fourier** transform in the variable $t = -\log r$:
$$\mathcal{M}u(\lambda i) = \int_0^\infty r^{\lambda i} u(r)\,\frac{dr}{r} = \int_{\mathbb{R}} e^{-t\lambda i} u(e^{-t})\,dt$$
- **Mellin inversion formula**
$$u(r) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} r^{-z}\mathcal{M}u(z)\,dz$$
- **Plancherel identity**
$$\int_{\mathbb{R}} \left|\mathcal{M}u(\tfrac{n}{2} + \lambda i)\right|^2 d\lambda = \int_0^\infty |u(r)|^2 r^{n-1}\,dr$$
- **Function spaces**
$$\|u\|_{H^\beta}^2 = \int_{\mathbb{R}} (1 + |\lambda|^2)^\beta \left|\mathcal{M}u(\tfrac{n}{2} + \lambda i)\right|^2 d\lambda$$

Mellin symbol of the fractional Laplacian

Proposition

$$\mathcal{M}\{(-\Delta)^s u\}(z) = \Theta_s(z)\mathcal{M}u(z-2s)$$

for the symbol

$$\Theta_s(z) = 2^{2s}\frac{\Gamma(\frac{z}{2})\Gamma(\frac{n+2s-z}{2})}{\Gamma(\frac{n-z}{2})\Gamma(\frac{z-2s}{2})}$$

- New proof: Hankel+Mellin
- **Cylindrical metric**
  $g = \frac{dr^2+r^2d\theta^2}{r^2} = dt^2 + d\theta^2$ (change $r = e^{-t}$)
- Conformal fractional Laplacian on the cylinder
  $$\mathcal{F}P_s v(z) = \Theta_s(z)\mathcal{F}v(z)$$
- Conformal law
  $$P_s v = r^{\frac{n+2s}{2}}(-\Delta)^s\left(r^{-\frac{n-2s}{2}}v\right)$$

Back to the eigenvalue problem

$$\boxed{(-\Delta)^s u - \frac{r}{2s}\partial_r u - \frac{n}{2}u = \nu u}$$

- Take Mellin transform
$$\Theta_s(z)\mathcal{M}u(z-2s) = \left(\nu + \frac{n-z}{2s}\right)\mathcal{M}u(z)$$
- Change $V(z) = 2^{-z}\frac{\Gamma(\frac{n-z}{2})}{\Gamma(\frac{z}{2})\Gamma(\nu+\frac{n-z}{2s})}\mathcal{M}u(z)$
$$V(z-2s) = V(z)$$
- Solution is $V(z) = e^{\frac{\pi i}{s}z\ell}$, $\ell \in \mathbb{Z}$
- The "eigenfunction" is
$$\mathcal{M}u_\nu(z) = 2^z\frac{\Gamma(\frac{z}{2})\Gamma(\nu+\frac{n-z}{2s})}{\Gamma(\frac{n-z}{2})} \qquad \to \qquad u_\nu(r)$$

Proposition

Under Fourier transform, $\quad \mathcal{F}u_\nu(\zeta) = \zeta^\nu e^{-\zeta^{2s}}$

Choosing the function space

The "eigenfunction" is

$$\mathcal{M}u_\nu(z) = 2^z \frac{\Gamma(\frac{z}{2})\Gamma(\nu + \frac{n-z}{2s})}{\Gamma(\frac{n-z}{2})} =: \Lambda_s(z)\mathcal{W}_\nu(z),$$

where

$$\Lambda_s(z) := 2^z \frac{\Gamma\left(\frac{z}{2}\right)}{\Gamma\left(\frac{n-z}{2}\right)} 2^{-\frac{z}{s}} \frac{\Gamma\left(\frac{z}{2s}\right)}{\Gamma\left(\frac{n-z}{2s}\right)} \quad \to \text{change of basis}$$

and

$$\mathcal{W}_\nu(z) := 4^{\frac{z}{2s}} \frac{\Gamma\left(\nu + \frac{n-z}{2s}\right)\Gamma\left(\frac{z}{2s}\right)}{\Gamma\left(\frac{n-z}{2s}\right)} \quad \to \quad w_k(r) = e^{-\frac{r^{2s}}{4}} L_k^{\left(\frac{n-2s}{2s}\right)}\left(\frac{r^{2s}}{4}\right)$$

Lemma

$\{w_k\}_k$ is a complete o.n. basis for the scalar product

$$\langle w, w' \rangle = \int_0^\infty w(r)w'(r)e^{-\frac{r^{2s}}{4}} r^{n-1}\, dr$$

Prove $\nu = k$, $k \in \mathbb{N}$

Theorem

If $\nu + \frac{n}{2s}$ is not a non-positive integer, then

$$w_\nu(r) = 2s\frac{\Gamma\left(\nu+\frac{n}{2s}\right)}{\Gamma\left(\frac{n}{2s}\right)} M\left(\nu + \frac{n}{2s}; \frac{n}{2s}; -\frac{r^{2s}}{4}\right)$$

$$+ 2s \sum_{\ell=0}^{\lfloor -\nu-\frac{n}{2s} \rfloor} \frac{\sin(\pi(\nu+\ell))}{\pi}\Gamma(\nu+1+\ell)\Gamma(\nu+\frac{n}{2s}+\ell)\frac{(-1)^\ell}{\ell!}\left(\frac{r^{2s}}{4}\right)^{-\nu-\frac{n}{2s}-\ell}$$

Moreover, $w_\nu \in L^2_w$ iff $\nu \in \mathbb{N}$. In fact, as $r \to +\infty$,

$$w_k(r) \sim e^{-\frac{r^{2s}}{4}}\left(-\frac{r^{2s}}{4}\right)^k \quad \text{for} \quad \nu = k \in \mathbb{N},$$

$$w_\nu(r) \sim \left(\frac{r^{2s}}{4}\right)^{-\nu-\frac{n}{2s}} \quad \text{for} \quad \nu \notin \mathbb{N}.$$

The change of basis

- Recall $\Lambda_s(z) := 2^z \frac{\Gamma\left(\frac{z}{2}\right)}{\Gamma\left(\frac{n-z}{2}\right)} 2^{-\frac{z}{s}} \frac{\Gamma\left(\frac{z}{2s}\right)}{\Gamma\left(\frac{n-z}{2s}\right)}$ $\rightarrow$ change of basis
- Define the convolution

$$\Phi_s w(r) := \varphi_s \star w(r) = \int_0^\infty \varphi_s\left(\tfrac{r}{r'}\right) w(r') \, \tfrac{dr'}{r'}$$

Lemma

$$\|\Phi_s w\|_{H^\beta} = 2^{\frac{n}{2}(1-\frac{1}{s})} \|w\|_{H^\beta}.$$

Regularity, $s \in (0,1)$

$$\boxed{(-\Delta)^s u + \frac{1}{2s} x \cdot \nabla u = f, \quad u = u(r)}$$

Theorem 2

Given f satisfying the compatibility condition

$$\int_{\mathbb{R}^n} f(x) e_s(x)\, dx = 0,$$

then there exists a solution u satisfying

$$\|u\|_{H^{\beta+2s}} \leq C \|f\|_{H^\beta}$$

Remark:

$$e_s(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} e^{-|\xi|^{2s}}\, d\xi \sim \frac{1}{(1+|x|)^{n+2s}}$$

Proof via Fourier

$$\boxed{(-\Delta)^s u + \frac{1}{2s} x \cdot \nabla u = f \qquad \text{in } \mathbb{R}^n}$$

- Fourier transform
$$|\xi|^{2s}\widehat{u} - \frac{n}{2s}\widehat{u} - \frac{1}{2s}\xi \cdot \nabla\widehat{u} = \hat{f}, \qquad \xi \in \mathbf{R}^n$$
- Radial variables $v(\zeta) = \widehat{w}(\zeta)$, $\zeta = |\xi|$
$$(2s\zeta^{2s} - n)v - \zeta v' = 2sh, \quad \text{for } \zeta \in (0, \infty)$$
- ODE
$$\left(\zeta^n e^{-\zeta^{2s}} v\right)' = -2sh\zeta^{n-1} e^{-\zeta^{2s}}$$
- Explicit solution
$$\begin{aligned} v(\zeta) &= -2s\zeta^{-n} e^{\zeta^{2s}} \int_\zeta^\infty h(\varrho)\varrho^{n-1} e^{-\varrho^{2s}}\, d\varrho \\ &= 2s\zeta^{-n} e^{\zeta^{2s}} \int_0^\zeta h(\varrho)\varrho^{n-1} e^{-\varrho^{2s}}\, d\varrho. \end{aligned}$$
- Estimate $\qquad \|u\|_{H^{\beta+2s-\epsilon}} \leq C\|f\|_{H^\beta}$

Proof

- Mellin transform
- Magic change
- Riemann-Hilbert problem

$$V(\lambda) - V(\lambda - 2si) = F(\lambda)$$

- Conformal mapping $\eta = e^{\pi\lambda/s}$

$$W(\eta + 0i) - W(\eta - 0i) = G(\eta)$$

- Cauchy formula

$$W(\eta) = -\frac{1}{2\pi i}\int_0^\infty \frac{G(\eta')}{\eta - \eta'}\, d\eta'$$

- Fredholm condition

$$\frac{1}{\eta - \eta'} = -\frac{1}{\eta'} - \frac{\eta}{(\eta - \eta')\eta'}$$

Regularity via Mellin

$$\begin{aligned}
\int_0^\infty \left| \eta^\delta \int_0^\infty \frac{\phi(\eta')\,d\eta'}{\eta-\eta'} \right|^2 d\eta &= \int_{-\infty}^\infty d\lambda \left| \int_0^\infty \eta^{i\lambda-\frac{1}{2}+\delta} \left(\int_0^\infty \frac{\phi(\eta')\,d\eta'}{\eta-\eta'} \right) d\eta \right|^2 \\
&= \int_{-\infty}^\infty d\lambda \left| \int_0^\infty \phi(\eta') \left(\int_0^\infty \frac{\eta^{i\lambda-\frac{1}{2}+\delta}}{\eta-\eta'}\,d\eta \right) d\eta' \right|^2 \\
&= \int_{-\infty}^\infty d\lambda \left| \int_0^\infty \eta'^{\,i\lambda+\delta-\frac{1}{2}}\,\phi(\eta') \left(\int_0^\infty \frac{\eta^{i\lambda+\delta-\frac{1}{2}}}{\eta-1}\,d\eta \right) d\eta' \right|^2 \\
&\leq \pi^2 \int_{-\infty}^\infty \left| \cot\left[\left(i\lambda+\delta-\tfrac{1}{2} \right) \pi \right] \right|^2 \left| \int_0^\infty \eta'^{\,i\lambda+\delta-\frac{1}{2}}\,\phi(\eta')\,d\eta' \right|^2 d\lambda \\
&\leq c \int_{-\infty}^\infty \left| \int_0^\infty \eta'^{\,i\lambda+\delta-\frac{1}{2}}\,\phi(\eta')\,d\eta' \right|^2 d\lambda = c \int_0^\infty \left| \phi(\eta') \right|^2 (\eta')^{2\delta}\,d\eta'.
\end{aligned}$$

Future work

- Add a critical Hardy potential

$$\boxed{L_\kappa u = (-\Delta)^s u - \frac{x}{2s}\cdot\nabla u - \frac{n}{2}u + \frac{\kappa}{|x|^{2s}} \quad \text{in } \mathbb{R}^n}$$

- To construct type II singularities for the fractional heat equation (Herrero-Velázquez)

Thank you!!