The fractional porous medium equation on classes of noncompact Riemannian manifolds

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The fractional porous medium equation

We consider the following Cauchy problem, that we refer to as fractional porous medium equation (WFPME for short):

$$\begin{cases} u_t = -(-\Delta_M)^s (u^m) & \text{in } M \times (0,\infty), \\ u = u_0 & \text{on } M \times \{0\}, \end{cases} \tag{1}$$

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Our goal will be to prove basic well–posedness results for solutions, in a suitable sense, provided M satisfies appropriate geometric assumptions, and to prove smoothing effects for data in a suitable class, larger than $L^1(M)$.

The Euclidean case

When $M = \mathbb{R}^N$, equation (1) have been introduced and thoroughly studied by de Pablo, Quiros, Rodriguez, Vázquez, and then by Bonforte and Vázquez in three seminal papers: Adv. Math. 2011, CPAM 2012, Adv. Math. 2014.

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- existence of a (strong) solution;
- conservation of mass;
- order preserving property of the evolution;
- smoothing effects, namely bounds of the form ($p \geq 1$)

$$\|u(t)\|_\infty \leq C\frac{\|u_0\|_p^{\alpha_p}}{t^{\delta_p}} \quad \forall t > 0.$$

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Methods rely on representation formulas, i.e. on the explicit expression of the fractional laplacian in terms of a kernel, and/or on the Caffarelli-Silvestre extension method.

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$$\begin{aligned}(-\Delta_M)^s f(x) &= \int_0^{+\infty} [T_t f(x) - f(x)] \frac{\mathrm{d}t}{t^{1+s}} \\ &= \int_0^{+\infty} \left(\int_M k_M(t,x,y)\,(f(y)-f(x))\,\mathrm{d}m(y) \right) \frac{\mathrm{d}t}{t^{1+s}},\end{aligned}$$

where m is the Riemannian measure, T_t is the heat semigroup and K_M the heat kernel on M.

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where m is the Riemannian measure, T_t is the heat semigroup and K_M the heat kernel on M. In fact, the second equality holds if

$$\int_M k_M(t,x,y)\,\mathrm{d}m(y) = 1, \quad \forall x \in M$$

which will follow under our assumptions on M (see below).

Assumptions on M

Assumption 1 (Ricci+Faber-Krahn)

We require that M is an N-dimensional and that:

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Besides, we require that $\exists c > 0$ s.t. the Faber-Krahn inequality holds:

$$\lambda_1(\Omega) \geq c\, m(\Omega)^{-\frac{2}{N}} \tag{3}$$

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Notice that (3) is equivalent to the Nash inequality

$$\|f\|_2^{1+\frac{2}{N}} \leq C\,\|f\|_1^{\frac{2}{N}}\,\|\nabla f\|_2$$

or to the Sobolev inequality, if $N \geq 3$.

It is known (see e.g. Hebey 99) that (3) implies, for all $\varepsilon > 0$, $x, y \in M$ and $t > 0$, Gaussian upper bounds ($r(x, y)$ is geodesic distance):

$$k_M(t, x, y) \leq \frac{C}{t^{\frac{N}{2}}} \, e^{-\frac{r(x,y)^2}{(4+\varepsilon)t}} \, .$$

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It follows that *M* is *s*-nonparabolic, in the sense that

$$\mathbb{G}_M^s(x,y) := \int_0^{+\infty} \frac{k_M(t,x,y)}{t^{1-s}}\,\mathrm{dt}$$

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We prove in Theorem 1 existence of a weak-dual solution under Assumption 1 and for a class of data larger than L^1, in Theorems 2 and 4 smoothing effects for different set of data.

Further assumptions on M

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If M is Cartan-Hadamard, the Faber-Krahn inequality is always true, but a lower Ricci bound need not be. Assumption 2 holds both in $\mathbb{R}^n$ and on hyperbolic space $\mathbb{H}^N$, the latter being the simply connected, N-dimensional manifold whose sectional curvatures are everywhere equal to -1.

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We prove, in Theorem 3, smoothing effects for large times, for a class of data larger than L^1 ,the bounds being stronger than the ones given in Theorem 2, and similar to the long time behaviour proved in Vázquez, JMPA 2015 on $\mathbb{H}^N$, and to the smoothing effect by G., Muratori, Nonlin. Anal. 2016 for general manifolds satisfying Assumption 3.

On the concept of solution

Let $\mathbb{G}^s_M$ be the fractional Green function on M. We define, for every fixed $x_0 \in M$, $B_1(x_0)$ denoting the Riemannian ball centered in x_0 of radius one:

$$\|f\|_{L^1_{x_0,\mathbb{G}^s_M}} := \int_{B_1(x_0)} |f(x)| \, \mathrm{d}m(x) + \int_{M \setminus B_1(x_0)} |f(x)| \, \mathbb{G}^s_M(x, x_0) \, \mathrm{d}m(x) \, .$$

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Accordingly, we introduce the following space:

$$L^1_{\mathbb{G}^s_M}(M) := \left\{ f : M \to \mathbb{R} \text{ measurable} : \sup_{x_0 \in M} \|f\|_{L^1_{x_0,\mathbb{G}^s_M}} < +\infty \right\},$$

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endowed with the norm

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The definition of weak dual solution is based on the observation that applying the operator $(-\Delta_M)^{-s}$ to both sides of the equation we would formally obtain the "dual equation"

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Definition 1

Let $u_0 \in L^1_{\mathbb{G}^s_M}(M)$, with $u_0 \geq 0$. We say that u is a Weak Dual Solution (WDS) to problem (1) *if, for every $T > 0$:*

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- $$\int_0^T \int_M \partial_t\psi\,(-\Delta_M)^{-s}u\,dm\,\mathrm{d}t - \int_0^T \int_M u^m\,\psi\,dm\,\mathrm{d}t = 0$$

  for every test function $\psi \in C^1_c((0,T); L^\infty_c(M))$;

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  for every test function $\psi \in C^1_c((0,T); L^\infty_c(M))$*;*
- $u(0,\cdot) = u_0$ *a.e. in* M*.*

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Under Assumption 1, it clearly holds $L^1(M) \subseteq L^1_{\mathbb{G}^s_M}(M)$, and $L^1_{\mathbb{G}^s_M}(M) \subseteq L^1_{x_0,\mathbb{G}^s_M}(M)$ is clear by definition. One may then wonder whether those spaces actually coincide. The answer is negative. In fact we prove what follows:

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Proposition

Let M satisfy Assumption 1. Then one has:

$$L^1(M) \subsetneq L^1_{\mathbb{G}^s_M}(M) \subsetneq L^1_{x_0,\mathbb{G}^s_M}(M) \quad \textit{for all } x_0 \in M,$$

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with strict inclusions.

The result is proven by providing explicit functions which belong to one space but not the other ones.

In order to highlight the admissible decay rate for the kind of initial data we deal with, we provide some sufficient conditions for a function to belong to $L^1_{\mathbb{G}^s_M}(M)$ in the very special cases of $M = \mathbb{R}^N$ or $M = \mathbb{H}^N$.

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- *$M = \mathbb{H}^N$ and $|u_0(x)| \leq \dfrac{C}{(r(x,o))^a}$ for all $r(x,o) \geq R$, for some $a > s$.*

In both cases, initial data are allowed to decay qualitatively quite slower than functions in $L^1(M)$: the requested bound is dimension independent when $M = \mathbb{R}^N$, whereas functions in $L^1(\mathbb{H}^N)$ are expected to decay faster than $e^{-r(x,o)(N-1)}$.

Main results

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Theorem 1 (Existence of a WDS for data in $L^1_{\mathbb{G}^s_M}$)

Let M satisfy Assumption 1, and let u_0 be any nonnegative initial datum such that $u_0 \in L^1_{\mathbb{G}^s_M}(M)$. Then there exists a weak dual solution to problem (1)*, in the sense of Definition 1.*

WDS are obtained as monotone limits of mild solutions in $L^1(M) \cap L^\infty(M)$ associated to a monotone sequence of initial data.

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Mild solution with "good" data enjoy well-known properties and, by adapting Bonforte-Vázquez, Nonlin. Anal. 2016, it can be shown that such solution are WDS. Fundamental properties of solutions are then proved.

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Theorem 2 (Smoothing effects for data in $L^1(M)$)

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If, in addition, M satisfies Assumption 2, then we have

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Assume that M also satisfies Assumption 3 (and $u_0 \not\equiv 0$). Then:

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In fact, the long-time behaviour even of the linear, non-fractional heat equation is faster than in $\mathbb{R}^N$ (exponential!).

A similar behaviour has been noticed on $\mathbb{H}^N$ and related manifolds, in the non-fractional, non-linear situation, by Vázquez, JMPA 2015, G., Muratori, Vázquez, Adv. Math. 2017, G., Muratori, Vázquez, Math. Ann. 2019. The corresponding bounds are sharp when $s = 1$. We don't know if they are here (no known Barenblatt, nor barriers!).

When enlarging the class of allowed initial data, i.e. when dealing with the space $L^1_{\mathbb{G}^s_M}(M)$ in place of $L^1(M)$, we obtain the following $L^1_{\mathbb{G}^s_M}$-L^∞ smoothing estimates.

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$$\|u(t)\|_{L^\infty(M)} \leq C_1 \left(\frac{\|u(t)\|_{L^1_{\mathbb{G}^s_M}}^{2s\vartheta_1}}{t^{N\vartheta_1}} \vee \|u_0\|_{L^1_{\mathbb{G}^s_M}} \right) \leq C_2 \left(\frac{\|u_0\|_{L^1_{\mathbb{G}^s_M}}^{2s\vartheta_1}}{t^{N\vartheta_1}} \vee \|u_0\|_{L^1_{\mathbb{G}^s_M}} \right)$$

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  This problem has been solved in the Euclidean, non-fractional case in Bénilan, Crandall, Pierre, Indiana 1984, and "almost solved" in certain class of manifolds in G., Muratori, Punzo, JMPA 2018. The fractional setting is still not completely solved even in the Euclidean case.

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It is impossible to enter into details of the proof. But it might be instructive to state a couple of crucial Lemmata, to have a hint of the necessary tools.

Assume Assumptions 2 or 3. It is first fundamental to compare the Green function on M (and its integrals over geodesic balls) with the Green function on the associated space form M_κ (i.e. the hyperbolic space of constant curvature $-\kappa$, or $\mathbb{R}^n$ if $\kappa = 0$).

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$$\int_{B_r(o)} \mathbb{G}^s_M(x, o)\, dm(x) \leq \int_{B_r(o_c)} \mathbb{G}^s_{M_\kappa}(x, o_c)\, dm_{M_\kappa}(x)\,,$$

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where o_κ stands for any pole in M_κ and $B_r(o_\kappa) \subset M_\kappa$ for the geodesic ball of radius r in centered at o_κ. Furthermore, we also have that

$$\mathbb{G}^s_M(x,y) \leq \mathbb{G}^s_{M_\kappa}(x_\kappa,y_\kappa)$$

for all $x, y \in M$ and their corresponding transplanted points $x_\kappa, y_\kappa \in M_c$ with respect to polar coordinates centered at o and o_κ, respectively.

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One then notice that $\int_{B_r(o)} k_M(t,y,o)\,\mathrm{d}m(y)$ solves

$$\begin{cases}\partial_t u = \Delta_M u & \text{in } M\times(0,+\infty)\,,\\ u(0,\cdot) = \chi_{B_r(o)} & \text{in } M\,.\end{cases}$$

and concludes using known Hessian comparison Theorems.

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Lemma

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$$\underline{C}\, \|\psi\|_1 \left(1 \wedge r(x_0, x)^{N-2s}\right) \mathbb{G}^s_M(x, x_0) \le$$
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The dependence on the radius σ is needed. The proof depends strongly on Li-Yau estimates: if v is a positive solution to the heat equation on M, then

$$v(t_1,x_1) \le c_0 \left(\frac{t_2}{t_1}\right)^\beta v(t_2,x_2)\, e^{c_1 \frac{r(x_1,x_2)}{t_2-t_1} + c_2(t_2-t_1)}$$

for all $0 < t_1 < t_2 < 3$ and all $x_1, x_2 \in M$.

Open problems

- Solutions that may change sign: Extend our results to signed solutions. Also investigate whether extension methods as in Banica, González, Sáez, Rev. Mat. Iberoam. 2015, can be applied.

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- Uniqueness: Show that WDS are unique, not only the ones obtained by limits of monotone approximations, as done here. Such result is known from G., Muratori, Punzo, Calc. Var. 2015 in the Euclidean case for very weak solutions.
- Mass conservation: For positive, integrable solutions to (1), prove that $\|u(t)\|_1 = \|u(0)\|_1$ for all such solutions and all $t > 0$. Precise bounds for the fractional Laplacian of a test function should be proved, which is not elementary on general manifolds.

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THANKS FOR YOUR ATTENTION!