Nonlocal Fast Diffusion Equation on Bounded Domains

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joint work with Matteo Bonforte and Mikel Ispizua

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Regularity for nonlinear diffusion equations.
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UAM UNIVERSIDAD AUTONOMA DE MADRID

Introduction

The Cauchy-Dirichlet problem of NFDE

$$\begin{cases} u_t = -\mathcal{L}[u^m] & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0 & \text{in } \Omega, \\ u(t,x) \equiv 0 & \text{on the lateral boundary.} \end{cases} \tag{NFDE}$$

- $\mathcal{L}$ is a nonlocal linear operator
- $m \in (0,1)$ Fast Diffusion range
- $\Omega \subset \mathbb{R}^N$ bounded and $C^{2,\alpha}$
- Nonnegative solutions $u \geq 0$

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Fractional Laplacians:

$\mathcal{L} = (-\Delta_\Omega)^s$

General Nonlocal operators:

$\mathcal{L} = (-\Delta_\Omega)^\alpha + (-\Delta_\Omega)^\beta$,

$\mathcal{L} = (-\Delta_\Omega)^s + b\cdot\nabla$,

$\mathcal{L} = (c^{1/s} - \Delta_\Omega)^s - c$

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Recall: $m = 1$ Heat Eq.
$m > 1$ PME

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Reminder about the fractional Laplacian on $\mathbb{R}^N$

We have several equivalent definitions of $(-\Delta_{\mathbb{R}^N})^s$:

$$\boxed{s \in (0,1)}$$

- Using **Fourier Transform**:

$$\left((-\Delta_{\mathbb{R}^N})^s f\right)^{\widehat{}}(\xi) = |\xi|^{2s} \hat{f}(\xi)$$

- As a **Hypersingular kernel**:

$$(-\Delta_{\mathbb{R}^N})^s f(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{f(x)-f(y)}{|x-y|^{N+2s}} \, \mathrm{d}y$$

  where $c_{N,s}$ is a normalization constant.

- **Spectral definition**, in terms of the heat semigroup associated to $-\Delta_{\mathbb{R}^N}$:

$$(-\Delta_{\mathbb{R}^N})^s f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} f(x) - f(x)\right) \frac{\mathrm{d}t}{t^{1+s}}$$

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Remark: All operators has an inverse and discrete spectrum $(\lambda_i, \Phi_i)_{i\geq 1}$ s.t.

$$\mathcal{L}^{-1} f(x) = \int_\Omega \mathbb{G}_\Omega(x,y) f(y)\,\mathrm{d}y \qquad \& \qquad \Phi_1(x) \asymp \mathrm{dist}(x, \partial\Omega)^\gamma$$

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$\mathcal{L}$ is linear and **sub-Markovian**. Moreover, the inverse operator is of the form

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with $\mathbb{G}_\Omega$ satisfying one of the followings

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WHY?

- $H(\Omega) = \left\{ v \in L^2(\Omega) : \int_\Omega v\mathcal{L}v\,\mathrm{d}x < +\infty \right\}$ with $\langle u,v\rangle_{H(\Omega)} = \int_\Omega u\mathcal{L}v\,\mathrm{d}x$
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Questions about $u_t = -\mathcal{L}[u^m]$

- Do solutions exist? Are they unique?

Gradient Flow in $H^*(\Omega)$

Weak Dual Solution in $L^1_{\Phi_1}(\Omega)$

$$\int u\mathcal{L}^{-1}\varphi_t = \int u^m \varphi$$

φ test function

- Are solutions bounded?

$$\left.\begin{array}{l} u_0 \in L^p(\Omega) \\ u_0 \in L^p_{\Phi_1}(\Omega) \\ u_0 \in H^*(\Omega) \end{array}\right\} \longrightarrow u(t) \in L^\infty(\Omega) \quad \text{for every} \quad t > 0$$

- How is the asymptotic behaviour?

$$\exists T < +\infty \quad \text{s.t.} \quad u(T) \equiv 0 \qquad \& \qquad \|u(t,\cdot)\| \asymp (T-t)^{\frac{1}{1-m}}$$

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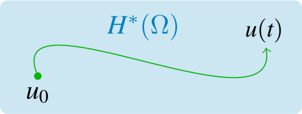

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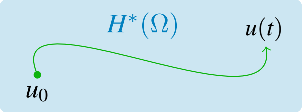

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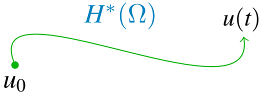

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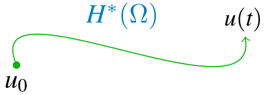

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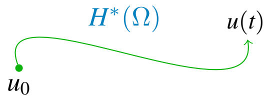

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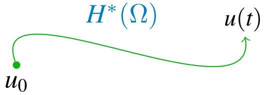


$\Longrightarrow$

Weak Dual Solution in $L^1_{\Phi_1}(\Omega)$

$$\int u\mathcal{L}^{-1}\varphi_t = \int u^m \varphi$$

φ test function

- Are solutions bounded? Smoothing effects

$$\left.\begin{array}{l} u_0 \in L^p(\Omega) \\ u_0 \in L^p_{\Phi_1}(\Omega) \\ u_0 \in H^*(\Omega) \end{array}\right\} \longrightarrow u(t) \in L^\infty(\Omega) \quad \text{for every} \quad t > 0$$

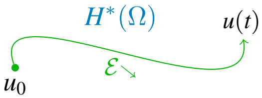
- How is the asymptotic behaviour? Extinction time and Sharp decay

$$\exists T < +\infty \quad \text{s.t.} \ \ u(T) \equiv 0 \qquad \& \qquad \|u(t,\cdot)\| \asymp (T-t)^{\frac{1}{1-m}}$$

Existence & Uniqueness

$\mathcal{E}$ convex and l.s.c. energy functional on $H^*(\Omega)$

$$\mathcal{E}(u) = \frac{1}{1+m}\int_\Omega |u|^{1+m}\,\mathrm{d}x \qquad m \in (0,1)$$



Theorem [Brezis-Komura 1967]

Given $u_0 \in H^*(\Omega)$, there exists only one Absolute Continuous Curve $u(t)$ in $H^*(\Omega)$ which minimizes the energy $\mathcal{E}$ along the flow.

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Let $u \in AC((0,T):H^*(\Omega))$ and $\partial\mathcal{E}[u]$ be the subdifferential $\mathcal{E}(u)$ in $H^*(\Omega)$. Then, u is a GF solution if

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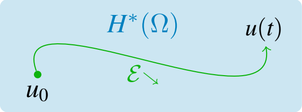

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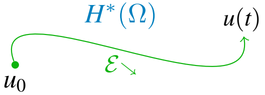

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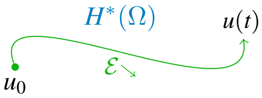

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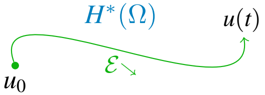

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More general nonnegative solutions of the **Dual Formulation** [Pierre-Vázquez]

$$\boxed{\mathcal{L}^{-1} u_t = -u^m}$$

in the space

$$L^1_{\Phi_1}(\Omega) := \left\{ f \in L^1_{loc}(\Omega) : \int_\Omega |f|\, \Phi_1 \,\mathrm{d}x < +\infty \right\}$$

Remark: if $u \geq 0$ then

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If $u_0 \in H^*(\Omega)$, then $\exists!\ u$ GF solution with $u(t) \xrightarrow{t\to 0^+} u_0$ in $H^*(\Omega)$.
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- For which u_0 do we have **bounded** solutions $u(t)$?

$$\mathbf{m_s} = \frac{N-2s}{N+2s} \qquad \mathbf{m_c} = \frac{N-2s}{N} \qquad \mathbf{p_c} = \frac{N(1-m)}{2s}$$

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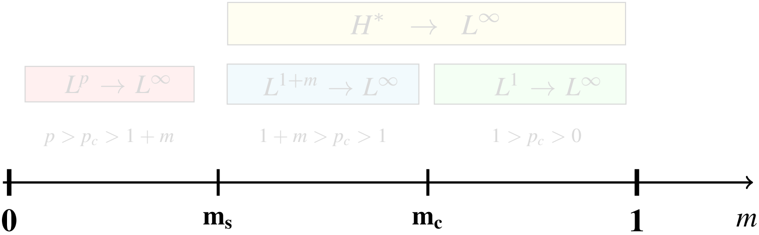

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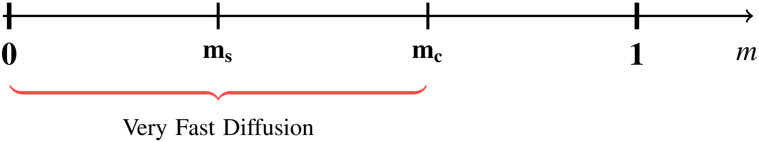

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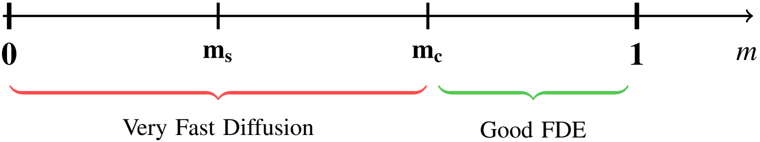

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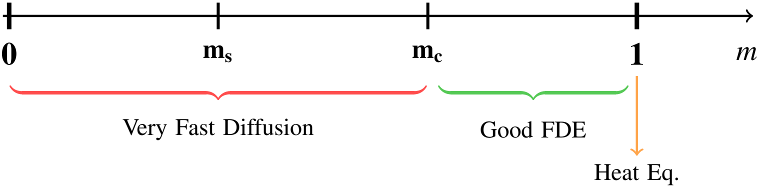

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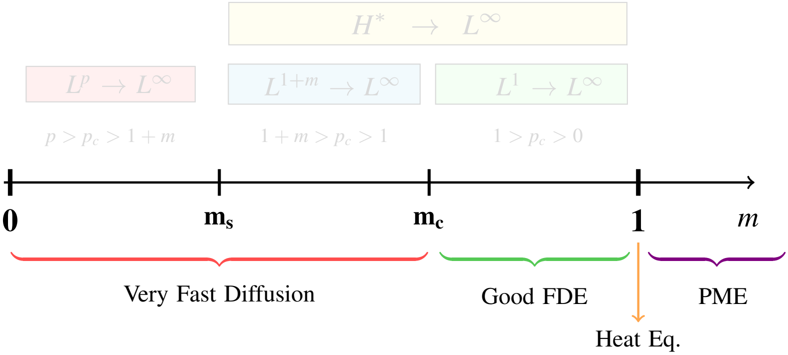

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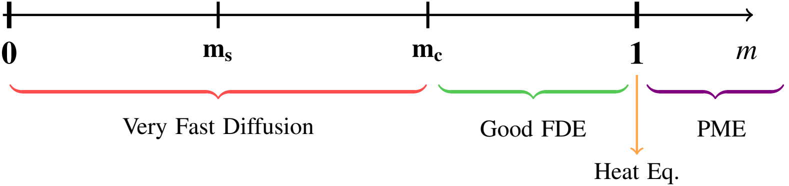

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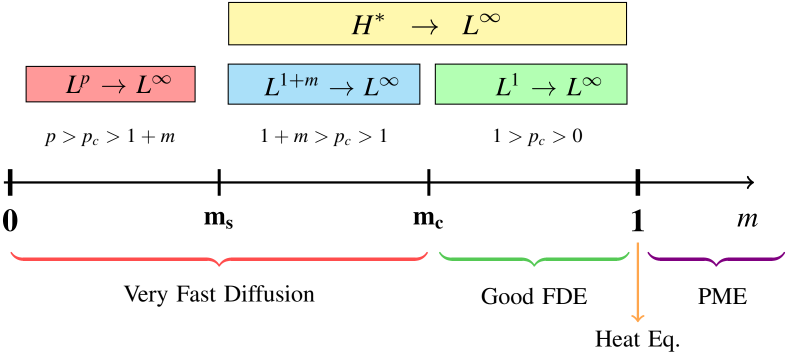

$L^p - L^\infty$ Smoothing effect

Let $N > 2s$ and assume (G1). Let u be a nonnegative WDS with $u_0 \in L^p(\Omega)$ and p admissible. Then, for every $t > 0$

$$\|u(t)\|_\infty \le \overline{\kappa}\,\frac{\|u_0\|_p^{2sp\vartheta_p}}{t^{N\vartheta_p}} \quad \text{with} \quad \vartheta_p = \frac{1}{2sp-N(1-m)}$$

The Green line in the (m,p)−plane

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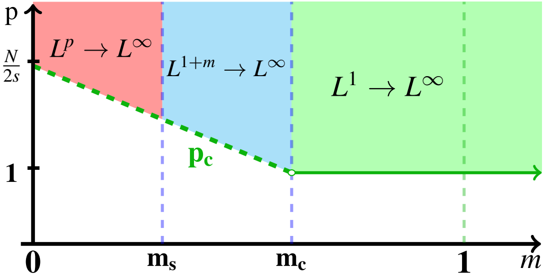

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$L^p_{\Phi_1} - L^\infty$ Smoothing effect

Let $N > 2s > \gamma$ and assume (G2). Let u be a nonnegative WDS with $u_0 \in L^p_{\Phi_1}(\Omega)$ and p admissible. Then, for every $t > 0$

$$\|u(t)\|_\infty \le \overline{\kappa}\, \frac{\|u_0\|_{L^p_{\Phi_1}}^{(2s-\gamma)p\,\vartheta_{p,\gamma}}}{t^{N\vartheta_{p,\gamma}}} \quad \text{with} \quad \vartheta_{p,\gamma} = \frac{1}{(2s-\gamma)p - N(1-m)}$$

$$\mathbf{p}_{\mathbf{c},\gamma} = \frac{N(1-m)}{2s-\gamma}$$

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The Weighted Green line in the $(m, p)-$plane

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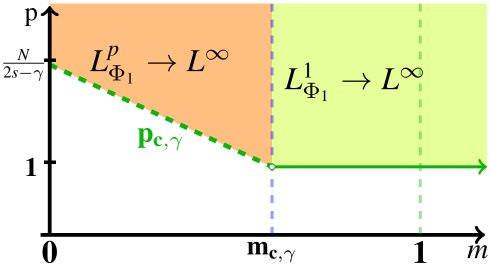

The Weighted Green line in the (m,p)−plane

Proof of Smoothing effects

Moser iteration VS Green Function method

Nonlinear Moser Iteration I. Main Ingredients [de Pablo, Quirós, Rodríguez, Vázquez'12]

- **GNS inequalities:** the following Sobolev-type ineq. holds

$$\|u\|_{2^*}^2 \le \mathcal{S}_{\mathcal{L}}^2 \, \|\mathcal{L}^{\frac{1}{2}} u\|_2^2 = \mathcal{S}_{\mathcal{L}}^2 \int_\Omega u \, \mathcal{L} u \, \mathrm{d}x \qquad \text{with} \quad 2^* = \frac{2N}{N-2s}$$

Interpolate to get a family of GNS inequalities: let $p > q > 0$ and

$$\frac{2q}{q+m-1} \le 2^* \qquad \text{and some} \qquad \theta \in (0,1)$$

so that

$$\|u\|_{\frac{2q}{q+m-1}} \le \|u\|_{2^*}^{\theta} \, \|u\|_{\frac{2p}{q+m-1}}^{1-\theta} \le \mathcal{S}_{\mathcal{L}}^{\theta} \, \|\mathcal{L}^{\frac{1}{2}} u\|_2^{\theta} \, \|u\|_{\frac{2p}{q+m-1}}^{1-\theta} \tag{GNS}$$

- **Stroock-Varopoulos inequality:** there exists a constant $c_{m,q} > 0$

$$\int_\Omega u^{q-1} \mathcal{L} u^m \, \mathrm{d}x \ge c_{m,q} \int_\Omega u^{\frac{q+m-1}{2}} \mathcal{L} u^{\frac{q+m-1}{2}} \, \mathrm{d}x = c_{m,q} \left\| \mathcal{L}^{\frac{1}{2}} u^{\frac{q+m-1}{2}} \right\|_2^2$$

Combining the two above inequalities, one gets

$$\int_\Omega u^{q-1} \mathcal{L} u^m \, \mathrm{d}x \ge c_{m,q} \left\| \mathcal{L}^{\frac{1}{2}} u^{\frac{q+m-1}{2}} \right\|_2^2 \ge c_{m,q} \mathcal{S}_{\mathcal{L}}^{-2} \frac{\|u\|_q^{\frac{q+m-1}{2\theta}}}{\|u\|_p^{\frac{1-\theta}{\theta}\,\frac{q+m-1}{2}}} \tag{M1}$$

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- **GNS inequalities:** the following Sobolev-type ineq. holds

$$\|u\|_{2^*}^2 \le \mathcal{S}_{\mathcal{L}}^2 \, \|\mathcal{L}^{\frac12} u\|_2^2 = \mathcal{S}_{\mathcal{L}}^2 \int_\Omega u\,\mathcal{L}u \,\mathrm{d}x \qquad \text{with} \quad 2^* = \frac{2N}{N-2s}$$

Interpolate to get a family of GNS inequalities: let $p > q > 0$ and

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Nonlinear Moser Iteration II. $L^p - L^q$ Smoothing Effects.

We shall prove first:

$$\|u(t)\|_q \le \overline{\kappa}_{p,q} \frac{\|u(t_0)\|_p^{\frac{p\vartheta_p}{q\vartheta_q}}}{(t-t_0)^{\frac{N(q-p)}{q}\vartheta_p}}, \qquad \text{with} \quad \vartheta_r = \frac{1}{2sr + N(1-m)} \tag{1}$$

The proof is formally simple:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q(t)\,\mathrm{d}x = q\int_\Omega u^{q-1}\partial_t u\,\mathrm{d}x = -q\int_\Omega u^{q-1}\mathcal{L}u^m\,\mathrm{d}x \le 0$$

$$(\mathrm{M1}) \longrightarrow \;\le -\mathcal{S}_{\mathcal{L}}^{-2}\frac{4q(q-1)m}{(q+m-1)^2}\frac{\|u(t)\|_q^{\frac{q+m-1}{2\theta}}}{\|u(t_0)\|_p^{\frac{1-\theta}{\theta}\frac{q+m-1}{2}}}$$

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Green Function method I. Main Ingredients.

Recall:

$$\mathcal{L}^{-1}u(x_0) = \int_\Omega u(x)\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x$$

Assumptions:

(G1) $$0 \le \mathbb{G}_\Omega(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}$$

(G2) $$c_{0,\Omega}\Phi_1(x)\Phi_1(y) \le \mathbb{G}_\Omega(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}\left(\frac{\Phi_1(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\Phi_1(y)}{|x-y|^\gamma}\wedge 1\right)$$

Benilan-Crandall Time Monotonicity estimate

$$u_t \le \frac{u}{(1-m)t} \qquad \text{in distributional sense,}$$

which is the weak version of

$$t \mapsto t^{-\frac{1}{1-m}}u(t,x) \qquad \text{is non-increasing for a.e. } x$$

Proof. Time-scailing + Comparison

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Green Function method II. "Almost" representation formula.

"Almost" representation formula [M. Bonforte, J.L. Vázquez 2015]

For all $x_0 \in \Omega$ and $0 < t_0 < t_1$, it holds that

$$\frac{u^m(t_1,x_0)}{t_1^{\frac{m}{1-m}}} \le \frac{1}{1-m}\int_\Omega \frac{u(t_0,x)-u(t_1,x)}{t_1^{\frac{1}{1-m}}-t_0^{\frac{1}{1-m}}}\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x \le \frac{u^m(t_0,x_0)}{t_0^{\frac{m}{1-m}}}$$

Proof. Choose $\varphi(t,x) := \chi_{[t_0,t_1]}(t)\delta_{x_0}(x)$ in the Weak Dual formulation:

$$\int_0^T\int_\Omega u\mathcal{L}^{-1}\varphi_t\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega u^m\varphi\,\mathrm{d}x\,\mathrm{d}t$$

to obtain

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Green Function method II. “Almost” representation formula.

“Almost” representation formula [M. Bonforte, J.L. Vázquez 2015]

For all $x_0 \in \Omega$ and $0 < t_0 < t_1$, it holds that

$$\frac{u^m(t_1,x_0)}{t_1^{\frac{m}{1-m}}} \le \frac{1}{1-m}\int_\Omega \frac{u(t_0,x)-u(t_1,x)}{t_1^{\frac{1}{1-m}}-t_0^{\frac{1}{1-m}}}\mathbb{G}_\Omega(x_0,x)\,\mathrm{d}x \le \frac{u^m(t_0,x_0)}{t_0^{\frac{m}{1-m}}}$$

Proof. Choose $\varphi(t,x) := \chi_{[t_0,t_1]}(t)\delta_{x_0}(x)$ in the Weak Dual formulation:

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We take $t_0 = 0$ in the "almost" representation formula to get

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$$(\mathrm{G1}) \Rightarrow \sup_{x_0 \in \Omega} \|\mathbb{G}_\Omega(x_0, \cdot)\|_{p'} < c_{p',\Omega} \quad \text{only if} \quad p' < \frac{N}{N-2s}$$

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Estimating both sides we get for $\varepsilon \in (0,m)$

$$u^m(t_1,x_0) \lesssim \left[\frac{\|u_0\|_p^{1-m+\varepsilon}}{t_1}\frac{1}{R^{\frac{2sp-N(1-m+\varepsilon)}{p}}}\right]^{\frac{m}{\varepsilon}} + \frac{\|u_0\|_p}{t_1}\frac{1}{R^{\frac{N}{p}+2s}}$$

Finally, optimizing in R and taking supremum in $x_0 \in \Omega$ we conclude that

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Consequences of a priori estimates

- **Boundary Behaviour**: "Almost" representation formula + Smoothings implies

$$\left\|\frac{u^m(t)}{\Phi_1}\right\|_\infty \le \overline{\kappa}\,\frac{\|u_0\|_p^{2sp\vartheta_p}}{t^{1+N\vartheta_p}} \qquad \forall t>0$$

and the analogous result with weighted norms.

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Extinction and Sharp Decay

Solutions vanish in finite time!!

- **Energy method**: Let us derive the energy and use Sobolev-type inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{1+m}\int_\Omega u^{1+m}(t)\,\mathrm{d}x\right)=\int_\Omega u^m u_t\,\mathrm{d}x=-\int_\Omega u^m\mathcal{L}u^m\,\mathrm{d}x=-\|u^m\|^2_{H(\Omega)}$$
$$\leq -\mathcal{S}_{\mathcal{L}}^2\|u\|^{2m}_{1+m}=-\mathcal{S}_{\mathcal{L}}^2\left(\int_\Omega u^{1+m}(t)\,\mathrm{d}x\right)^{1-\varepsilon}$$

with $\varepsilon=\frac{1-m}{1+m}$. Then, we integrate on $[0,t]$ to obtain

$$\|u(t)\|^{1-m}_{1+m}\leq\|u_0\|^{1-m}_{1+m}-\mathsf{c}_m\,t\Rightarrow\quad\boxed{\exists T>0 \text{ s.t. } \|u(T)\|_{1+m}=0}$$

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- **Other norms:** similar argument for H^*, L^p and $L^p_{\Phi_1}$ norms

Solutions vanish in finite time!!

- **Energy method**: Let us derive the energy and use Sobolev-type inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{1+m}\int_\Omega u^{1+m}(t)\,\mathrm{d}x\right) = \int_\Omega u^m u_t\,\mathrm{d}x = -\int_\Omega u^m \mathcal{L} u^m\,\mathrm{d}x = -\|u^m\|^2_{H(\Omega)}$$
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Sharp extinction rate

- $\mathbf{L^{1+m}}$ **sharp decay**: Combining previous estimates we have

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Thank You!!

M. Bonforte, PI, M. Ispizua - "The Cauchy-Dirichlet Problem for Singular Nonlocal Diffusions on Bounded Domains" https://arxiv.org/abs/2203.12545