A Steklov Version of the Torsional Rigidity

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Workshop: Regularity for nonlinear diffusion equations. Green functions and functional inequalities

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Introduction

From now on we set: $\Omega \subset \mathbb{R}^N$ with $\partial\Omega$ Lipschitz and normal vector ν_Ω, and $\delta > 0$.

- *Classical torsional rigidity*:

$$T(\Omega) = \sup_{\varphi \in W^{1,2}(\Omega)\setminus\{0\}} \frac{\left(\int_\Omega \varphi\, dx\right)^2}{\int_\Omega |\nabla \varphi|^2\, dx} \quad \rightarrow \quad \begin{cases} -\Delta v_\Omega = 1, & \text{in } \Omega, \\ v_\Omega = 0, & \text{on } \partial\Omega. \end{cases}$$

- *Boundary torsional rigidity functional*

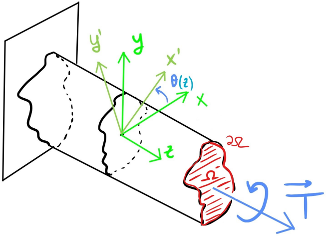
$$T(\Omega;\delta) = \sup_{\varphi \in W^{1,2}(\Omega)\setminus\{0\}} \frac{\left(\int_{\partial\Omega} \varphi\, d\mathcal{H}^{N-1}\right)^2}{\int_\Omega |\nabla \varphi|^2\, dx + \delta^2 \int_\Omega \varphi^2\, dx} \quad \rightarrow$$

$$\rightarrow \quad \begin{cases} -\Delta u + \delta^2 u = 0, & \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega \rangle = 1, & \text{on } \partial\Omega. \end{cases}$$

Motivation

Motivation: Torsion Problem by Saint Venant (1797-1886)

- *(i) Rotation* of the cross-sections as rigid bodies.
- *(ii) Warping* phenomena, equal for all the cross-sections.



- $$\begin{cases} -\Delta v_\Omega = 1, & \text{in } \Omega, \\ v_\Omega = 0, & \text{on } \partial\Omega, \end{cases}$$
  where v_Ω is the so called *stress-function*.
- The resultant torque $T(\Omega)$, *torsional rigidity*, can be expressed as

$$T(\Omega) = \int_\Omega v_\Omega \, dx$$

- Let Ω^* be any circle having the same area as Ω. Then ([Pòlya, 50][Makai, 66])

$$T(\Omega) \leq T(\Omega^*)$$

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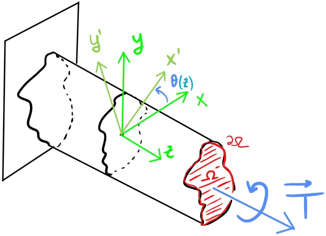

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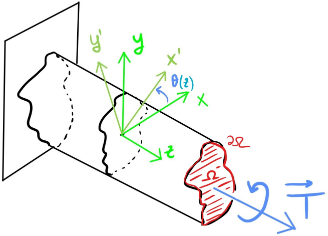

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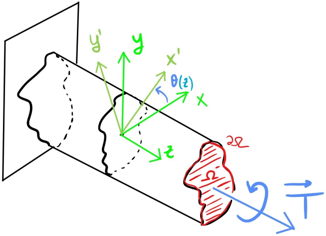

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Steklov Eigenvalue Problem

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega \rangle = \sigma\, u, & \text{on } \partial\Omega. \end{cases}$$

Due to the compact embedding $W^{1,2}(\Omega) \hookrightarrow L^q(\partial\Omega) \implies$ increasing sequence of eigenvalues $\{\sigma_n(\Omega;\delta)\}_{n\in\mathbb{N}}$.

Applications [N. Kuznetsov, T. Kulczycki, M. Kwásnicki, A. Nazarov, S. Poborchi, I. Polterovich, B. Siudeja, 2014]:

- In engineering or physics: sloshing problem, electric impedance tomography, stationary heat distribution with flux at the boundary dependent of temperature...
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Motivation: Limit of a Stationary Reaction-Diffusion Problem

Let $\Omega \subset \mathbb{R}^N$ with C^2 boundary $\partial\Omega$. We define the strip

$$w_\varepsilon = \{x - \alpha\nu_\Omega(x), x \in \partial\Omega, \alpha \in [0, \varepsilon)\}$$

and we call χ_ε its characteristic function. Then, present the following problem:

$$\begin{cases} -\nabla \cdot (a(x)\nabla u^\varepsilon(x)) + \lambda u^\varepsilon(x) + c(x)u^\varepsilon = \frac{1}{\varepsilon}\chi_\varepsilon f_\varepsilon & \text{in } \Omega, \\ a(x)\langle \nabla u^\varepsilon, \nu_\Omega\rangle + b(x)u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (1)$$

Theorem [J. M. Arrieta, A. Jiménez-Casas, A. Rodríguez-Bernal, 2008]

Set

$$a = 1 \quad b = c = 0 \quad \lambda = \delta^2 \quad \text{and} \quad f_\varepsilon = 1.$$

Then, by taking the limit as $\varepsilon \to 0$ we get that the unique solution u_ε of (1) converges to the solution of

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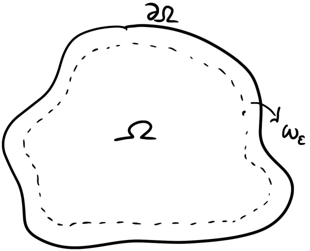

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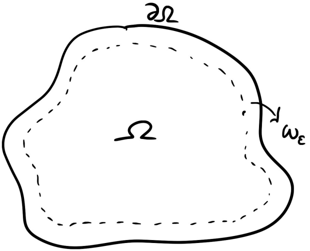

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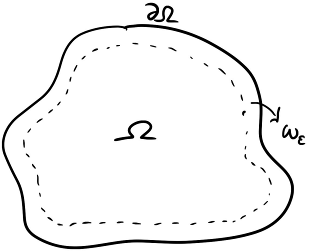

Boundary Torsional Rigidity

First properties

Scaling laws:

- $u_{t\,\Omega,\delta/t}(x) = t\, u_{\Omega,\delta}\left(\frac{x}{t}\right)$.
- $T\left(t\,\Omega;\, \frac{\delta}{t}\right) = t^N\, T(\Omega;\delta)$.

Relation to **Sobolev constant:**

$$W^{1,2}(\Omega) \hookrightarrow L^q(\partial\Omega),$$

$1 \le q \le 2^\#$, compact when $q \ne 2^\#$.

$$2^\# = \begin{cases} \dfrac{2N-2}{N-2}, & \text{if } N \ge 3, \\ \text{finite}, & \text{if } N = 2, \end{cases}$$

We set

$$\eta_q(\Omega) = \inf_{\varphi \in W^{1,2}(\Omega)} \left\{ \|\varphi\|^2_{W^{1,2}(\Omega)} \, : \, \|\varphi\|_{L^q(\partial\Omega)} = 1 \right\} > 0,$$

which is the sharp constant for the embedding.

Lemma

The supremum in the torsion functional is attained and

$$\frac{1}{\delta^2} \frac{(\mathcal{H}^{N-1}(\partial\Omega))^2}{|\Omega|} \le T(\Omega;\delta) \le \frac{1}{\min\{1,\delta^2\}} \frac{1}{\eta_1(\Omega)}.$$

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Proposition: *Unconstrained* concave problem

We reformulate the functional characterization of $T(\Omega, \delta)$ as

$$T(\Omega;\delta) = \sup_{\varphi\in W^{1,2}(\Omega)} \left\{ 2\int_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1} - \int_{\Omega} |\nabla\varphi|^2 \, dx - \delta^2 \int_{\Omega} \varphi^2 \, dx \right\}.$$

The supremum above is uniquely attained by a **non-negative** function $u_{\Omega,\delta} \in W^{1,2}(\Omega)$, which is the weak solution of the Neumann boundary value problem

$$\begin{cases} -\Delta u + \delta^2 u = 0, & \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega \rangle = 1, & \text{on } \partial\Omega. \end{cases}$$

This is, it satisfies the following **Weak Boundary Torsion Problem**:

$$\int_\Omega \langle \nabla u_{\Omega,\delta}, \nabla\varphi \rangle \, dx + \delta^2 \int_\Omega u_{\Omega,\delta} \, \varphi \, dx = \int_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1}, \quad \text{for } \forall \varphi \in W^{1,2}(\Omega).$$

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Constrained convex problem

Dual formulation [L. Brasco, 2021]

We set

$$\mathcal{A}^+(\Omega) = \left\{ (\phi, g) \in L^2(\Omega; \mathbb{R}^N) \times L^2(\Omega) \ : \ \begin{array}{rr} -\mathrm{div}\,\phi + \delta^2\, g \geq 0, & \text{in } \Omega \\ \langle \phi, \nu_\Omega \rangle \geq 1, & \text{on } \partial\Omega \end{array} \right\},$$

whith the conditions intended in weak sense. Then, we have

$$T(\Omega; \delta) = \min_{(\phi,g)\in\mathcal{A}^+(\Omega)} \left\{ \int_\Omega |\phi|^2\, dx + \delta^2 \int_\Omega g^2\, dx \right\}, \qquad (2)$$

and the minimum is uniquely attained by the pair $(\nabla u_{\Omega,\delta}, u_{\Omega,\delta})$.

Proof. For every non-negative $\varphi \in W^{1,2}(\Omega)$ and every $(\phi, g) \in \mathcal{A}^+(\Omega)$, we get that

$$2\int_{\partial\Omega} \varphi\, d\mathcal{H}^{N-1} - \left(\int_\Omega |\nabla\varphi|^2\, dx + \delta^2 \int_\Omega \varphi^2\, dx \right) \leq \int_\Omega |\phi|^2\, dx + \delta^2 \int_\Omega g^2\, dx,$$

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Relation to the δ-Steklov Eigenvalue

Recall:

$$\sigma_1(\Omega;\delta) = \min_{\varphi\in W^{1,2}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla\varphi|^2\,dx + \delta^2\int_\Omega \varphi^2\,dx}{\displaystyle\int_{\partial\Omega} u^2\,\mathcal{H}^{N-1}}$$

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Notice that by choosing φ to be the characteristic function of Ω, we obtain

$$\sigma_1(\Omega;\delta) \le \delta^2 \frac{|\Omega|}{\mathcal{H}^{N-1}(\partial\Omega)} \qquad \text{and thus} \qquad \lim_{\delta\to 0^+} \sigma_1(\Omega;\delta) = 0.$$

Remark: Pòlya type inequality

Applying Hölder's inequality on the boundary integral we get

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Applying Hölder's inequality on the boundary integral we get

$$\frac{\sigma_1(\Omega;\delta)\, T(\Omega;\delta)}{\mathcal{H}^{N-1}(\partial\Omega)} \le 1,$$

Relation to the δ-Steklov Eigenvalue

Recall:

$$\sigma_1(\Omega;\delta) = \min_{\varphi\in W^{1,2}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla\varphi|^2\,dx + \delta^2\int_\Omega \varphi^2\,dx}{\displaystyle\int_{\partial\Omega} u^2\,\mathcal{H}^{N-1}}$$

$$\begin{cases} -\Delta u + \delta^2 u = 0, & \text{in } \Omega, \\ \langle \nabla u, \nu_\Omega\rangle = \sigma\, u, & \text{on } \partial\Omega. \end{cases}$$

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Some Properties of the Torsion Function

Quantitative properties of $u_{\Omega,\delta}$

Regularity of $u_{\Omega,\delta}$

(i) The $L^1(\Omega)$ norm of $u_{\Omega,\delta}$ is given by

$$\int_\Omega u_{\Omega,\delta}\,dx = \frac{\mathcal{H}^{N-1}(\partial\Omega)}{\delta^2}.$$

(ii) Its trace is in $L^\infty(\partial\Omega)$, with the following estimate: for every $2 < q < 2^\#$,

$$\|u_{\Omega,\delta}\|_{L^\infty(\partial\Omega)} \le C_q \left(\frac{T(\Omega;\delta)^{\frac{q-2}{q}}}{\min\{1,\delta^2\}\,\eta_q(\Omega)} \right)^{\frac{q}{2\,(q-1)}},$$

where $C_q > 0$ is a constant only depending on q, which blows-up as $q \searrow 2$.

(iii) *Maximum principle.* We have $u_{\Omega,\delta} \in L^\infty(\Omega)$ and it holds

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$$\int_\Omega |\nabla(u_{\Omega,\delta} - L)_+|^2\, dx + \delta^2 \int_\Omega u_{\Omega,\delta}\,(u_{\Omega,\delta} - L)_+\, dx = 0.$$

Since both terms are non-negative:

$$\nabla(u_{\Omega,\delta} - L)_+ = 0, \qquad \text{and} \qquad u_{\Omega,\delta}\,(u_{\Omega,\delta} - L)_+ = 0, \qquad \text{a. e. in } \Omega.$$

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A few more properties

Monotonicity

For every $0 < \delta_0 < \delta_1$, we have that: $u_{\Omega,\delta_0} > u_{\Omega,\delta_1}$, in Ω

Asymptotics for $\delta \to 0$

Under the previous assumptions we have

$$\lim_{\delta\to 0+} \|\nabla(\delta^2 u_{\Omega,\delta})\|_{L^2(\Omega)} = 0 \quad \text{and} \quad \lim_{\delta\to 0+} \left\| \delta^2 u_{\Omega,\delta} - \frac{\mathcal{H}^{N-1}(\partial\Omega)}{|\Omega|} \right\|_{L^m(\Omega)} = 0,$$

for every $2 \leq m < +\infty$. Moreover,

$$\lim_{\delta\to 0+} \delta^2\, T(\Omega;\delta) = \frac{(\mathcal{H}^{N-1}(\partial\Omega))^2}{|\Omega|}.$$

Remark. With previous estimates we can prove that the "naive" lower bound given before,

$$\frac{1}{\delta^2}\, \frac{(\mathcal{H}^{N-1}(\partial\Omega))^2}{|\Omega|} \leq T(\Omega;\delta),$$

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Exact Solutions in Some Special Sets

Solution in the Ball

We indicate by I_α and K_α, the *modified Bessel functions with index* α of first and second kind, respectively, and by Γ the usual Gamma function. We recall that these functions have the following asymptotic behavior for z converging to 0:

$$I_\alpha(z) \sim \frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^\alpha \quad \text{and} \quad K_\alpha(z) \sim \begin{cases} -\log\left(\frac{z}{2}\right), & \text{for } \alpha = 0, \\ \frac{\Gamma(\alpha)}{2}\left(\frac{2}{z}\right)^\alpha, & \text{otherwise,} \end{cases}$$

Exact solution on the ball

Let $\delta > 0$ and let $B \subset \mathbb{R}^N$ be the ball of radius 1, centered at the origin. Then,

$$u_{B,\delta}(x) = \mathcal{U}_\delta(|x|) \qquad \text{where} \quad \mathcal{U}_\delta(\varrho) = \frac{\varrho^{1-N/2}\, I_{N/2-1}(\delta\,\varrho)}{\delta\, I_{N/2}(\delta)}.$$

and $u_{B,\delta}$ is a radially symmetric increasing function. Accordingly, we get

$$T(B;\delta) = \int_{\partial B} u_{B,\delta}\, d\mathcal{H}^{N-1} = \frac{N\,\omega_N\, I_{N/2-1}(\delta)}{\delta\, I_{N/2}(\delta)}.$$

Finally, the function $\varrho \mapsto \mathcal{U}'_\delta(\varrho)/\varrho$ is monotone non-decreasing

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Other exact solutions

Spherical Shells. Let $\delta > 0$. For $0 < r < R$, we consider the spherical shell

$$\Omega = \Big\{x \in \mathbb{R}^N \ : \ r < |x| < R\Big\}.$$

Then $u_{\Omega,\delta}$ is a radially symmetric function. This is explicitly given by

$$u_{\Omega,\delta}(x) = \mathcal{V}_{r,R,\delta}(|x|), \qquad \text{for } x \in \Omega,$$

where

$$\mathcal{V}_{r,R,\delta}(\varrho) = C_0\, \varrho^{1-\frac{N}{2}}\, I_{N/2-1}(\delta\, \varrho) + D_0\, \varrho^{1-\frac{N}{2}}\, K_{N/2-1}(\delta\, \varrho),$$

and the constants $C_0 = C_0(r,R,\delta) \neq 0$ and $D = D_0(r,R,\delta) \neq 0$ are explicit. Accordingly, we get

$$T(\Omega;\delta) = \frac{\Big[r^{1-\frac{N}{2}}\, K_{N/2}(\delta\, r) + R^{1-\frac{N}{2}}\, K_{N/2}(\delta\, R)\Big]\, \Big[R^{1-\frac{N}{2}} I_{1-N/2}(\delta\, R) + r^{1-\frac{N}{2}}\, I_{1-N/2}(\delta\, r)\Big]}{\delta\, r^{1-\frac{N}{2}}\, R^{1-\frac{N}{2}}\, \Big[I_{N/2}(R)\, K_{N/2}(r) - I_{N/2}(r)\, K_{N/2}(R)\Big]}$$
$$+ \frac{\Big[r^{1-\frac{N}{2}}\, I_{N/2}(\delta\, r) + R^{1-\frac{N}{2}}\, I_{N/2}(\delta\, R)\Big]\, \Big[R^{1-\frac{N}{2}}\, K_{1-N/2}(\delta\, R) + r^{1-\frac{N}{2}}\, K_{1-N/2}(\delta\, r)\Big]}{\delta\, r^{1-\frac{N}{2}}\, R^{1-\frac{N}{2}}\, \Big[I_{N/2}(r)\, K_{N/2}(R) - I_{N/2}(R)\, K_{N/2}(r)\Big]}.$$

Other exact solutions

Hyperrectangle. Let $\delta > 0$ and let $\ell_1, \ell_2, \dots, \ell_N > 0$. If we set

$$\Omega = \prod_{i=1}^{N} (-\ell_i, \ell_i),$$

then we have

$$u_{\Omega,\delta}(x) = \sum_{i=1}^{N} \frac{\cosh(\delta\, x_i)}{\delta\ \sinh(\delta\, \ell_i)}, \qquad \text{for every } x = (x_1, \dots, x_N) \in \Omega. \tag{3}$$

Its boundary torsional rigidity is given by

$$T(\Omega;\delta) = \sum_{k=1}^{N} \left[\frac{1}{\delta\ \tanh(\delta\, \ell_k)}\, \mathcal{H}^{N-1}(\Sigma_k) + \sum_{i \neq k} \frac{1}{\delta^2}\, \mathcal{H}^{N-2}(\Sigma_{k,i}) \right],$$

where

$$\Sigma_k = \left\{ x \in \overline{\Omega} \,:\, |x_k| = \ell_k \right\} \qquad \text{and} \qquad \Sigma_{k,i} = \left\{ x \in \overline{\Omega} \,:\, |x_k| = \ell_k,\ |x_i| = \ell_i \right\}.$$

Geometric Estimates

- Lower bound in dim 2.
- Lower bound in dim N (Convex sets).
- Upper bound in dim N (Convex sets).

Geometric Properties in dim 2

Let $\mathbb{D} = \{x \in \mathbb{R}^2 \, : \, |x| < 1\}$ and $\Omega \subsetneq \mathbb{R}^2$. Given a point $x_0 \in \Omega$,

- *Riemann Mapping Theorem* states that there exists a unique (up to a rotation) holomorphic isomorphism

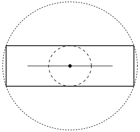
$$f_{x_0} : \mathbb{D} \to \Omega, \qquad \text{with } f_{x_0}(0) = x_0.$$

Furthermore, when $\partial\Omega$ is $C^{1,\alpha}$, we know that this is C^1 in $\overline{\mathbb{D}}$ and

$$f'_{x_0}(x) \neq 0, \qquad \text{for every } x \in \partial\mathbb{D}.$$ [S. E. Warschawski, 1961]

- We define the *boundary distortion radius* of Ω by

$$\dot{\mathcal{R}}_\Omega := \inf_{x_0 \in \Omega} \left(\frac{1}{2\pi} \int_{\partial\mathbb{D}} |f'_{x_0}|^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}},$$



- and the *inradius* : $\dot{r}_\Omega := \sup_{x_0 \in \Omega} |f'_{x_0}(0)|$.

Remark: As $|f'_{x_0}|^2$ is a subharmonic function, due to G. H. Hardy (1915), we have that

$$|f'_{x_0}(0)| \leq \left(\frac{1}{2\pi\varrho} \int_{\{|x|=\varrho\}} |f'_{x_0}|^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}} \leq \left(\frac{1}{2\pi} \int_{\partial\mathbb{D}} |f'_{x_0}|^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}}.$$

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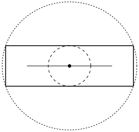

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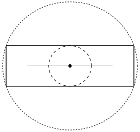

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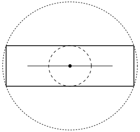

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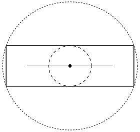

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Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected open set, with $\partial\Omega \in C^{1,\alpha}$. Then,

$$|\Omega| \leq \pi\, \dot{\mathcal{R}}_\Omega^2.$$

In particular, if Ω is a disk of radius R, we have $\dot{\mathcal{R}}_\Omega = R$ and equality holds.

Proof. Write

$$\int_{\mathbb{D}} |f'_{x_0}|^2\, dw = \int_0^1 \left(\int_{\{|x|=\varrho\}} |f'_{x_0}|^2\, d\mathcal{H}^1 \right) d\varrho.$$

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Let $\delta > 0$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected open set, with $C^{1,\alpha}$ boundary, for some $0 < \alpha \leq 1$. Then,

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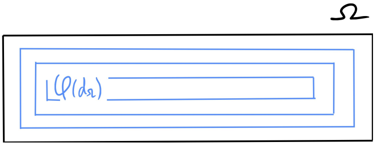
Method of interior parallels [E. Makai, 1954][G. Pòlya, 1960]



1D Lemma

Lemma

For every $\delta > 0$, we have

$$\alpha(\delta) := \sup_{\varphi \in W^{1,2}(I)} \frac{(\varphi(0))^2}{\int_I |\varphi'|^2 \, dt + \delta^2 \int_I \varphi^2 \, dt} = \frac{1}{\delta \tanh(\delta)}.$$

Moreover, the maximum is attained by

$$u_I(t) = \frac{1}{\delta} \left(\frac{\cosh(\delta t)}{\tanh(\delta)} - \sinh(\delta t) \right),$$

Proof. We rephrase the maximization problem to

$$\alpha(\delta) = \sup_{\varphi \in W^{1,2}(I)} \left\{ 2\varphi(0) - \int_I |\varphi'|^2 \, dt - \delta^2 \int_I \varphi^2 \, dt \right\},$$

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Proof Theorem: Lower bounds for convex sets.

Without loss of generality we prove it for $r_\Omega = 1$.

We introduce $\varphi(x) = u_I(d_\Omega(x))$ as test function in the torsion functional, where u_I is the solution of 1 dimensional problem. Then, by Coarea Formula and the fact that $|\nabla d_\Omega| = 1$ a.e. in Ω,

$$T(\Omega;\delta) \geq \frac{\left(\int_{\partial\Omega} \varphi \, d\mathcal{H}^{N-1}\right)^2}{\int_\Omega |\nabla\varphi|^2 \, dx + \delta^2 \int_\Omega \varphi^2 \, dx} = \frac{(u_I(0))^2 \left(\mathcal{H}^{N-1}(\partial\Omega)\right)^2}{\int_0^1 \left[|u_I'(t)|^2 + \delta^2 (u_I(t))^2\right] \mathcal{H}^{N-1}(\partial\Omega_t) \, dt}$$

$$\geq \frac{(u_I(0))^2}{\int_0^1 \left[|u_I'(t)|^2 + \delta^2 (u_I(t))^2\right] dt} \mathcal{H}^{N-1}(\partial\Omega).$$

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Convex sets: upper bounds N-dim

Recall the dual formulation of the torsion problem:

$$T(\Omega;\delta) = \min_{(\phi,g)\in\mathcal{A}^+(\Omega)} \left\{ \int_\Omega |\phi|^2\,dx + \delta^2 \int_\Omega g^2\,dx \right\},$$

$$\text{where, } \mathcal{A}^+(\Omega) = \left\{ (\phi,g)\in L^2(\Omega;\mathbb{R}^N)\times L^2(\Omega) \ : \ \begin{array}{ll} -\operatorname{div}\phi + \delta^2 g \ge 0, & \text{in } \Omega \\ \langle \phi, \nu_\Omega\rangle \ge 1, & \text{on } \partial\Omega \end{array} \right\}$$

For an open bounded convex set $\Omega \subset \mathbb{R}^N$, we define its *proximal radius* by

$$L_\Omega := \inf\left\{ R>0 \ : \ \exists x_0 \in M(\Omega) \text{ such that } \Omega \subset B_R(x_0) \right\}.$$

Theorem: Upper bound N-dim

Let $\delta > 0$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded **convex** set. Then,

$$T(\Omega;\delta) \le \left(\frac{r_\Omega}{L_\Omega}\right)^{N-2} \left(\frac{I_{N/2}(\delta L_\Omega)}{I_{N/2}(\delta r_\Omega)}\right)^2 T(B_{L_\Omega};\delta).$$

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We assume that the proximal center of Ω coincides with the origin. Then, we set

$$C_{\Omega,\delta} = \mathcal{U}'_{\delta\, L_\Omega}\left(\frac{r_\Omega}{L_\Omega}\right), \quad \phi_0 = \frac{1}{C_{\Omega,\delta}} \nabla u_{B_{L_\Omega},\delta} \quad \text{and} \quad g_0 = \frac{1}{C_{\Omega,\delta}} u_{B_{L_\Omega},\delta}.$$

Notice that by construction we have that $B_{r_\Omega}(x_\Omega) \subset \Omega \subset B_{L_\Omega}(x_\Omega)$. Then,

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The equality case. It is clear that the equality holds for any ball. Then, we assume that the inequality holds. In particular, it must be true that

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$$|B_{L_\Omega} \setminus \Omega| = 0.$$

Convexity implies that Ω must coincide with the ball B_{L_Ω}. □

From the previous result, we can get the following sharp geometric estimate, involving four geometric quantities.

Corollary: Sharp geometric estimate

Let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. Then, we have

$$\frac{(\mathcal{H}^{N-1}(\partial\Omega))^2}{|\Omega|} \leq N^2 \omega_N \left(\frac{L_\Omega}{r_\Omega}\right)^2 L_\Omega^{N-2}.$$

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Muchas gracias!