Madrid 2022

Existence and uniqueness results for Mean Field Game systems with fractional or nonlinear diffusions

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Joint work with

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Plan:

1. Introduction
2. MFG with uncontrolled Levy noise
3. MFG with controlled noise
4. Other results:
    - Numerics
    - Master eq'n
5. List of papers

1. Introduction: Mean Field Games (MFGs)

- ∞-player limit of games of identical players observing only the distribution of the other players.

- Nash-equilibria given by MFG system, e.g.

$$\begin{cases} -u_t - \mathcal{L}u + H(x, Du) = F(x, m(t)) & ; \quad u(t=T) = G(x, m(T)) \\ m_t - \mathcal{L}^* m + \operatorname{div}(m D_2 H(x, Du)) = 0 & ; \quad m(t=0) = m_0 \end{cases}$$

- u value func'n of generic player, m distribution of players.

- Backward HJB eq'n, forward Fokker-Planck (FP) eq'n

1. Heuristics: HJB eq'n

Minimization problem:

density of other players

$$u(x,t) = \inf_{\alpha} E\left\{ \int_t^T \left(F(X_s, \tilde{m}_s) + \tfrac{1}{2}\alpha_s^2 \right) ds + G(X_T, \tilde{m}_T) \right\}$$

control running cost terminal cost

where $X_s = x + \int_t^s \alpha_r \, dr + \int_t^s dL_r$

ctrl'ed SDE

Proc. w. generator $\mathcal{L}$

Dynamic programing:

$$-u_t - \mathcal{L}u + \underbrace{\max_{\alpha}\left\{ \alpha Du - \tfrac{1}{2}\alpha^2 \right\}}_{= \frac{1}{2}|Du|^2} = F(x, \tilde{m}(t)) \quad ; \quad u(t=T) = G(x, \tilde{m}(T))$$

Opt. feedback ctrl.: $\alpha^*(x,t) = Du(x,t)$

1. Heuristics: FP-eq'n

Opt. ctrl'ed process:

$$X^*_s = x + \int_t^s \alpha^*(X^*_r, r)\, dr + \int_t^s dL_r$$

PDF for X^*_t:

$m(x,t) =$ "$\mathrm{Prob}(X^*_t = x)$" $\rightsquigarrow$ FP eq'n:

$$m_t - \mathcal{L}^* m + \mathrm{div}\big(m \underbrace{\alpha^*(x,t)}_{= Du(x,t)}\big) = 0 \quad ; \quad m(t=0) = m_0$$

All players play optimally:

$\tilde{m} = m$, where m_0 init. distr. of players $\rightsquigarrow$ coupled syst.

1. Introduction: Diffusion and coupling

- $\mathcal{L}$ = diffusion operator, e.g. $\mathcal{L} = \Delta$ (noise = B.M.)

- Nonlocal MFG: noise = Levy jump process,

$$\mathcal{L}\varphi(x) = \int_{\mathbb{R}^d} [\varphi(x+z) - \varphi(x) - \mathbb{1}_{|z|<1}\, z \cdot D\varphi(x)]\, d\mu(z)$$ hypersingular integral

where $\mu \geq 0$ measure $\int |z|^2 \wedge 1 \, d\mu(z) < \infty$

ex. $\mathcal{L} = -(-\Delta)^{\frac{\sigma}{2}}$

- Couplings F, G:

Smoothing/nonlocal $\quad F, G : \mathbb{R}^d \times \underbrace{P(\mathbb{R}^d)}_{\text{prob. meas.}} \mapsto C_b^k(\mathbb{R}^d) \quad$ [e.g. $F(m) = m * K$]

Local $\quad F, G : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R} \quad$ [e.g. $F(m) = m^3$]

1. Introduction: Some litterature

- Origin: Lasry-Lions 2006, Huang-Malhame-Cains 2006

- Books:

  PDE approach (Bellman/DPP): Achdou et al., Springer 2020 ...

  Probabilistic (Pontryagin): Carmona-Delarue, Springer 2018
  FBSDEs

- Nonlocal MFG:

  Cesaroni et al. 2019, Cirant-Goffi 2019, Ersland-J. 2021

- MFG with controlled diffusion:

  Ricciardi PhD 2020, Andrade-Pimentel 2020, Benazoli et al. 2019/20, Lacker '15

1. Introduction: MFG models in this talk

Case 1: Uncontrolled nonlocal noise → linear diffusion (nonloc)

$$\text{(MFG 1)} \quad \begin{cases} -u_t - \mathcal{L}u + H(x, Du) = f(x, m(t)) & ; \; u(t=T) = g(x, m(T)) \\ m_t - \mathcal{L}^* m - \operatorname{div}\big(m\, D_2 H(x, Du)\big) = 0 & ; \; m(t=0) = m_0(x) \end{cases}$$

linear diffusion

Case 2: Controlled noise → nonlinear diffusion (loc. or nonloc.)

$$\text{(MFG 2)} \quad \begin{cases} -u_t - F(\mathcal{L}u) = f(x, m(t)) & ; \; u(t=T) = g(x, m(T)) \\ m_t - \mathcal{L}^*\big(F'(\mathcal{L}u)\, m\big) = 0 & ; \; m(t=0) = m_0(x) \end{cases}$$

nonlinear diffusion

2. MFG with uncontrolled Levy noise

$$(\mathrm{MFGT})\quad \begin{cases} -u_t - \mathcal{L}u + H(x, Du) = F(x, m(t)) & ; \quad u(t=T) = G(x, m(T)) \\ m_t - \mathcal{L}^* m - \operatorname{div}\left(m\, D_2 H(x, Du)\right) = 0 & ; \quad m(t=0) = m_0 \end{cases}$$

Well-posedness - classical sol'ns

$\mathcal{L} = 0$: 1st order, "variational" sol'ns in L^p [Cardaliaguet-Graber 2014]

$\mathcal{L} = \Delta$: 2nd order [Lasry-Lions ...]

$\mathcal{L} = -(-\Delta)^{\frac{\sigma}{2}}$ on $\mathbb{T}^n$: [Cesaroni et al. 2019], [Cirant - Goffi 2019]

$\mathcal{L} =$ "general" nondegenerate nonlocal Levy op. on $\mathbb{R}^n$: [Ersland - J. JDE 2021]

2. MFG with uncontrolled Levy noise

Conditions on $\mathcal{L}$:

(Levy) $\boxed{\mathcal{L}\varphi(x) = \int_{\mathbb{R}^d} \left[\varphi(x+z) - \varphi(x) - D\varphi(x)\,\mathbb{1}_{|z|<1}\right] d\mu(z)}$ where $\mu \geq 0$, $\int |z|^2 \wedge 1 \, d\mu(z) < \infty$

NON-DEGENERATE:

(Non-deg.') $\boxed{\frac{1}{C}\frac{1}{|z|^{d+\sigma}} \leq \frac{d\mu}{dz} \leq C\frac{1}{|z|^{d+\sigma}}}$ for $|z| \leq 1$, $\sigma \in (1,2)$ order of $\mathcal{L}$

$\mathcal{L} \sim \Delta^{\frac{\sigma}{2}}$

OR MUCH MORE GENERAL

(Non-deg.) Heat kernel bnd's + moment cond'n:

$$\boxed{\|D^\beta K_t\|_p \,,\ \|D^\beta K_t^*\|_p \leq c_T\, t^{-\frac{1}{\sigma}\left(|\beta| - (1-\frac{1}{p})d\right)}} + \boxed{r^\sigma \int_{|z|<1} \frac{|z|^2}{r^2} \wedge 1 \, d\mu(z) \leq c}$$

NO COND'N AT ∞ !

2. MFG with uncontrolled Levy noise

Examples of $\mathcal{L}$: $\sigma, \sigma_1, \dots, \sigma_2, Y \in (1,2)$

(i) $\mathcal{L} = -(-\Delta)^{\frac{\sigma}{2}}$ μ abs. cont.

(ii) $\mathcal{L} = -(-\partial_{x_1})^{\frac{\sigma_1}{2}} - \dots - (\partial_{x_n})^{\frac{\sigma_n}{2}}$ μ singular, different orders

(iii) $\mathcal{L}\varphi(x) = \int_0^\infty \left[\varphi(x+z\cdot e_1) - \varphi(x) - \partial_1\varphi(x)\cdot z\,\mathbb{1}_{|z|<1}\right] \frac{dz}{|z|^{1+\sigma}}$ μ spectrally one-sided

(iv) $\mathcal{L}\varphi(x) = \int_{\mathbb{R}} \left[\varphi(x+z\cdot e_1) - \varphi(x) - \partial_1\varphi(x)\mathbb{1}_{|z|<1}\right] \frac{Ce^{-Gz^+ - Mz^-}}{|z|^{1+Y}}dz$ CGMY non-symmetric

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Tempered $\mathcal{L}$, $\mathcal{L}$ in Finance, no moments at ∞, sums of $\mathcal{L}$, ...

(MFGT) $\boxed{\begin{array}{ll} -u_t - \mathcal{L}u + H(x,Du) = F(x,m(t)) & ;\ u(t=T) = G(x,m(T)) \\ m_t - \mathcal{L}m - \operatorname{div}(m\, D_2 H(x,Du)) = 0 & ;\ m(t=0) = m_0 \end{array}}$

2. MFG with uncontrolled Levy noise

Smoothing couplings:

(I) $F, G : \mathbb{R}^d \times P(\mathbb{R}^d) \longrightarrow \mathbb{R}$ Lipschitz ; $F(\cdot,m) \in C_b^2$, $G(\cdot,m) \in C_b^3$ indep. m

(II) $|H| + \dots + |D^3 H| \leq C_R \quad \forall\, x \in \mathbb{R}^d,\ |p| \leq R$; $|H(x,p) - H(y,p)| \leq C(1+|p|)|x-y|$

"$C^3_{b,loc}$" "x-Lipschitz"

(III) $m_0 \in C_b^2 \cap P(\mathbb{R}^d)$

NO MOMENT COND.'N

Theorem [Ersland - J. JDE 2021]

$\boxed{\exists \text{ cl. sol'n } u \in C_{b\ t,x}^{1,3},\ m \in C_{b\ t,x}^{1,2} \cap C([0,T]; P(\mathbb{R}^d)) \text{ of (MFGT) in } [0,T] \times \mathbb{R}^d}$

$$(\mathrm{MFGT}') \quad \boxed{\begin{array}{ll} -u_t - \mathcal{L}u + H(x,Du) = F(x,m(x,t)) & ; \; u(t=T) = G(x) \\ m_t - \mathcal{L}m - \mathrm{div}\big(m\, D_2H(x,Du)\big) = 0 & ; \; m(t=0) = m_0 \end{array}}$$

2. MFG with uncontrolled Levy noise

Local couplings:

(I) $F = F(x,s) \in C^2 \cap C_b(\underset{x}{\mathbb{R}^d} \times \underset{s}{\mathbb{R}}) \cap C(\underset{s}{\mathbb{R}}; \underset{x}{C^2_{b,x}}(\mathbb{R}^d))$, $\quad G = G(x) \in C^3_b(\mathbb{R}^d)$

(II) $|H| + \dots + |D^3H| \le C_R \quad \forall\, x \in \mathbb{R}^d,\ |p| \le R$; $\quad |H(x,p) - H(y,p)| \le C(1+|p|)|x-y|$

"$C^3_{b,loc}$" "x-Lipschitz"

(III) $m_0 \in C^2_b \cap P(\mathbb{R}^d)$

NO MOMENT COND.'N

Theorem [Ersland - J. JDE 2021]

$\exists$ cl. sol'n $u \in C^{1,3}_{b\; t,x}$, $m \in C^{1,2}_{b\; t,x} \cap C([0,T]; P(\mathbb{R}^d))$ of (MFGT') in $[0,T] \times \mathbb{R}^d$

(MFGT) $\boxed{\begin{array}{lll} -u_t - \mathcal{L}u + H(x, Du) = F(x, m(t)) & ; & u(t=T) = G(x, m(T)) \\ m_t - \mathcal{L}m - \mathrm{div}(m D_2 H(x, Du)) = 0 & ; & m(t=0) = m_0 \end{array}}$

2. MFG with uncontrolled Levy noise

Mixed local - nonlocal operators:

$$\mathcal{L}_{mix} = tr[aa^T D^2] + b \cdot D + \mathcal{L}_{nonloc}$$

Smoothing couplings:

(I) $F, G : \mathbb{R}^d \times P(\mathbb{R}^d) \longrightarrow \mathbb{R}$

....

Theorem [J - Rutkowski, in progress]

$\exists$ cl. sol'n (u, m) of (MFGT) in $[0,T] \times \mathbb{R}^d$

2. MFG with uncontrolled Levy noise

Existence ok – what about uniqueness ??

Uniqueness holds in all cases under :

(i) Monotonicity of F and G, e.g.

$$\int (F(m_1) - F(m_2))\, d(m_1 - m_2) \geq 0 \quad \forall m_1, m_2 \in P$$

(ii) Convexity of H (in gradient)

Proof: As in local case

(MFGT) $\boxed{\begin{array}{ll} -u_t - \mathcal{L}u + H(x,Du) = F(x,m(t)) & ;\ u(t=T) = G(x,m(T)) \\ m_t - \mathcal{L}m - \operatorname{div}(m\, D_2H(x,Du)) = 0 & ;\ m(t=0) = m_0 \end{array}}$

2. MFG with uncontrolled Levy noise

New contributions:

1.) General and mixed operators

2.) Local coupling

3.) Noncompact space $\mathbb{R}^n$

4.) No(!) moment assumptions, no P_1 or P_2, no $\mathcal{W}_1$

Wasserstein 1

5.) New regularity theory for nonlocal HJB eq'ns

(6.) Reduced regularity requirements)

$$\text{(MFGT)} \quad \boxed{\begin{array}{ll} -u_t - \mathcal{L}u + H(x,Du) = F(x,m(t)) & ; \ u(t=T) = G(x,m(T)) \\ m_t - \mathcal{L}m - \operatorname{div}(m\, D_2 H(x,Du)) = 0 & ; \ m(t=0) = m_0 \end{array}}$$

2. MFG with uncontrolled Levy noise

Existence of smooth solutions (smoothing couplings)

1) Heat kernel estimates : $K_t - \mathcal{L}K = 0$, $K(t=0) = \delta_0$

$$\|D^\alpha K(t,\cdot)\|_{L^1} \leq C \cdot t^{-\frac{|\alpha|}{\sigma}} \quad \text{for all } \alpha \in \mathbb{N}_0^d$$

2) Short time existence HJB (freeze m) : via Duhamel formula

$$\partial_i u(x,t) = K(t) * \partial_i u_0 + \int_0^t \partial_i K(t-s) * (H(Du) - F)\, ds$$

... existence in C^1_x by fixed pt. arg. (Banach) ... higher derivatives C^k_x ...

3) Long time existence HJB : via uniform Lipschitz bnd's

viscosity sol'n arguments [~ Droniou - Imbert $\mathcal{L} = -(-\Delta)^{\frac{\sigma}{2}}$]

(MFGT) $\boxed{\begin{array}{ll} -u_t - \mathcal{L}u + H(x, Du) = F(x, m(t)) & ;\ u(t=T) = G(x, m(T)) \\ m_t - \mathcal{L}m - \operatorname{div}(m\, D_2 H(x, Du)) = 0 & ;\ m(t=0) = m_0 \end{array}}$

2. MFG with uncontrolled Levy noise

4.) HJB: Time-regularity estimates

5.) FP (freeze u): Regularity – using results for HJB

6.) FP: A priori estimates

≥ 0, L^∞, mass-preserving, tightness, equi-cont. in R-K metric d_0

$$d_0(\mu_1, \mu_2) = \inf_{\substack{\varphi \in W^{1,\infty} \\ \|\varphi\|_\infty + \|D\varphi\|_\infty \leq 1}} \int \varphi(x)\, d(\mu_1 - \mu_2)(x) \quad ; \quad \mu_1, \mu_2 \in P(\mathbb{R}^n)$$

d_0- conv. $\Longleftrightarrow$ w. conv. of meas.

Previously: Stronger $d_1 = W_1$ conv. + moment assumptions!

(MFGT) $\boxed{\begin{array}{lll} -u_t - \mathcal{L}u + H(x, Du) = F(x, m(t)) & ; & u(t=T) = G(x, m(T)) \\ m_t - \mathcal{L}m - \mathrm{div}\left(m D_2 H(x, Du)\right) = 0 & ; & m(t=0) = m_0 \end{array}}$

2. MFG with uncontrolled Levy noise

7.) MFG : Solve coupled system by fixed pt. argument

Convex, compact set: $\mathcal{C} = \Big\{ \mu \in C([0,T]; P) : \underbrace{\int \varphi \, d\mu \le C_1}_{\text{tight}} ; \underbrace{\frac{d_0(\mu_s, \mu_t)}{|t-s|^{\frac{1}{\sigma}}} \le C_2}_{\text{equi-cont.}} \Big\}$

Continuous map in $\mathcal{C}$: $T : \mu \xmapsto[\text{HJB w. } m = \mu]{\text{cl. sol'n}} u(\mu) \xmapsto[\text{FP w. } u = u(\mu)]{\text{cl. sol'n}} m$

Schauder fixed pt. theorem

$\Longrightarrow$ $\exists$ $m \in \mathcal{C}$ such that $Tm = m$

$\overset{\text{def. of } T}{\Longleftrightarrow}$ $\exists$ (m, u) cl. sol'n of (MFG)

2. MFG with uncontrolled Levy noise

Remarks: i) "Lions fixed pt. argument"

ii) Local couplings more complicated [Ersland - J. JDE 2021]

... approximate w. nonlocal couplings ...

... uniform a priori estimates ...

... pass to the limit

Problem: F less regular in proof ($\sim$ regularity of m)

$\rightarrow$ new long time estimates

$\rightarrow$ no decoupling, fractional improvement, bootstrapping

3. MFG with controlled noise

$$(\text{MFG 2}) \begin{cases} -u_t - F(\mathcal{L}u) = f(x, m(t)) & ; \quad u(t=T) = g(x, m(T)) \\ m_t - \mathcal{L}^*\big(F'(\mathcal{L}u)\, m\big) = 0 & ; \quad m(t=0) = m_0(x) \end{cases}$$

Very resent direction, only initial results:

Ricciardi PhD 2020, Andrade-Pimentel 2020, Benazoli et al. 2019/20, Lacker '15

some overlap, much different — different setting — probabilistic, simpler setting

We will discuss:

Chowdhury - J. - Krupski arXiv: 2105:00073

(MFG 2) $\boxed{\begin{array}{ll} -u_t - F(\mathcal{L}u) = f(x,m(t))\,; & u(t=T) = g(x,m(T)) \\ m_t - \mathcal{L}^*\big(F'(\mathcal{L}u)\,m\big) = 0 \,; & m(t=0) = m_0(x) \end{array}}$

3. MFG with controlled noise

Sample assumptions:

(I) F convex, $F' \in W^{1,\infty}(\mathbb{R})$, $F' \geq 0$

(II) $f, g : \mathbb{R}^d \times P(\mathbb{R}^d) \to \mathbb{R}$ cont.; $f(\cdot, m(\cdot,\cdot)) \in C^1_b$, $g(\cdot,m) \in C^2_b$ indep. $m \in C([0,T],P)$

smoothing couplings

(III) $m_0 \in P(\mathbb{R}^d)$

(IV) $\int_{\mathbb{R}^d} \big[g(x,m_1(t)) - g(x,m_2(t))\big]\, d(m_1 - m_2)(x) \leq 0$

$\int_0^T \int_{\mathbb{R}^d} \big[f(x,m_1(t)) - f(x,m_2(t))\big]\, d(m_1 - m_2)(t,x)\, dt \leq 0$

Lions monotonicity cond'ns

(MFG 2) $\boxed{\begin{array}{ll} -u_t - F(\mathcal{L}u) = f(x, m(t)) \,; & u(t=T) = g(x, m(T)) \\ m_t - \mathcal{L}^*\big(F'(\mathcal{L}u)m\big) = 0 \,; & m(t=0) = m_0(x) \end{array}}$

3. MFG with controlled noise

Degenerate, low order:

(L1) $\mathcal{L}\varphi(x) = \int [\varphi(x+z) - \varphi(x)]\, d\mu(z)$, $\mu \geq 0$, $\sigma \in (0,1)$, $\alpha \in (\sigma, 1)$

$$0 \leq \int 1 \wedge \frac{|z|^\alpha}{r^\alpha}\, \mu(dz) \leq \frac{K}{\alpha - \sigma} r^{-\sigma} \quad \text{for } |r| \leq 1.$$

no lower bnd.

order $\leq \sigma < 1$, can be degenerate

Theorem 1: [Chowdhury - J. - Krupski]

If (L1) holds with $\sigma \leq 0{,}29$, then

$\exists!$ classical - very weak sol'n (u, m) of (MFG 2) in $[0,T] \times \mathbb{R}^d$

(MFG 2) $\boxed{\begin{array}{ll} -u_t - F(\mathcal{L}u) = f(x, m(t)) \,; & u(t=T) = g(x, m(T)) \\ m_t - \mathcal{L}^*\big(F'(\mathcal{L}u)\, m\big) = 0 \,; & m(t=0) = m_0(x) \end{array}}$

3. MFG with controlled noise

Nondegenerate, order 2:

($\mathcal{L}$2) $\mathcal{L} = \Delta$

(ND) $F' \geq K > 0$

Theorem 2: [Chowdhury - J. - Krupski]

$\exists!$ classical - very weak sol'n (u, m) of (MFG 2) in $[0,T] \times \mathbb{R}^d$

(MFG 2) $\boxed{\begin{array}{ll} -u_t - F(\mathcal{L}u) = f(x,m(t))\,; & u(t=T) = g(x,m(T)) \\ m_t - \mathcal{L}^*\big(F'(\mathcal{L}u)\,m\big) = 0 \quad ; & m(t=0) = m_0(x) \end{array}}$

3. MFG with controlled noise

Nondegenerate, order $\sigma \in (0,2)$:

($\mathcal{L}$3) $\quad \mathcal{L} \simeq -(-\Delta)^{\frac{\sigma}{2}}$

(ND) $\quad F' \geq K > 0$

(Schauder HJB) $\quad g, f \in C^{\alpha, \frac{\alpha}{\sigma}}_{x,t} \quad \xrightarrow[u(t=0)=g(x)]{u_t - F(\mathcal{L}u) = f(x,t)} \quad u_t, \mathcal{L}u \in C^{\alpha, \frac{\alpha}{\sigma}}_{x,t}$

Then we prove:

$\exists !$ classical – very weak sol'n (u,m) of (MFG 2) in $[0,T] \times \mathbb{R}^d$

$\boxed{\text{Q to YOU: Do you have references for (Schauder HJB)? Fully nonlin.!}}$

3. MFG with controlled noise

Remarks:

i) More general results in Chowdhury - J. - Krupski arXiv: 2105:00073

... less regularity ... more $\mathcal{L}$'s ... abstract theory ... "derivation"

ii) Very weak sol'ns of FP: $m \in C([0,T]; P(\mathbb{R}^d))$ solving

$$\int m(t)\,\varphi\,dx = \int m_0\,\varphi\,dx + \int_0^t \int F'(\mathcal{L}u)\,m(s)\,\mathcal{L}\varphi\,dx\,ds \qquad \forall\,\varphi \in C_c^\infty$$

iii) Existence: Variant of Lions fixed pt. argument

iv) Uniqueness: Variant of Lions cross-multiplication method

Uses monotonicity and convexity, but no strict mon. or convexity!

3. MFG with controlled noise

v) Uniqueness:

Requires ! of FP eq'n $\rightsquigarrow$ new results for degenerate eq'ns

Prop.: If $(\mathcal{L}1)$ holds, $m_0 \in P(\mathbb{R}^d)$, and
order $\sigma < 1$

$$0 \leq b \in C([0,T]; C_b^\beta(\mathbb{R}^d)) \quad \text{for} \quad \beta > \sigma + \frac{\sigma}{1-\sigma},$$

then there is a unique very weak sol'n of

$$m_t - \mathcal{L}^*(bm) = 0, \quad m(t=0) = m_0.$$

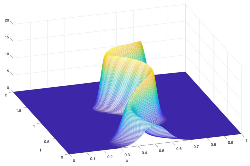
non-Lipschitz

degen.

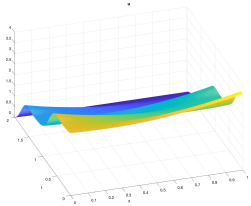
duality arg.
+ visc. sol'ns

iv) Semi-heuristic derivation of MFG: New ctrl. problem,
ctrl'ed time change rates.

4. Other results: Numerical schemes (MFG1)



(A) $m(t,x)$



(B) $u(t,x)$

- Nonlocal, smoothing coupling
- SL schemes:

  HJB: discrete time DPP eq'n, interpolation

  Opt. ctrl.: Approx. by smoothing

  FP: FP for discr. time proc., duality, P1 test-func'ns

- Stability, compactness, sometimes full conv.:

  dim $= 1$, degenerate PDEs OR dim ≥ 1, non-degen. PDEs

Chowdhury, Ersland, and Jakobsen.
Numerical Approximations of Fractional and Nonlocal Mean Field Games.
Found. Comput. Math. (2022), https://doi.org/10.1007/s10208-022-09572-w.

4. Other results: Master eq'n, conv. $N \mapsto \infty$ players

Define $\mathcal{U} : (t_0, x, m_0) \longmapsto u(x,t)$, (u,m) solves $\begin{cases} (\text{MFG 1}) \\ u(T,x) = G(x, m(T)) \\ m(t_0) = m_0 \end{cases}$

prob. meas

Smooth $\mathcal{U}$ satifies MASTER EQ'N:

$$(\text{ME}) \quad \begin{cases} \mathcal{U}_t - \mathcal{L}_x \mathcal{U} + H(D\mathcal{U}) - \int \left(\Delta_y + H'(D_x \mathcal{U}) \nabla_y \right) \frac{\delta \mathcal{U}}{\delta m}(t,x,y,m)\, dm(y) = F(x,m) \\ \mathcal{U}(T,x,m) = G(x,m) \end{cases}$$

in $(0,T) \times \mathbb{R}^n \times P(\mathbb{R}^n)$

Work in progress w. A. Rutkowski:

- $\exists !$ of cl. sol'ns of (ME) when $\mathcal{L}$ nonlocal / mixed
- Next: N-player games $\xrightarrow{N \to \infty}$ MFG

Thank you for the attention!

Ersland, Jakobsen.

On fractional and nonlocal parabolic Mean Field Games in the whole space.

J. Differential Equations 301, 2021.

Chowdhury, Jakobsen, Krupski.

On fully nonlinear parabolic mean field games with examples of nonlocal and local diffusions.

Preprint: https://arxiv.org/abs/2104.06985

Chowdhury, Ersland, and Jakobsen.

Numerical Approximations of Fractional and Nonlocal Mean Field Games.

Found. Comput. Math. (2022), https://doi.org/10.1007/s10208-022-09572-w.