UNIVERSITÄT
BIELEFELD
Faculty of Mathematics

The Wiener criterion for nonlocal Dirichlet problems

joint work with Ki-Ahm Lee and Se-Chan Lee

Minhyun Kim

Workshop: Regularity for nonlinear diffusion equations.
Green functions and functional inequalities
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Dirichlet problem for the Laplace equation

- Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Given $g \in C(\partial\Omega)$, the Perron's solution

$$u(x) = \sup\{v(x) : v \in C(\overline{\Omega}) \text{ is subharmonic in } \Omega,\ v \leq g \text{ on } \partial\Omega\}$$

  solves the Dirichlet problem for Δ.
- However, it does not imply

$$\lim_{\Omega \ni x \to x_0} u(x) = g(x_0) \tag{1}$$

  for boundary points $x_0 \in \partial\Omega$.

Definition

A boundary point $x_0 \in \partial\Omega$ is called *regular w.r.t.* Δ if (1) holds $\forall g \in C(\partial\Omega)$.

- (1) is connected to the geometric properties of the boundary through the concept of barrier function.

Examples of regular and irregular boundaries

Two dimensional case

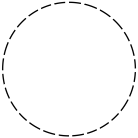

Figure: Regular

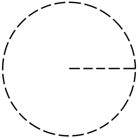

Figure: Regular

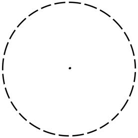

Figure: Irregular at 0

n-dimensional case, $n > 2$

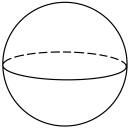

Figure: Regular

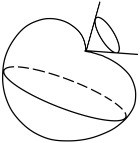

Figure: Regular

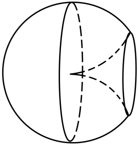

Figure: Irregular at 0

Other sufficient conditions

- Measure density condition: $\inf_{0<r<r_0} \frac{|B_r(x_0)\setminus\Omega|}{r^n} \geq c$.
- Exterior corkscrew condition, exterior Reifenberg flat condition, ...

Wiener criterion

Definition (Capacity)

Let Ω be an open set and $K \subset \Omega$ a compact set. The *capacity of* K *in* Ω is defined by

$$\mathrm{cap}(K, \Omega) = \inf \left\{ \int_\Omega |\nabla v|^2 dx : v \in C_c^\infty(\Omega), v \geq 1 \text{ on } K \right\}.$$

- Note that $\mathrm{cap}(\overline{B_\rho}, B_{2\rho}) \sim \rho^{n-2}$.

Theorem (Wiener '24)

A boundary point $x_0 \in \partial\Omega$ is regular w.r.t. Δ if and only if

$$\int_0 \frac{\mathrm{cap}(\overline{B_\rho(x_0)} \setminus \Omega, B_{2\rho}(x_0))}{\rho^{n-2}} \frac{d\rho}{\rho} = +\infty.$$

Theorem (Littman–Stampacchia–Weinberger '63)

A boundary point $x_0 \in \partial\Omega$ is regular w.r.t. Δ if and only if it is regular w.r.t. any uniformly elliptic operator.

Quasilinear elliptic equations

- Given $g \in W^{1,p}(\Omega)$, there exists a unique weak solution $u \in W^{1,p}(\Omega)$ of

$$Qu := -\mathrm{div}\mathcal{A}(x, \nabla u) = 0 \text{ in } \Omega$$

with $u - g \in W_0^{1,p}(\Omega)$, where $\mathcal{A}(x,\xi) \cdot \xi \approx |\xi|^p$, $p \in (1, \infty)$.
- In particular, u has a representative that is continuous on Ω.
- A boundary point $x_0 \in \partial\Omega$ is said to be *regular w.r.t. Q* if

$$\lim_{\Omega \ni x \to x_0} u(x) = g(x_0)$$

for all $g \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.

Theorem (Maz'ya '70, Gariepy–Ziemer '77, Lindqvist–Martio '85, and Kilpeläinen–Malý '94)

A boundary point $x_0 \in \partial\Omega$ is regular w.r.t. Q if and only if

$$\int_0 \left(\frac{\mathrm{cap}_p(\overline{B_\rho(x_0)} \setminus \Omega, B_{2\rho}(x_0))}{\rho^{n-p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} = +\infty.$$

Nonlocal Dirichlet problems

- Goal: find a necessary and sufficient condition for a boundary point to be regular w.r.t. a nonlinear nonlocal operator

$$\mathcal{L}u(x) = 2\text{p.v.} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))k_{s,p}(x, y)dy,$$

where $s \in (0, 1)$, $p \in (1, \infty)$, and $k_{s,p}$ is a measurable function satisfying $k_{s,p}(x, y) = k_{s,p}(y, x)$ and

$$\frac{\Lambda^{-1}}{|x-y|^{n+sp}} \leq k_{s,p}(x, y) \leq \frac{\Lambda}{|x-y|^{n+sp}}, \quad \Lambda \geq 1.$$

Nonlocal Dirichlet problems

- Function spaces:

$$V^{s,p}(\Omega|\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \to \mathbb{R} : u|_\Omega \in L^p(\Omega), \frac{|u(x) - u(y)|}{|x - y|^{n/p+s}} \in L^p(\Omega \times \mathbb{R}^n) \right\},$$

$$W_0^{s,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{V^{s,p}(\Omega|\mathbb{R}^n)}.$$

- Let $g \in V^{s,p}(\Omega|\mathbb{R}^n)$. There exists a unique weak solution $u \in V^{s,p}(\Omega|\mathbb{R}^n)$ of $\mathcal{L}u = 0$ in Ω with $u - g \in W_0^{s,p}(\Omega)$.
- In particular, u has a representative that is continuous on Ω.

Definition

A boundary point $x_0 \in \partial\Omega$ is said to be *regular w.r.t.* $\mathcal{L}$ if

$$\lim_{\Omega \ni x \to x_0} u(x) = g(x_0)$$

for each $g \in V^{s,p}(\Omega|\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

Main theorem

Definition (Capacity)

Let Ω be an open set and $K \subset \Omega$ a compact set. The *(s,p)-capacity of K in Ω* is defined by

$$\mathrm{cap}_{s,p}(K,\Omega) = \inf\left\{ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\,dydx : v \in C_c^\infty(\Omega), v \geq 1 \text{ on } K \right\}.$$

Theorem (**K.**–Lee–Lee, '22)

A boundary point $x_0 \in \partial\Omega$ is regular w.r.t. $\mathcal{L}$ if and only if

$$\int_0 \left(\frac{cap_{s,p}(\overline{B_\rho(x_0)} \setminus \Omega, B_{2\rho}(x_0))}{\rho^{n-sp}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} = +\infty.$$

- See [Eilertsen '00 and Björn '21] for $\mathcal{L} = (-\Delta)^s$.

Corollary (**K.**–Lee–Lee, '22)

The regularity of a boundary point depends only on n, s, and p, not on the operator $\mathcal{L}$ itself.

Idea of proof

Sufficiency (x_0 irregular $\implies$ Wiener integral $< \infty$)

- View the solution u as an admissible function for $\mathrm{cap}_{s,p}(\overline{B_\rho(x_0)} \setminus \Omega, B_{2\rho}(x_0))$.
- Use local boundedness and Weak Harnack inequality up to the boundary.
- $\implies$ Wiener integral is finite.

Necessity (Wiener integral $< \infty \implies x_0$ irregular)

- Consider the $\mathcal{L}$-potential u_ρ of $\overline{B_\rho(x_0)} \setminus \Omega$ in $B_{8\rho}(x_0)$.
- If there exists $\rho > 0$ such that $u_\rho(x_0) < 1$, then x_0 is irregular.
- Wolff potential estimate $\implies u_\rho(x_0) < 1$.

Sufficiency (1/3)

Theorem (Local boundedness up to boundary)

Let $p \in (1, n/s]$ and $B_R(x_0) \subset \mathbb{R}^n$. If u is a weak subsolution of $\mathcal{L}u = 0$ in Ω, then

$$\sup_{B_{R/2}(x_0)} u_M^+ \leq \delta \, Tail(u_M^+; x_0, R/2) + C(\delta) \left(\fint_{B_R(x_0)} (u_M^+)^p \, dx \right)^{1/p},$$

where $M = \sup_{B_R(x_0) \setminus \Omega} u_+$, $u_M^+ = \max\{u, M\}$, and

$$Tail(v; x_0, r) = \left(r^{sp} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|v(y)|^{p-1}}{|y - x_0|^{n+sp}} \, dy \right)^{\frac{1}{p-1}}.$$

Theorem (Weak Harnack inequality up to boundary)

Let $p \in (1, n/s]$, $t \in (0, \frac{n(p-1)}{n-sp})$ and $B_R(x_0) \subset \mathbb{R}^n$. If u is a weak supersolution of $\mathcal{L}u = 0$ in Ω such that $u \geq 0$ in $B_R(x_0)$, then

$$\left(\fint_{B_{R/2}(x_0)} (u_m^-)^t \, dx \right)^{1/t} \leq C \inf_{B_{R/4}(x_0)} u_m^- + C \, Tail((u_m^-)_-; x_0, R),$$

where $m = \inf_{B_R(x_0) \setminus \Omega} u$ and $u_m^- = \min\{u, m\}$.

Sufficiency (2/3)

- If x_0 is irregular, then

$$\lim_{\rho\to 0} \sup_{\Omega\cap B_\rho(x_0)} u > g(x_0) \quad \text{or} \quad \lim_{\rho\to 0} \inf_{\Omega\cap B_\rho(x_0)} u < g(x_0).$$

  Assume WLOG

$$L := \lim_{\rho\to 0} \sup_{\Omega\cap B_\rho(x_0)} u > g(x_0)$$

  and choose $l \in \mathbb{R}$ such that $L > l > g(x_0)$.
- Find $r_* > 0$ such that $l \geq \sup_{\overline{B_r(x_0)}\setminus\Omega} g$ for any $r \in (0, r_*)$ by continuity of g.
- Consider $u_r := M(r) - (u - l)_+$, where $M(r) = \sup_{B_r(x_0)} (u - l)_+$.
- Then, $(u_r)_m^- = u_r$.

Sufficiency (3/3)

- Let $\rho \in (0, r_*/4)$ and $\eta \in \mathrm{cutoff}(\overline{B_\rho(x_0)}, B_{2\rho}(x_0))$. Then,

$$\frac{u_{4\rho}\eta}{M(4\rho)}$$

is admissible for $\mathrm{cap}_{s,p}(\overline{B_\rho(x_0)} \setminus \Omega, B_{2\rho}(x_0))$.

- By the weak Harnack inequality local boundedness up to boundary, we have

$$\begin{aligned}
&\int_0^{r_*/4} \left(\frac{\mathrm{cap}_{s,p}(\overline{B_\rho(x_0)} \setminus \Omega, B_{2\rho}(x_0))}{\rho^{n-sp}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \\
&\leq C \int_0^{r_*/4} \left(\inf_{B_\rho} u_{4\rho} + \mathrm{Tail}(u_{4\rho}^-; x_0, 4\rho) \right) \frac{d\rho}{\rho} \\
&= C \int_0^{r_*/4} \big(M(4\rho) - M(\rho) + \mathrm{Tail}(u_{4\rho}^-; x_0, 4\rho) \big) \frac{d\rho}{\rho} \\
&\leq C \left(\sup_{B_{4r_*}(x_0)} u + |l| + \mathrm{Tail}(u; x_0, r_*) \right) < \infty.
\end{aligned}$$

Necessity (1/4)

Definition

Let $\psi \in C_c^\infty(\Omega)$ be such that $\psi \equiv 1$ on K. The $\mathcal{L}$-harmonic function in $\Omega \setminus K$ with $u - \psi \in W_0^{s,p}(\Omega \setminus K)$ is called the *$\mathcal{L}$-potential of K in Ω.*

Lemma

Let u_ρ be the $\mathcal{L}$-potential of $\overline{B_\rho(x_0)} \setminus \Omega$ in $B_{8\rho}(x_0)$. If there exists $\rho > 0$ such that $u_\rho(x_0) = \liminf_{\Omega \ni x \to x_0} u_\rho(x) < 1$, then x_0 is irregular.

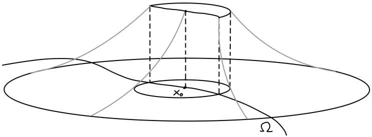

Figure: $u_\rho(x_0) = 1$

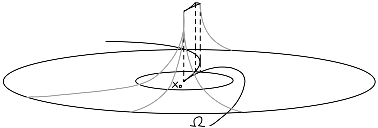

Figure: $u_\rho(x_0) < 1$

Necessity (2/4)

Definition

A function $u : \mathbb{R}^n \to (-\infty, +\infty]$ is said to be *$\mathcal{L}$-superharmonic in* Ω if it satisfies the following properties:

- $u < +\infty$ a.e. in $\mathbb{R}^n$.
- u is lower semicontinuous in Ω.
- for each $\Omega' \Subset \Omega$ and each weak solution $v \in C(\overline{\Omega'})$ of $\mathcal{L}v = 0$ in Ω' with $v_+ \in L^\infty(\mathbb{R}^n)$ such that $u \geq v$ on $\partial\Omega'$ and a.e. on $\mathbb{R}^n \setminus \Omega'$, it holds that $u \geq v$ in Ω'.
- $u_- \in L^{p-1}_{sp}(\mathbb{R}^n)$.

Theorem (Korvenpää–Kuusi–Palatucci '17)

- *If an $\mathcal{L}$-superharmonic function is of $L^\infty_{loc}(\Omega)$ or $W^{s,p}_{loc}(\Omega)$, then it is a weak supersolution.*
- *If a weak supersolution u is lower semicontinuous in Ω and satisfies $u(x) = \operatorname{ess\,lim\,inf}_{y \to x} u(y)$ for all $x \in \Omega$, then u is $\mathcal{L}$-superharmonic.*

Necessity (3/4)

Theorem (Wolff potential estimate)

Let $p \in (1, n/s]$. Let u be an $\mathcal{L}$-superharmonic function in $B_{8\rho}(x_0)$, which is nonnegative in $B_{8\rho}(x_0)$. If $\mu = \mathcal{L}u$ exists, then

$$u(x_0) \leq C \left(\inf_{B_{2\rho}(x_0)} u + \mathbf{W}^{\mu}_{s,p}(x_0, 4\rho) + Tail(u_\rho; x_0, 2\rho) \right),$$

where

$$\mathbf{W}^{\mu}_{s,p}(x_0, 4r) = \int_0^{4r} \left(\frac{\mu(B_\rho(x_0))}{\rho^{n-sp}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}$$

is the Wolff potential of μ.

- When $p > 2 - \frac{s}{n}$, it is known for SOLA, see [Kuusi–Mingione–Sire '15].
- The existence of μ for general $\mathcal{L}$-supreharmonic function is open.
- However, it exists for $u = u_\rho$.

Necessity (4/4)

- Assume that

$$\int_0 \left(\frac{\mathrm{cap}_{s,p}(\overline{B_r(x_0)} \setminus \Omega, B_{2r}(x_0))}{r^{n-sp}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.$$

- It is enough to find small $\rho > 0$ so that $u(x_0) < 1$. Indeed, we have

$$\begin{aligned} u_\rho(x_0) &\leq C \left(\mathbf{W}^{\mu}_{s,p}(x_0, 4\rho) + \inf_{B_{2\rho}(x_0)} u_\rho + \mathrm{Tail}(u_\rho; x_0, 2\rho) \right) \\ &\leq C \int_0^{4\rho} \left(\frac{\mathrm{cap}_{s,p}(\overline{B_r(x_0)} \setminus \Omega, B_{2r}(x_0))}{r^{n-sp}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\quad + C \left(\varepsilon^p + \varepsilon^{-\frac{p}{p-1}} \frac{\mathrm{cap}_{s,p}(\overline{B_r(x_0)} \setminus \Omega, B_{2r}(x_0))}{\rho^{n-sp}} \right)^{\frac{1}{p-1}} \end{aligned}$$

for any $\varepsilon > 0$. Take sufficiently small ε and then send $\rho \to 0$.

Thank you for your attention!