Interacting helical traveling waves for the Gross-Pitaevskii equation

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Regularity for nonlinear diffusion equations. Green functions and functional inequalities (June 13-17, 2022)

The Gross-Pitaevskii equation.

Joint work with J. Dávila, M. del Pino (University of Bath) and Rémy Rodiac (Universitè d'Orsay).

We consider

$$i\partial_t \psi + \Delta\psi + (1 - |\psi|^2)\psi = 0, \quad \psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}. \qquad \text{(GP)}$$

$\rightsquigarrow$ Nonlinear Schrödinger equation with a Ginzburg-Landau potential.

$\rightsquigarrow$ Bose-Einstein condensate theory, nonlinear optics, superfluidity.

Two conserved quantities:

- The **energy**: $E(\psi) = \frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla\psi|^2 + \frac{1}{2}(1 - |\psi|^2)^2\right]dx.$
- The **momentum**: $P(\psi) = \int_{\mathbb{R}^3}(i\psi, \nabla\psi)dx.$

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Traveling wave solutions.

We are interested in special solutions of the form

$$\psi(t,x) = u(x_1, x_2, x_3 - Ct), \quad u : \mathbb{R}^3 \to \mathbb{C},$$

where $C \in \mathbb{R}$ is a constant $\rightsquigarrow$ Traveling wave.

If ψ solves (GP) then u satisfies

$$iC\partial_{x_3} u = \Delta u + (1 - |u|^2)u \quad \text{in } \mathbb{R}^3. \qquad \text{(GP-TW)}$$

Jones-Putterman-Roberts program ('86): existence of finite energy solutions if and only if $C \in (0, \sqrt{2})$ $\rightsquigarrow$ subsonic range.

$\rightsquigarrow$ Nonexistence for $C > \sqrt{2}$ and $n \geq 3$, and for $C \geq \sqrt{2}$ and $n = 2$, [Gravejat '03, '04].

$\rightsquigarrow$ Existence for $C \in (0, \sqrt{2})$ and $n \geq 3$, [Béthuel-Orlandi-Smets, '04], [Maris, '13].

$\rightsquigarrow$ Existence for almost every $C \in (0, \sqrt{2})$ and $n = 2$, [Béthuel-Gravejat-Saut, '09], [Bellazzini-Ruiz, '20].

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Traveling wave solutions.

Question: Location and dynamics of vortices (the zeroes of the wave function u).

Let $\varepsilon > 0$ small, and consider $C = c\varepsilon|\log\varepsilon|$, with $c \in \mathbb{R}$ fixed. Defining $u_\varepsilon(x) = u\left(\frac{x}{\varepsilon}\right)$, it solves

$$i\varepsilon^2|\log\varepsilon|\partial_{x_3}u_\varepsilon = \varepsilon^2\Delta u_\varepsilon + (1-|u_\varepsilon|^2)u_\varepsilon \quad \text{in } \mathbb{R}^3.$$

Motivation: The study of the equation

$$i\varepsilon^2|\log\varepsilon|\partial_t\psi + \varepsilon^2\Delta\psi + (1-|\psi|^2)\psi = 0 \text{ in } \mathbb{R}\times\Omega.$$

For initial data *concentrating* near a 1D-curve then ψ also concentrates near a 1D curve evolving through the binormal curvature flow

$$\partial_t\gamma = \partial_s\gamma \wedge \partial^2_{ss}\gamma, \qquad \text{(BCF)}$$

[Jerrard, '02], [Jerrard-Smets, '18].

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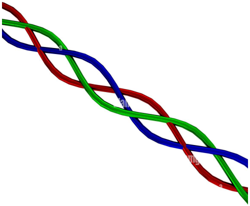
Special solutions of (BCF):

• Stationary straight line $\rightsquigarrow$ Standard GL vortex of degree 1 in $\mathbb{R}^2$.

• Translating circle $\rightsquigarrow$ Trav. waves with vortex rings [Béthuel-Orlandi-Smets, '04], [Chiron, '04], [Lin-Wei-Yang, '13].

• Translating rotating helix $\rightsquigarrow$ Trav. waves with helical vortex set [Chiron, '05].

Goal: to construct solutions with velocity $C = c\varepsilon|\log \varepsilon|$ and a special form in the vortex set.



Consider the Klein-Majda-Damodaran system

$$-i\partial_t f_k(t,z) - \partial_{zz} f_k(t,z) - 2\sum_{j\neq k} d_j d_k \frac{f_k - f_j}{|f_k - f_j|^2} = 0, \quad k = 1,\dots,n. \quad \text{(KMD)}$$

$\rightsquigarrow$ Derived in fluid mechanics [Klein-Majda-Damodaran, '95].

For well-prepared initial data, the vortex set of solutions to (GP) converges, as $\varepsilon \to 0$, towards n almost parallel filaments solutions to the (KMD) system [Jerrard-Smets, '21].

Solutions: for $k = 1,\dots,n$,

$$f_k(t,z) := \hat{d} e^{i(z-\nu t)} e^{\frac{2i(k-1)\pi}{n}}, \text{ with } \hat{d} := \sqrt{\frac{n-1}{1-\nu}}, \quad \nu < 1.$$

Observation: The curves $z \mapsto (f_k(t,z), z)$ are helices arranged with polygonal symmetry.

Question: Can we construct a solution with a vortex set of multiple helices?

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The Ginzburg-Landau vortex in $\mathbb{R}^2$.

The equation

$$\Delta w + (1 - |w|^2)w = 0 \text{ in } \mathbb{R}^2, \qquad \text{(GL)}$$

has a solution $w : \mathbb{R}^2 \to \mathbb{C}$ that can be written as

$$w(z) = \rho(r)e^{i\theta} \quad \text{with} \quad \rho(0) = 0, \quad \rho(+\infty) = 1.$$

Observation: This provides a solution in $\mathbb{R}^3$ for (GL) and (GP) $\rightsquigarrow$ vortex set along a straight line.

Idea: To glue copies of this vortex in an appropriate way to construct the helices.

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The construction.

Theorem (Dávila-del Pino-M.-Rodiac, '21)

For each $n \geq 2$ and for every $-\infty < c < 1$, there exists $\varepsilon 0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exists u_ε which solves (GP-TW) with $C = c\varepsilon|\log \varepsilon|$. The solution u_ε can be written as

$$u_\varepsilon(r, \theta, x_3) = \prod_{k=1}^{n} w\left(re^{i\theta} - d_\varepsilon e^{i\varepsilon x_3} e^{2ik\pi/n}\right) + \varphi_\varepsilon$$

with

$$\|\varphi_\varepsilon\|_{L^\infty} \leq \frac{M}{|\log \varepsilon|} \text{ for some constant } M > 0,$$

and $d_\varepsilon = \frac{\hat{d}_\varepsilon}{\varepsilon\sqrt{|\log \varepsilon|}}$ *with* $\hat{d}_\varepsilon = \sqrt{\frac{n-1}{1-c}} + o_\varepsilon(1)$.

Technique: Lyapunov-Schmidt reduction method.

Reduction to a 2D equation.

A function u is screw-symmetric if

$$u(r, \theta + h, x_3 + h) = u(r, \theta, x_3)$$

for any $h \in \mathbb{R}$. Equivalently $u(r, \theta, x_3) = u(r, \theta - x_3, 0) =: U(r, \theta - x_3)$.

Observation: $u_d(r, \theta, x_3) := \prod_{k=1}^{n} w\left(\frac{r}{\varepsilon} e^{i\theta} - d_\varepsilon e^{ix_3} e^{2ik\pi/n}\right)$ is not symmetric, since

$$u_d(r, \theta, x_3) = e^{inx_3} u_d(r, \theta - x_3, 0),$$

but so it is $v_d(r, \theta, x_3) = e^{-inx_3} u_d(r, \theta, x_3)$.

$\rightsquigarrow$ We look for solutions in the form

$$u(r, \theta, x_3) = e^{inx_3} U(r, \theta - x_3),$$

being $U : \mathbb{R}^+ \times \mathbb{R}$ a 2π-periodic function in the second variable.

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Reduction to a 2D equation.

Denoting $U = U(r,s)$ and $V(r,s) := U(\varepsilon r, s)$ the equation becomes

$$\Delta V + \varepsilon^2(\partial_{ss}^2 V - 2in\partial_s V - n^2 V) - ic|\log\varepsilon|\varepsilon^2(inV - \partial_s V) + (1 - |V|^2)V = 0$$

in $\mathbb{R}^2$.

Observation: It is a perturbation of the (GL) equation.

Approximation:

$$V_d(z) = \prod_{j=1}^{n} w(z - \xi_j), \quad \xi_j := d_\varepsilon e^{2i\pi(j-1)/n}, \quad d_\varepsilon := \frac{d}{\varepsilon\sqrt{|\log\varepsilon|}}.$$

Notice that

$$V_d(\overline{z}) = \overline{V_d}(z), \quad V_d(e^{2i\pi/n}z) = V_d(z).$$

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The Lyapunov-Schmidt reduction method.

Let S be a differential operator.

Given an approximation V_d satisfying $S(V_d) = o_\varepsilon(1)$ we want to find V such that

$$S(V) = 0.$$

If $V = V_d + \phi$, then

$$0 = S(V) = S(V_d) + L_d(\phi) + N(\phi).$$

$E := S(V_d)$: **error term**.

$L_d(\phi)$: **linearized operator** of S around V_d.

$N(\phi)$: **nonlinear term**.

The Lyapunov-Schmidt reduction method.

Step 1: Linear theory. Let $\ker\{L_d\} = \text{span}\{w_1, \dots, w_m\}$.
For any h we solve

$$\begin{cases} L_d(\phi) = h - \sum_{i=1}^m c_i(h) w_i, \\ \phi \perp \{w_1, \dots, w_m\}, \end{cases}$$

with

$$c_i(h) = \langle h, w_i \rangle \quad \text{and} \quad \|\phi\|_* \leq C\|h\|_{**}.$$

Step 2: Fixed point argument. We set $h = -E - N(\phi)$ and we obtain a solution of

$$S(V_d + \phi) = \sum_{i=1}^m c_i(d) w_i.$$

Step 3: Reduction procedure. We choose d so that

$$c_i(d) = 0 \qquad \forall\, i = 1, \dots, m.$$

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Key points:

- $\ker\{L_d\} \rightsquigarrow$ nondegeneracy.
- Linear theory $\rightsquigarrow$ a priori estimates, norms.
- Good approximation $\rightsquigarrow$ error size.
- Reduction $\rightsquigarrow$ size of the projections.

The linearized Ginzburg-Landau operator.

The linearized operator around the standard vortex w,

$$L^0(\phi) := \Delta\phi + (1 - |w|^2)\phi - 2\mathrm{Re}(\overline{w}\phi)w,$$

has a kernel:

$$\ker(L^0) = \mathrm{span}\{w_{x_1}, w_{x_2}, iw\}.$$

Observation: Thanks to the symmetries of the construction, we can consider only w_{x_1} in the projections.

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The form of the approximation.

Since the functions are complex-valued, and due to the form of the non-linear term, we cannot use a perturbation of the form

$$V = V + \phi.$$

We should use

$$V = \eta(V_d + \phi) + (1-\eta)V_d e^{i\psi}, \quad \phi = iV_d\psi.$$

- Additive form close to the vortices.
- Multiplicative form far form the vortices.

$\rightsquigarrow$ First for the GL equation in [del Pino-Kowalczyk-Musso, '06].

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Problem in the projections.

For this ansatz we have

$$\|S(V_d)\|_{**} \leq \frac{C}{|\log \varepsilon|} \quad \text{and consequently} \quad \|\psi\|_* \leq \frac{C}{|\log \varepsilon|}.$$

Thus,

$$\mathrm{Re}\int_{\{l_1<d\}} S(V_d)\bar{w}_{x_1} = \varepsilon\sqrt{|\log \varepsilon|}\left(\frac{a_1}{d} - a_2 d\right) + o_\varepsilon(\varepsilon\sqrt{|\log\varepsilon|}),$$

$$\mathrm{Re}\int_{\{l_1<d\}} N(\phi)\bar{w}_{x_1} = O\left(\frac{1}{|\log \varepsilon|^2}\right),$$

since

$$S(V_d) = \frac{d^2}{|\log \varepsilon|} w_{x_2 x_2} + \frac{d\varepsilon}{\sqrt{|\log \varepsilon|}} w_{x_1} + \varepsilon\sqrt{|\log \varepsilon|}E_0 + O(\varepsilon^2),$$

and

$$w_{x_2 x_2} \perp w_{x_1} !!$$

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Decomposition in Fourier modes.

Idea: Decompose the error in Fourier modes centered at the vortices, and separate the odd and the even parts.

Let us call $E = S(V_d)$. We write

$$E = \sum_{k=0}^{\infty} E^k = \sum_{k=0}^{\infty} E_1^k(r)\cos(k\theta) + iE_2^k(r)\sin(k\theta),$$

and

$$E^o := \sum_{k \text{ odd}} E^k, \qquad E^e := \sum_{k \text{ even}} E^k.$$

Then

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If we write $\psi = \psi^o + \psi^e$, refining the linear theory we may get

$$|\psi^o|_\sharp \leq C\varepsilon\sqrt{|\log\varepsilon|},$$

and

$$|N(\phi)^o| \leq C(\|\psi^e\|_*|\psi^o|_\sharp + |\psi^o|_\sharp^2) \leq \frac{C\varepsilon}{\sqrt{|\log\varepsilon|}}.$$

Therefore

$$\mathrm{Re}\int_{\{l_1<d\}} N(\phi)\bar{w}_{x_1} = \mathrm{Re}\int_{\{l_1<d\}} N(\phi)^o\,\bar{w}_{x_1} = o_\varepsilon(\varepsilon\sqrt{|\log\varepsilon|}),$$

and the adjustment is just

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$$\varepsilon\sqrt{|\log\varepsilon|}\left(\frac{a_1}{d} - a_2 d\right) + o_\varepsilon(\varepsilon\sqrt{|\log\varepsilon|}) = 0.$$

Open problems.

• Consider another set of solutions of the (KMD) system and understand if they are related to solutions of (GP) $\rightsquigarrow$ challenge with the collision solutions.

• Similar constructions for other equations: Euler and Schrödinger maps $\rightsquigarrow$ Single helix in [Dávila-del Pino-Musso-Wei, '20], [Lin-Wei, '03].

Muchas gracias!