Nonexistence of solutions to quasilinear parabolic equations with a potential in bounded domains

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Regularity for nonlinear diffusion equations. Green functions and functional inequalities

Madrid, June 13-17th 2022

Statement of the problem

We investigate *nonexistence of nonnegative global in time solutions* to a quasilinear parabolic problem

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \geq V\, u^q & \text{in } \Omega \times (0,\tau) \\ u \geq 0 & \text{on } \partial\Omega \times (0,\tau) \\ u \geq u_0 & \text{in } \Omega \times \{0\}\,; \end{cases} \tag{1}$$

where

- $p > 1,\ q > \max\{p-1, 1\},\ \tau > 0,$
- Ω is an open bounded connected subset of $\mathbb{R}^N$,
- $V \in L^1_{loc}(\Omega \times [0, +\infty)),\ V > 0$ a.e. in $\Omega \times (0, +\infty)$,
- $u_0 \in L^1_{loc}(\Omega),\ u_0 \geq 0$ a.e. in Ω.

D.D. Monticelli, G. M., F. Punzo, *Nonexistence of solutions to quasilinear parabolic equations with a potential in bounded domains*, Calc. Var. and PDEs (2022).

Introduction to the problem

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- suitable a priori estimates.

Nonexistence of global solutions for problems closed to problem (1) has been deeply studied both in the Euclidean setting and on Riemannian manifolds of infinite volume.

Similarly, let me mention that also the elliptic counterpart of equation in (1) has a long history. I would like to mention some of the work that are mostly connected to our.

Parabolic problems - Euclidean setting

A corner stone is the fundamental work of [Fujita, J. Fac. Sci. Univ. Tokyo 1966].

He considers equation

$$u_t = \Delta u + u^q \quad \text{in } \mathbb{R}^N \times (0, \tau)$$

which corresponds to problem (1) with $p = 2$, $V \equiv 1$, $\Omega = \mathbb{R}^N$. Moreover, $u_0 \in L^\infty(\mathbb{R}^N)$.

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which corresponds to problem (1) with $p = 2$, $V \equiv 1$, $\Omega = \mathbb{R}^N$. Moreover, $u_0 \in L^\infty(\mathbb{R}^N)$. He shows that

- blow-up of solutions in finite time prevails, for *all* nontrivial nonnegative initial data, for any

$$1 < q < 1 + \frac{2}{N};$$

- global in time solutions exist, for sufficiently small nonnegative initial data, for any

$$q > 1 + \frac{2}{N}.$$

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The exponent $1 + \frac{2}{N} =: q_c$ is called *Fujita exponent*. The critical value $q = q_c$ is addressed in [Hayakawa, Proc. Jap. Acc. 1973] where it is shown that it belongs to the blow-up case.

Parabolic problems - Euclidean setting

We now mention problem (1) with $\Omega = \mathbb{R}^N$ and $V \equiv 1$, i.e.

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \geq u^q \quad \text{in } \mathbb{R}^N \times (0, \tau).$$

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In [Mitidieri, Pohozaev, 2001, 2004], the authors show nonexistence of global weak solution when

$$p > \frac{2N}{N+1}; \quad \max\{1, p-1\} < q \leq p - 1 + \frac{p}{N}.$$

The powerful role of properly chosen test functions has been deeply explained by these authors.

Parabolic problems - Riemannian manifolds

Problem (1) with $p = 2$ has also been studied in [Bandle, Pozio, Tesei, JDE 2011] with M being the hyperbolic space $\mathbb{H}^N$ and u_0 bounded and nonnegative.

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The authors show that, if

$$V = V(t) = e^{\alpha t} \quad (\alpha > 0),$$

then blow-up can occur. More precisely, they prove that

- if $1 < q < q_c$, every nontrivial solution blows up in finite time;
- if $q > q_c$, the problem possesses global solutions for small initial data;

with

$$q_c := 1 + \frac{\alpha}{\Lambda},$$

where $\Lambda = \frac{(N-1)^2}{4}$ is the bottom of the L^2 spectrum of $-\Delta$ in $\mathbb{H}^N$.

Parabolic problems - Riemannian manifolds

In [Mastrolia, Monticelli, Punzo, Math. Ann. 2017] it is studied problem

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) \geq V(x,t)u^q & \text{in } M \times (0,\infty), \\ u = u_0 \geq 0 & \text{in } M \times \{0\}, \end{cases} \tag{2}$$

where M is N–dimensional, complete, noncompact Riemannian manifold. Here $p > 1$, $q > \max\{p-1, 1\}$,

$$V = V(x,t) > 0 \text{ a.e. in } M \times (0,\infty),\ V \in L^1_{\text{loc}}(M \times (0,\infty))$$

and the initial condition $u_0 \in L^1_{\text{loc}}$ is nonnegative.

Parabolic problems - Riemannian manifolds

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- In particular, there exists no *global* nonnegative weak solution for $p = 2$, i.e. for the Laplace operator, if $V \equiv 1$ and

$$\mathrm{Vol}(B_R) \leq CR^{\frac{2}{q-1}} (\log R)^{\frac{1}{q-1}} .$$

Let us mention some of the results in literature where the elliptic counterpart of equation in (1) has been considered.

Both in the Euclidean setting and on Riemannian manifolds, the parabolic case presents substantial differences with respect to the elliptic one. In fact, different test functions have to be used, as well as different a priori estimates.

Elliptic equations - Euclidean setting

In [Mitidieri, Pohozaev, Milan J. Math. 2004] the following class of inequalities is studied:

$$\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + V(x)u^q \leq 0, \quad \text{in } \mathbb{R}^N \tag{3}$$

where

$$V > 0 \text{ a.e. on } \mathbb{R}^N,\ V \in L^1_{loc}(\mathbb{R}^N),\ p > 1,\ q > \max\{1, p-1\}.$$

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where

$$V > 0 \text{ a.e. on } \mathbb{R}^N,\ V \in L^1_{loc}(\mathbb{R}^N),\ p > 1,\ q > \max\{1, p-1\}.$$

They show that it does not admit any global nontrivial nonnegative solution

- provided that

$$\liminf_{R\to+\infty} R^{-\frac{2q}{q-1}} \int_{B_{\sqrt{2}R}\setminus B_R} V^{-\frac{1}{q-1}}\,dx \ < \ \infty;$$

- or, if $V \equiv 1$, provided that

$$N > p \quad \text{and} \quad 0 < p-1 < q \leq \frac{N(p-1)}{N-p}.$$

Observe that this can be read as a condition relating the volume growth of Euclidean balls, which depends on N, and the exponent of the nonlinearity.

Elliptic equations - Riemannian manifolds

The same equation has been considered in the case of general complete, noncompact Riemannian manifolds M of infinite volume and dimension N.

In [Mastrolia, Monticelli, Punzo, Calc. Var. PDEs, 2015] it is investigated the influence of the geometry of the underlying manifold and of the potential V on the *existence of positive global solutions*.

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The authors show *nonexistence of global positive solutions* under suitable weighted volume growth conditions with weight V. E.g. nonexistence holds provided that there exist $C_0 > 0$, $k \in [0, \beta)$ such that, for every $R > 0$ sufficiently large and every $\varepsilon > 0$ sufficiently small,

$$\int_{B_R \setminus B_{R/2}} V^{-\beta+\varepsilon} \, d\mu \leq C \, R^{\alpha + C_0 \varepsilon} (\log R)^k;$$

where

$$\alpha = \frac{pq}{q-p+1}, \quad \beta = \frac{p-1}{q-p+1}.$$

Elliptic equations - Riemannian manifolds

Finally let me mention the work of [Monticelli, Punzo, Sciunzi, J. Geom. Anal. 2017]. The aim of this paper is to study *nonexistence of stable, possibly sign changing solutions* of the elliptic equation (3) on Riemannian manifolds.

Elliptic equations - Riemannian manifolds

Finally let me mention the work of [Monticelli, Punzo, Sciunzi, J. Geom. Anal. 2017]. The aim of this paper is to study *nonexistence of stable, possibly sign changing solutions* of the elliptic equation (3) on Riemannian manifolds.

The authors show that no nontrivial stable solution exists, provided that a condition on the growth of V is satisfied, i.e. there exist $C, C_0, R_0, \varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ and $R \in (R_0, \infty)$ one has

$$\int_{B_R \setminus B_{R/2}} V^{-\beta+\varepsilon} \, d\mu \leq \, C \, R^{\alpha + C_0 \varepsilon} (\log R)^{-1};$$

for some positive α and β depending on p, q.

The proof of such result rely on the construction of an appropriate family of test functions, depending on two parameters. (This choice is similar to our).

Our problem

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \geq V\, u^q & \text{in } \Omega \times (0,\tau) \\ u \geq 0 & \text{on } \partial\Omega \times (0,\tau) \\ u \geq u_0 & \text{in } \Omega \times \{0\}\,; \end{cases}$$

- $u_0 \in L^1_{loc}(\Omega)$, nonnegative;
- $V \in L^1_{loc}(\Omega \times [0,+\infty))$, $V > 0$ a.e. in $\Omega \times [0,+\infty)$;
- $p > 1$, $q > \max\{1, p-1\}$.

Goal: prove nonexistence of global solutions under suitable integral conditions involving V, p and q.

Integral conditions on V

Let $\theta_1 \geq 1$, $\theta_2 \geq 1$, for each $\delta > 0$ we define

$$E_\delta := \left\{ (x,t) \in \Omega \times [0,\infty) \,:\, d(x)^{-\theta_2} + t^{\theta_1} \leq \delta^{-\theta_2} \right\}.$$

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Moreover let

$$\begin{aligned} \bar{s_1} &:= \frac{q}{q-1}\theta_2\,, \quad \bar{s_2} := \frac{1}{q-1}\,, \\ \bar{s_3} &:= \frac{pq}{q-p+1}\theta_2\,, \quad \bar{s_4} := \frac{p-1}{q-p+1}\,. \end{aligned} \tag{4}$$

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Let $\theta_1 = \theta_2 = 1$ then, for any $\delta > 0$ sufficiently small

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So that we can write: $d(x) \geq \dfrac{\delta}{1 - \delta\, t}, \quad (0 \leq t < \frac{1}{\delta})$.

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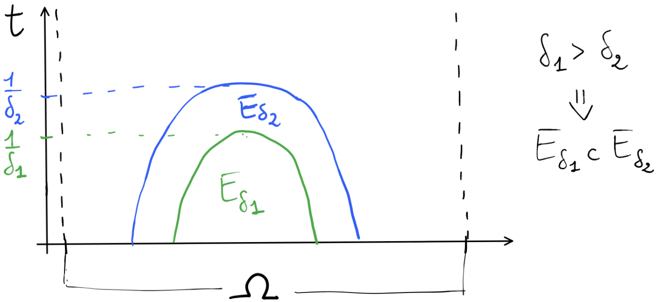

Integral conditions on V

We say that condition (HP1) holds if:

there exist $\theta_1 \geq 1$, $\theta_2 \geq 1$, $C_0 \geq 0$, $C > 0$, $\delta_0 \in (0,1)$ and $\varepsilon_0 > 0$:

- for some $0 < s_2 < \bar{s}_2$

$$\iint_{E_{\frac{\delta}{2}} \setminus E_\delta} t^{(\theta_1 - 1)\left(\frac{q}{q-1} - \varepsilon\right)} V^{-\frac{1}{q-1} + \varepsilon} \, dxdt \; \leq \; C\delta^{-\bar{s}_1 - C_0\varepsilon} \left|\log(\delta)\right|^{s_2}$$

  for any $\delta \in (0, \delta_0)$ and for any $\varepsilon \in (0, \varepsilon_0)$;

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  for any $\delta \in (0, \delta_0)$ and for any $\varepsilon \in (0, \varepsilon_0)$;

- for some $0 < s_4 < \bar{s}_4$

$$\iint_{E_{\frac{\delta}{2}} \setminus E_\delta} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1} - \varepsilon\right)} V^{-\frac{p-1}{q-p+1} + \varepsilon} \, dxdt \leq C\delta^{-\bar{s}_3 - C_0\varepsilon} \left|\log(\delta)\right|^{s_4}$$

  for any $\delta \in (0, \delta_0)$ and for any $\varepsilon \in (0, \varepsilon_0)$.

Integral conditions on V

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We say that condition (HP2) holds if: there exist $\theta_1 \geq 1$, $\theta_2 \geq 1$, $C_0 \geq 0$, $C > 0$, $\delta_0 \in (0,1)$ and $\varepsilon_0 > 0$:

- for any $\delta \in (0, \delta_0)$ and for any $\varepsilon \in (0, \varepsilon_0)$

$$\iint_{E_{\frac{\delta}{2}} \setminus E_\delta} t^{(\theta_1 - 1)\left(\frac{q}{q-1} \mp \varepsilon\right)} V^{-\frac{1}{q-1} \pm \varepsilon} \, dxdt \; \leq \; C\delta^{-\bar{s}_1 - C_0\varepsilon} \left|\log(\delta)\right|^{\bar{s}_2} \; ;$$

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$$\iint_{E_{\frac{\delta}{2}} \setminus E_\delta} t^{(\theta_1 - 1)\left(\frac{q}{q-1} \mp \varepsilon\right)} V^{-\frac{1}{q-1} \pm \varepsilon} \, dxdt \; \leq \; C \delta^{-\bar{s}_1 - C_0 \varepsilon} \left|\log(\delta)\right|^{\bar{s}_2} ;$$

- for any $\delta \in (0, \delta_0)$ and for any $\varepsilon \in (0, \varepsilon_0)$

$$\iint_{E_{\frac{\delta}{2}} \setminus E_\delta} d(x)^{-(\theta_2 + 1)p\left(\frac{q}{q-p+1} \mp \varepsilon\right)} V^{-\frac{p-1}{q-p+1} \pm \varepsilon} \, dxdt \; \leq \; C \delta^{-\bar{s}_3 - C_0 \varepsilon} \left|\log(\delta)\right|^{\bar{s}_4} .$$

Definition 1

Let $p > 1$, $q > \max\{p-1, 1\}$, $V \in L^1_{loc}(\Omega \times [0, +\infty))$, $V > 0$ a.e. in $\Omega \times (0, +\infty)$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω. We say that $u \in W^{1,p}_{loc}(\Omega \times [0, +\infty)) \cap L^q_{loc}(\Omega \times [0, +\infty), Vdxdt)$ is a weak solution of problem (1) if $u \geq 0$ a.e. in $\Omega \times (0, +\infty)$ and for every $\varphi \in \mathrm{Lip}(\Omega \times [0, \infty))$, $\varphi \geq 0$ and with compact support in $\Omega \times [0, \infty)$, one has

$$\begin{aligned}\int_0^\infty \int_\Omega V u^q \varphi \, dxdt \leq & \int_0^\infty \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \, dxdt \\ & - \int_0^\infty \int_\Omega u \, \varphi_t \, dxdt - \int_\Omega u_0 \, \varphi(x, 0) \, dx.\end{aligned}$$

Main result

We can now state our main results.

Theorem 1 (M., Monticelli, Punzo, Calc. Var. and PDEs (2022))

Let $p > 1$, $q > \max\{p-1, 1\}$, $V \in L^1_{loc}(\Omega \times [0, +\infty))$, $V > 0$ a.e. in $\Omega \times (0, +\infty)$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω. Assume that either condition (HP1) or condition (HP2) hold.
If u is a nonnegative weak solution of problem (1), then $u = 0$ a.e. in $\Omega \times (0, +\infty)$.

As a consequence of Theorem 1 we can state the following Corollary.

Further results

We consider those potentials V such that $V \in L^1_{loc}(\Omega \times [0, +\infty))$ and satisfy

$$V(x,t) \geq h(x)g(t) \quad \text{for a.e. } (x,t) \in \Omega \times (0, +\infty), \tag{5}$$

where $h : \Omega \to \mathbb{R}$ and $g : (0, +\infty) \to \mathbb{R}$ are such that

$$h(x) \geq C\, d(x)^{-\sigma_1} \left(\log\left(1 + d(x)^{-1}\right)\right)^{-\delta_1} \qquad \text{for a.e. } x \in \Omega, \tag{6}$$

$$0 < g(t) \leq C\,(1+t)^{\alpha} \qquad \text{for a.e. } t \in (0, +\infty), \tag{7}$$

with $\sigma_1, \delta_1, \alpha \geq 0,\ C > 0$.

Further results

Corollary 1 (M., Monticelli, Punzo, Calc. Var. and PDEs (2022))

Let $p > 1$, $q > \max\{p-1, 1\}$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω. where h and g satisfy (6) and (7) respectively. Let V be as in (5) and suppose that

$$\int_0^T g(t)^{-\frac{1}{q-1}}\, dt \leq C T^{\sigma_2} \left(\log T\right)^{\delta_2},$$

$$\int_0^T g(t)^{-\frac{p-1}{q-p+1}}\, dt \leq C T^{\sigma_4},$$

for $T > 1$, σ_2, σ_4, $\delta_2 \geq 0$ and $C > 0$.

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for $T > 1$, σ_2, σ_4, $\delta_2 \geq 0$ and $C > 0$. Finally assume that

- $\sigma_1 > q + 1$;

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for $T > 1$, σ_2, σ_4, $\delta_2 \geq 0$ and $C > 0$. Finally assume that

- $\sigma_1 > q + 1$;
- $0 \leq \sigma_2 \leq \frac{q}{q-1}$;
- $\delta_1 < 1$ and $\delta_2 < \frac{1-\delta_1}{q-1}$.

If u is a nonnegative weak solution of problem (1), then $u = 0$ a.e. in $\Omega \times [0, \infty)$.

Sketch of the proof

We have said that the proof relies on the method of properly constructed test functions. In particular, these test functions depend on two parameters. We consider the case of assumption (HP1).

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Step 1)

- For any $\delta > 0$ sufficiently small, let $\alpha := \frac{1}{\log \delta} < 0$. We define for any $(x, t) \in \Omega \times [0, +\infty)$

$$\varphi_\delta(x,t) := \begin{cases} 1 & \text{in } E_\delta \\ \left[\dfrac{d(x)^{-\theta_2} + t^{\theta_1}}{\delta^{-\theta_2}}\right]^{C_1\alpha} & \text{in } (E_\delta)^C \end{cases} .$$

where

$$C_1 > \frac{2(C_0 + \theta_2 + 1)}{\theta_2 q}$$

with $C_0 \geq 0$, $\theta_1, \theta_2 \geq 1$ as in the growth conditions on V and

$$E_\delta := \left\{(x,t) \in \Omega \times [0, +\infty) \ : \ d(x)^{-\theta_2} + t^{\theta_1} \leq \delta^{-\theta_2}\right\} .$$

Sketch of the proof

- Moreover, for any $n \in \mathbb{N}$ we define

$$\eta_n(x,t) := \begin{cases} 1 & \text{in } E_{\frac{\delta}{n}} \\ \frac{2^{\theta_2}}{2^{\theta_2}-1} - \frac{1}{2^{\theta_2}-1}\left(\frac{\delta}{n}\right)^{\theta_2}\left[d(x)^{-\theta_2} + t^{\theta_1}\right] & \text{in } E_{\frac{\delta}{2n}} \setminus E_{\frac{\delta}{n}} \\ 0 & \text{in } E^C_{\frac{\delta}{2n}} \end{cases}.$$

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- Then finally we take

$$\psi_{\delta,n}(x,t) := \eta_n(x,t)\,\varphi_\delta(x,t).$$

Sketch of the proof

Step 2)

- Let $s \geq \max\left\{1, \frac{q}{q-1}, \frac{pq}{q-p+1}\right\}$. We consider the quantity

$$\int_0^\infty \int_\Omega V\, u^{q+\alpha}\, \psi_{\delta,n}^s\, dx\, dt.$$

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$$\int_0^{\infty} \int_{\Omega} V\, u^{q+\alpha}\, \psi_{\delta,n}^{s}\, dx\, dt.$$

- By means of suitable a priori estimates we show that

$$\int_0^{\infty} \int_{\Omega} V\, u^{q+\alpha}\, \psi_{\delta,n}^{s}\, dxdt \leq C|\alpha|^{-\frac{(p-1)q}{q-p+1}} \left[|\alpha|^{\frac{pq}{q-p+1}-s_4-1} + n^{-\frac{|\alpha|}{q-p+1}} \left|\log\left(\frac{\delta}{n}\right)\right|^{s_4}\right] + C\left[|\alpha|^{\frac{1}{q-1}-s_2} + n^{-\frac{|\alpha|}{q-1}} \left|\log\left(\frac{\delta}{n}\right)\right|^{s_2}\right],$$

Sketch of the proof

Step 3)

- By taking the limit as $n \to \infty$ for fixed small enough $\delta > 0$, we get

$$0 \leq \iint_{E_\delta} V\, u^{q+\alpha}\, dxdt \leq \int_0^\infty \int_\Omega V\, u^{q+\alpha}\, \varphi_\delta^s\, dxdt$$
$$\leq C\left[|\alpha|^{\frac{p-1}{q-p+1}-s_4} + |\alpha|^{\frac{1}{q-1}-s_2}\right].$$

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Step 4)

- Due to the definitions of s_2 and s_4 in the growth conditions on V, we observe that

$$\frac{1}{q-1} - s_2 > 0\,, \quad \frac{p-1}{q-p+1} - s_4 > 0\,.$$

Hence we can take the limit as $\delta \to 0$, and thus $\alpha \to 0^-$, obtaining by Fatou's Lemma

$$\int_0^\infty \int_\Omega V\, u^q\, dxdt = 0,$$

which concludes the proof.

Further results

As a special case, we have also considered the semilinear parabolic problem obtained from problem (1) for $p = 2$

$$\begin{cases} u_t - \Delta u = V u^q & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u = u_0 & \text{in } \Omega \times \{0\}. \end{cases} \tag{8}$$

Here $\Omega \subset \mathbb{R}^N$, $N \geq 3$, $q > 1$. The previous assumptions on u_0 and V are made.

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Here $\Omega \subset \mathbb{R}^N$, $N \geq 3$, $q > 1$. The previous assumptions on u_0 and V are made.

Depending on the rate of blowing up of V at the boundary $\partial\Omega$, we show nonexistence of nonnegative global solutions to (8) versus existence of a global solution.

Further results

Results are new also for $p = 2$, indeed from our general result we get the following

Corollary 2 (M., Monticelli, Punzo, Calc. Var. and PDEs (2022))

Let $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω. Suppose that $V \in L^1_{loc}(\Omega)$ satisfies, for some $C > 0$ and for a.e. $x \in \Omega$

$$V(x) \geq C d(x)^{-\sigma} \quad \text{with } \sigma > q + 1. \tag{9}$$

If u is a nonnegative weak solution of problem (8), then $u = 0$ a.e. in S.

We aim at investigating optimality of condition (9).

Our results - Problem III

Theorem 2 (M., Monticelli, Punzo, Calc. Var. and PDEs (2022))

Suppose that $\partial\Omega$ is smooth and let $u_0 \in C(\Omega)$, $u_0 \geq 0$ in Ω, be sufficiently small. Moreover let $V = V(x) \in C(\Omega)$, $V > 0$ in Ω and assume that for some $C > 0$, for any $x \in \overline{\Omega}$

$$V(x) \leq C\, d(x)^{-\sigma} \quad \text{with } 0 < \sigma < q + 1.$$

Then problem (8) admits a classical solution u in $\Omega \times (0, +\infty)$.

Sketch of the proof

The proof relies on the construction of sub- and supersolution.

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Step 1)

- We construct a supersolution $\overline{u}$ to problem (8).

  Let us define, for any $\lambda > 0$

  $$S_\lambda = \{v \in C(\bar{\Omega}) \,:\, 0 \le v(x) \le \lambda\, d(x),\ \forall x \in \Omega\};$$

  and $T : S_\lambda \to S_\lambda$ such that

  $$Tv(x) = \lambda^q \int_\Omega G(x,y)\, dy + \int_\Omega G(x,y) V(y) v(y)^q\, dy.$$

  where $G(x,y)$ is the Green function associated to the Laplacian operator $-\Delta$.

Sketch of the proof

Step 2)

- Let $\psi(x) := \int_\Omega G(x,y)\, d(y)^\beta\, dy$, then, by the regularity of $\partial\Omega$, we prove

Lemma 1

Suppose that

$$\beta > -1.$$

Then, there exists $c > 0$ such that

$$0 \le \psi(x) \le c\, d(x) \quad \text{for any } \; x \in \Omega.$$

By means of Lemma 1, we prove that

- $Tv : \bar{\Omega} \to \mathbb{R}$ is continuous, thus $T : S_\lambda \to S_\lambda$ is well defined;

Sketch of the proof

Step 3)

Lemma 2

Suppose that

$$\beta > -2.$$

Then, there exist $M > 0$ such that

$$0 \leq \psi(x) \leq M \quad \text{for any } \; x \in \Omega.$$

By means of Lemma 2, we prove that

- T is a contraction map for $\lambda > 0$ small enough.

Sketch of the proof

Therefore, there exists $\varphi \in S_\lambda$ such that $\varphi = T\varphi$, i.e.

- $0 \leq \varphi(x) \leq \lambda\, d(x) \quad$ for any $x \in \bar{\Omega}$;
- φ is a solution of

$$\begin{cases} -\Delta\varphi = \lambda^q + V\varphi^q & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

- $\varphi > 0$ in Ω due to $\lambda > 0$.

Sketch of the proof

Step 4)

- We show that $\overline{u} = \varphi$ is a supersolution to problem (8) provided that u_0 is small enough.
- We set the subsolution $\underline{u} \equiv 0$.
- We conclude that there exists a solution $u : \Omega \times [0, +\infty) \to \mathbb{R}$ of problem (8) such that
$$0 \leq u(x) \leq \overline{u}(x) \quad \text{for any } x \in \bar{\Omega}.$$

This concludes the proof.

Further results

Our results do not cover the case of *critical rate* of growth, i.e.

$$V(x) = d(x)^{-q-1} \quad \text{for all} \quad x \in \Omega,$$

but we conjecture that also in this case no nonnegative nontrivial solution of problem (8) exists.

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However, we study the *slightly supercritical* case

$$V(x) = d(x)^{-q-1} f(d(x)) \quad \text{for all } \; x \in \Omega,$$

where f is such that $\lim_{\varepsilon \to 0^+} f(\varepsilon) = +\infty$, for which we prove nonexistence of solutions.

Further results

Theorem 3 (M., Monticelli, Punzo, Calc. Var. and PDEs (2022))

Suppose that $u_0 \in L^1_{loc}(\Omega)$ with $u_0 \geq 0$ a.e. in Ω. Assume that V satisfies for some $C > 0$

$$V(x) \geq C d(x)^{-q-1} f(d(x)) \quad \text{for a.e. } x \in \Omega,$$

where $f : (0, +\infty) \to [1, +\infty)$ is such that $\lim_{\varepsilon \to 0^+} f(\varepsilon) = +\infty$. If u is a nonnegative weak solution of problem (8), then $u = 0$ a.e. in $\Omega \times (0, +\infty)$.

The proof of Theorem 3 is different to the previous nonexistence result.

Further results: sketch of the proof

- We introduce the *Whitney distance* $\delta : \Omega \to \mathbb{R}^+$, it is a $C^\infty(\Omega)$ function regardless of the regularity of $\partial\Omega$ such that

$$\begin{aligned} &c_0^{-1}\, d(x) \le \delta(x) \le c_0\; d(x)\,, \\ &|\nabla\delta(x)| \le c_0\,, \\ &|\Delta\delta(x)| \le c_0\, \delta^{-1}(x)\,, \quad \text{for all } x \in \Omega. \end{aligned} \tag{10}$$

- We use the test functions

$$\phi_\varepsilon(x,t) := g_\varepsilon(\delta(x))\, \eta(t),$$

where $0 \le \eta \le 1$, $\eta \equiv 1$ in $[0, \frac{1}{2} f(\varepsilon)]$, $\operatorname{supp}\eta \subset [0, f(\varepsilon))$ and $-\frac{C}{f(\varepsilon)} \le \eta' \le 0$ and $g_\varepsilon : [0,\infty) \to \mathbb{R}$ such that $0 \le g_\varepsilon \le 1$, $g_\varepsilon \equiv 1$ in $[\varepsilon, +\infty)$, $\operatorname{supp} g_\varepsilon \subset [\frac{\varepsilon}{2}, +\infty)$, $0 \le g_\varepsilon' \le \frac{C}{\varepsilon}$ and $|g_\varepsilon''| \le \frac{C}{\varepsilon^2}$ for some constant $C > 0$.

Further results: sketch of the proof

- By using ϕ_ε as test function, we show that

$$\int_0^{+\infty} \int_\Omega u^q V \, dxdt < +\infty.$$

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- Define

$$\tilde{\Omega}_\varepsilon = \{x \in \Omega | \delta(x) \geq \varepsilon\}; \;\; K_\varepsilon = \tilde{\Omega}_\varepsilon \times \left[0, \frac{f(\varepsilon)}{2}\right]; \; S_{K_\varepsilon} := \big(\Omega \times \big[0, +\infty)\big) \setminus K_\varepsilon \, .$$

Observe that $\phi_\varepsilon \equiv 1$ on K_ε.

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- By means of suitable a priori estimates we show that

$$\iint_{K_\varepsilon} u^q\,V\,dxdt \;\leq\; C\left(\iint_{S_{K_\varepsilon}} u^q V\,dxdt\right)^{\frac{1}{q}}\;.$$

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$$\iint_{K_\varepsilon} u^q\,V\,dxdt\;\leq\;C\left(\iint_{S_{K_\varepsilon}} u^q V\,dxdt\right)^{\frac{1}{q}}\;.$$

- Finally, letting $\varepsilon\to 0$ we obtain

$$\int_0^{+\infty}\int_\Omega u^q\,V\,dxdt\;=0\,.$$

Thank you!