The fast diffusion equation on Riemannian manifolds with nonpositive curvature and related elliptic problems

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"REGULARITY FOR NONLINEAR DIFFUSION EQUATIONS. GREEN FUNCTIONS AND FUNCTIONAL INEQUALITIES"

Outline of the problem

We investigate the following Cauchy problem for the fast diffusion equation:

$$\begin{cases} u_t = \Delta(u^m) & \text{in } M \times (0, +\infty)\,, \\ u = u_0 & \text{on } M \times \{0\}\,, \end{cases} \tag{FDE}$$

where $m \in (0, 1)$, M is a complete, connected, noncompact n-dimensional Riemannian manifold and $u_0 \in L^1_{\mathrm{loc}}(M)$. When dealing with sign-changing solutions, we mean $u^m := \mathrm{sign}(u)|u|^m$.

We are mainly interested in setting up a well-posedness theory having in mind the case $M \equiv \mathbb{R}^n$, both for sign-changing and nonnegative solutions, under "mild" curvature assumptions on the manifold M.

As we will see, this is strictly related to nonexistence results for the semilinear elliptic equation

$$\Delta W = \alpha\, W\, |W|^{p-1} \qquad \text{in } M\,, \tag{ELL}$$

where $p > 1$ and $\alpha > 0$.

Previous results by Herrero and Pierre

In a breakthrough paper, M.A. Herrero and M. Pierre [TAMS, 1985] studied the problem in the Euclidean space $\mathbb{R}^n$. They proved that solutions exist without further assumptions on u_0, i.e. for merely locally integrable data. Their argument relies on the following key estimate:

$$\begin{aligned}&\left[\int_{B_R} |v(x,t)-u(x,t)|\,dx\right]^{1-m}\\ \leq &\left[\int_{B_{2R}} |v(x,s)-u(x,s)|\,dx\right]^{1-m} + \mathcal{C}\,R^{n(1-m)-2}\,|t-s|\,,\end{aligned} \tag{HP}$$

which can be shown by smart cut-off arguments. Here $R>0$, $t,s\geq 0$ and u,v are any two (regular enough) solutions of the FDE. The constant $\mathcal{C}>0$ only depends on n,m.

As concerns uniqueness, they establish it in the class of locally strong solutions, i.e. those satisfying

$$u \in C\left([0,+\infty); L^1_{\mathrm{loc}}(\mathbb{R}^n)\right) \quad \text{and} \quad u_t \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times (0,+\infty))\,.$$

Indeed, due to Kato's inequality, one sees that the integral function

$$w(x) := \int_0^t |v^m(x,s) - u^m(x,s)|\, ds$$

is subharmonic for all $t > 0$ if $u_0 = v_0$. By using the mean-value inequality, Hölder's inequality and (HP) letting $R \to +\infty$, it turns out that $w \equiv 0$.

We stress that the assumption $m \in (0,1)$ is crucial. In fact, if $m \geq 1$ it is by now well known (see Tychonoff, Widder, Bénilan-Crandall-Pierre) that the maximum allowed growth is

$$|u_0(x)| \sim e^{c|x|^2} \quad \text{if } m = 1 \text{ (for any } c > 0)\,, \qquad |u_0(x)| \sim |x|^{\frac{2}{m-1}} \quad \text{if } m > 1\,.$$

Our Riemannian setting: the notions of solution

We will mostly work with the so-called very weak solutions.

Definition

Let $m \in (0,1)$ and $u_0 \in L^1_{\mathrm{loc}}(M)$. We say that a function $u \in L^1_{\mathrm{loc}}(M \times [0,+\infty))$ is a very weak solution of the Cauchy problem (FDE) *if it satisfies*

$$u_t = \Delta(u^m) \qquad \text{in } \mathcal{D}'(M \times (0,+\infty))$$

and (in the sense of essential limits)

$$\lim_{t \to 0^+} \int_M u(x,t)\,\phi(x)\,d\mu(x) = \int_M u_0\,\phi\,d\mu \qquad \forall \phi \in C^\infty_c(M)\,,$$

where μ stands for the volume measure on M.

Note that such a definition does make sense since $u \in L^1_{\mathrm{loc}}(M \times [0,+\infty))$ implies $u^m \in L^1_{\mathrm{loc}}(M \times [0,+\infty))$.

Moreover, we will say that u is a strong solution if in addition

$$u_t \in L^1_{\mathrm{loc}}(M \times (0, +\infty)) \, .$$

As concerns the semilinear elliptic equation (ELL), solutions are still meant in the sense of distributions, so we will ask that W belongs to $L^p_{\mathrm{loc}}(M)$ and satisfies

$$\int_M W \, \Delta\phi \, d\mu = \alpha \int_M W^p \, \phi \, d\mu \qquad \forall \phi \in C^\infty_c(M) \, .$$

Note, however, that by elliptic regularity and bootstrap a solution in the above sense always turns out to be $W^{2,q}_{\mathrm{loc}}(M)$ for all $q \in [1, \infty)$.

This is in general not the case for subsolutions, namely functions $\underline{W} \in L^p_{\mathrm{loc}}(M)$ satisfying

$$\int_M \underline{W} \, \Delta\phi \, d\mu \geq \alpha \int_M \underline{W^p} \, \phi \, d\mu \qquad \forall \phi \in C^\infty_c(M) : \ \phi \geq 0 \, .$$

Main results: existence

All the results I will present are based on a joint work with G. Grillo and F. Punzo [TAMS, 2021].

Theorem ES (Existence of sign-changing solutions)

Let $m \in (0,1)$ and $u_0 \in L^1_{\mathrm{loc}}(M)$. Then there exists a very weak solution u of problem (FDE)*. In addition $u \in C\big([0,+\infty); L^1_{\mathrm{loc}}(M)\big)$.*

In general, we are not able to guarantee that the constructed solution is strong, but we will come back to this point later.

Theorem EM (Existence of the minimal solution)

Let $m \in (0,1)$ and $u_0 \in L^2_{\mathrm{loc}}(M)$, with $u_0 \geq 0$. Then there exists a nonnegative very weak solution $u \in L^2_{\mathrm{loc}}(M \times [0,+\infty))$ of problem (FDE)*, which is the minimal one. That is, if $v \in L^2_{\mathrm{loc}}(M \times [0,+\infty))$ is another nonnegative very weak solution of the same problem, then $u \leq v$.*

The L^2_{loc} assumption is purely technical and will be discussed below.

Main results: uniqueness and nonexistence

Note that above existence results hold regardless of curvature assumptions. However, for uniqueness the following additional hypothesis on M will be crucial: there exists $o \in M$ such that

$$\mathrm{Ric}_o(x) \geq -(n-1)\,\frac{\psi''(r(x))}{\psi(r(x))} \qquad \forall x \in M \setminus (\{o\} \cup \mathrm{cut}(o)), \tag{C1}$$

where $\mathrm{cut}(o)$ is the cut locus of o and $r(x) := \mathrm{dist}(x, o)$, for some function ψ satisfying

$$\psi \in C^\infty((0,+\infty)) \cap C^1([0,+\infty)), \quad \psi' \geq 0, \; \psi(0) = 0, \; \psi'(0) = 1 \tag{C2}$$

and

$$\int_0^{+\infty} \frac{\int_0^r \psi^{n-1}(\rho)\, d\rho}{\psi^{n-1}(r)}\, dr = +\infty. \tag{C3}$$

We stress that (C1) is in fact an identity on model manifolds, namely spherically symmetric Riemannian manifolds whose metric reads

$$g \equiv dr^2 + \psi^2(r)\, g_{\mathbb{S}^{n-1}}.$$

Assumption (C3) is related to stochastic completeness.

Theorem US (Uniqueness of strong solutions)

Let $m \in (0,1)$, $u_0 \in L^1_{\mathrm{loc}}(M)$ and (C1)–(C3) hold. Let u and v be any two strong solutions of problem (FDE) such that $|u(\cdot,t) - v(\cdot,t)| \to 0$ in $L^1_{\mathrm{loc}}(M)$ as $t \to 0^+$. Then $u = v$.

Theorem UN (Uniqueness of nonnegative very weak solutions)

Let $m \in (0,1)$, $u_0 \in L^2_{\mathrm{loc}}(M)$ with $u_0 \geq 0$ and (C1)–(C3) hold. Let $u \in L^2_{\mathrm{loc}}(M \times [0,+\infty))$ be a nonnegative very weak solution of problem (FDE). Then u coincides with minimal solution.

Theorem NE (Nonexistence for the semilinear elliptic equation)

Let $p > 1$, $\alpha > 0$ and (C1)–(C3) hold. Then:
(i) There exists no nonnegative nontrivial subsolution of (ELL);
(ii) There exists no nontrivial solution of (ELL).

Note that well-posedness results on manifolds in $L^1(M)$ and $H^{-1}(M)$ were previously studied by Bonforte-Grillo-Vázquez [JEE, 2008].

A (formal) strategy of proof

In the sequel, for simplicity we will assume that $B_R(o)$ is a smooth domain for all $R > 0$, which need not be true in general: geodesic balls should be replaced by sublevels of a regular exhaustion function on M.

• Theorem ES

The proof is very similar to that of Herrero-Pierre: one exploits the well-posedness of the $L^1(M)$ theory and picks the truncations

$$u_{0,k,h} := u_0^+ \, \chi_{B_k(o)} - u_0^- \, \chi_{B_h(o)} \in L^1(M) \, , \qquad \forall k, h \in \mathbb{N} \, ,$$

solving (FDE) with such data replacing u_0. Thanks to the monotonicity w.r.t. k and h, along with the modified Herrero-Pierre estimates

$$\begin{aligned} & \left[\int_{B_R(o)} \left| u_{k,h}(x,t) - u_{k',h'}(x,t) \right| d\mu(x) \right]^{1-m} \\ \leq & \left[\int_{B_{2R}(o)} \left| u_{k,h}(x,s) - u_{k',h'}(x,s) \right| d\mu(x) \right]^{1-m} + \mathcal{C}_R \left| t-s \right| , \end{aligned}$$

one can pass to the limit as $k, h \to \infty$ and obtain a solution of (FDE).

• Theorem EM

When $u_0 \geq 0$, in order to construct the candidate minimal solution first of all we solve the "localized" Dirichlet problems

$$\begin{cases} \partial_t u_k = \Delta(u_k^m) & \text{in } B_{R_k}(o) \times (0,+\infty)\,, \\ u_k = 0 & \text{on } \partial B_{R_k}(o) \times (0,+\infty)\,, \\ u_k = u_0 & \text{on } B_{R_k}(o) \times \{0\}\,. \end{cases} \tag{DP}$$

By standard comparison the sequence $\{u_k\}$ is monotone increasing, with $u_k \geq 0$. Moreover, still the Herrero-Pierre estimates ensure uniform L^1 local boundedness, so the pointwise limit

$$u := \lim_{k\to\infty} u_k$$

exists finite and solves (FDE). If $v \geq 0$ is another solution of (FDE), then it satisfies the localized problem with boundary datum

$$v|_{\partial B_k(o)\times(0,+\infty)} \geq 0\,,$$

thus it is a supersolution of (DP) and hence

$$u_k \leq v \quad \Rightarrow \quad u \leq v \quad \Rightarrow \quad u \text{ is minimal.}$$

However, the above inequalities are a consequence of a nonstandard comparison principle, that we will discuss later.

• Theorem US

For an arbitrary (but fixed) $t > 0$, we consider the function

$$\underline{W}(x) := \int_0^t |v^m(x,s) - u^m(x,s)|\, e^{-s}\, ds\,,$$

where u and v are any two strong solutions of (FDE) taking the same initial datum u_0. Kato's inequality entails

$$\frac{\partial}{\partial t}|v-u| \le \Delta\, |v^m - u^m| \qquad \text{in } \mathcal{D}'(M \times (0,+\infty))\,. \tag{K}$$

Since we assume that $|v(\cdot,t) - u(\cdot,t)| \to 0$ in $L^1_{\mathrm{loc}}(M)$ as $t \to 0^+$, upon multiplying (K) by e^{-t} and integrating in time we readily obtain

$$\int_0^t |v(x,s) - u(x,s)|\, e^{-s}\, ds \le \Delta \underline{W}(x) \qquad \text{in } \mathcal{D}'(M)\,.$$

Taking advantage of Hölder's inequality and the numerical inequality $2^{m-1}|v^m - u^m| \leq |v-u|^m$, we deduce that

$$\underbrace{\left(2-2e^{-t}\right)^{-\frac{1-m}{m}}}_{=:\alpha} \left(\int_0^t |v^m(x,s) - u^m(x,s)|\, e^{-s}\, ds \right)^{\frac{1}{m}} \leq \int_0^t |v(x,s) - u(x,s)|\, e^{-s}\, ds\,,$$

hence

$$\alpha\, \underline{W}^{\frac{1}{m}} \leq \Delta \underline{W} \qquad \text{in } \mathcal{D}'(M)\,.$$

In view of Theorem NE with $p = 1/m$, it follows that $\underline{W} \equiv 0$, that is $u = v$ due to the arbitrariness of t.

• Theorem UN

In the case of nonnegative solutions, we can repeat exactly the same computations without moduli, picking u as the minimal solution and v as any other nonnegative solution: this is the reason why we can drop the strong-solution requirement.

• Theorem NE

We exploit a strategy similar to the one originally developed by R. Osserman [Pacific J. Math., 1957] and J.B. Keller [CPAM, 1957].

Lemma

Let $p > 1$ and $\alpha > 0$. Let (C1)–(C3) hold, and let

$$H(r) := \int_0^r \frac{\int_0^\rho \psi(\zeta)^{n-1}\, d\zeta}{\psi(\rho)^{n-1}}\, d\rho \qquad \forall r \geq 0\,.$$

Given $R > 0$, there exists a constant $\mathrm{C} > 0$, depending only on p and α, such that the function

$$\overline{W}_R(x) := \mathrm{C}\, \frac{[H(R)]^{\frac{1}{p-1}}}{[H(R) - H(r(x))]^{\frac{2}{p-1}}} \qquad \forall x \in B_R(o)$$

fulfills

$$\Delta \overline{W}_R \leq \alpha\, \overline{W}_R^{\,p} \qquad \text{in } \mathcal{D}'(B_R(o))\,,$$

A crucial point in the proof of the above lemma is the following Laplacian-comparison inequality, consequence of the curvature bound:

$$\Delta r(x) \le (n-1)\,\frac{\psi'(r(x))}{\psi(r(x))} \qquad \text{in } \mathcal{D}'(M)\,.$$

If $\underline{W}$ is a nonnegative and nontrivial subsolution of (ELL), then by local comparison it turns out that

$$\underline{W} \le \overline{W}_R \qquad \text{in } B_R(o)\,,$$

because by construction $\lim_{r(x)\to R^-} \overline{W}_R(x) = +\infty$ and $\underline{W}$ is locally bounded (to be proved . . .). As a result, we end up with

$$\underline{W} \le \lim_{R\to+\infty} \overline{W}_R = 0 \qquad \text{in } M\,,$$

since $\{\overline{W}_R\}$ vanishes locally uniformly as $R \to +\infty$.

The nonexistence result for sign-changing solutions follows again by Kato's inequality: if W is a solution then $|W|$ is a subsolution.

About the local comparison principle

In order to rigorously justify the fact that $u_k \leq v$ in $B_k(o) \times (0, +\infty)$ in the proof of Theorem EM, we take advantage of a well-established duality method. The basic (formal) idea is to test the very weak formulation solved by $(v - u_k)$ with the solution of

$$\begin{cases} \partial_t \xi_h + a_h \, \Delta \xi_h = 0 & \text{in } B_k(o) \times (0, T)\,, \\ \xi_h = 0 & \text{on } \partial B_k(o) \times (0, T)\,, \\ \xi_h = \omega & \text{on } B_k(o) \times \{T\}\,, \end{cases}$$

for any $T > 0$ and any function $\omega \in C_c^\infty(B_k(o))$ with $\omega \geq 0$. Here $\{a_h\}$ is a smooth approximation of the ratio

$$a(x,t) := \begin{cases} \frac{v^m(x,t) - u_k^m(x,t)}{v(x,t) - u_k(x,t)} & \text{if } v(x,t) \neq u_k(x,t)\,, \\ 0 & \text{if } v(x,t) = u_k(x,t)\,. \end{cases}$$

However, there are some technical issues: since $u \mapsto u^m$ is not a Lipschitz map at $u = 0$, the function a is (in principle) not even locally finite. To overcome it, one should actually solve the "lifted" problems

$$\begin{cases} \partial_t u_{k,\epsilon} = \Delta\left(u_{k,\epsilon}^m\right) & \text{in } B_{R_k}(o) \times (0, +\infty) \,, \\ u_{k,\epsilon} = \epsilon & \text{on } \partial B_{R_k}(o) \times (0, +\infty) \,, \\ u_{k,\epsilon} = u_0 + \epsilon & \text{on } B_{R_k}(o) \times \{0\} \,, \end{cases}$$

instead of (DP) and eventually let $\epsilon \to 0^+$ to recover u_k.

Moreover, the "only" good estimate on $\Delta\xi_h$ is of the form

$$\int_0^T \int_{B_k(o)} a_h \left(\Delta\xi_h\right)^2 d\mu \, dt \leq \frac{1}{2} \int_{B_k(o)} |\nabla\omega|^2 \, d\mu \,,$$

and this is the main reason why we have to work in $L^2_{\mathrm{loc}}(M \times [0, +\infty))$.

Finally, to handle boundary terms, we need to ensure that v has a suitable notion of trace on $\partial B_R(o) \times (0, +\infty)$: this can be shown to hold at least for almost every $R > 0$.

About strong solutions

In the Euclidean case $M \equiv \mathbb{R}^n$, Bonforte and Vázquez [Adv. Math., 2010] proved a quantitative modification of the Herrero-Pierre estimate that in particular entails, for every $p \in [1, \infty)$,

$$u_0 \in L^p_{\mathrm{loc}}(\mathbb{R}^n) \implies u(\cdot, t) \in L^p_{\mathrm{loc}}(\mathbb{R}^n) \qquad \forall t \in (0, +\infty).$$

Moreover, if $p > p_c := 1 \vee \frac{n(1-m)}{2}$, they also show a quantitative local smoothing effect that yields

$$u_0 \in L^p_{\mathrm{loc}}(\mathbb{R}^n) \implies u(\cdot, t) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n) \qquad \forall t \in (0, +\infty).$$

These results have a purely local nature, so they can reasonably be extended to the manifold setting with little effort.

Taking advantage of these properties and local energy estimates, a formal computation gives

$$\partial_t \left(u^{\frac{m+1}{2}} \right) \in L^2_{\mathrm{loc}}(M \times (0, +\infty))$$

at least for $m > 1/2$, showing that u is in fact a strong solution.

A brief digression on stochastic completeness

For an n-dimensional model manifold M_ψ, we have a dichotomy:

$$\int_0^{+\infty} \frac{\int_0^r \psi^{n-1}(\rho)\, d\rho}{\psi^{n-1}(r)}\, dr \begin{cases} = +\infty \ \Rightarrow \ M_\psi \text{ is stochastically complete,} \\ < +\infty \ \Rightarrow \ M_\psi \text{ is stochastically incomplete.} \end{cases}$$

In general, stochastic completeness is a property related to the lifetime of the Brownian motion on M. However, a fully analytic characterization is also available, see e.g. the monograph by Grigor'yan [AMS/IP, 2009].

Theorem

Let $\alpha > 0$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following assertions are equivalent:

- *M is stochastically complete;*
- *The Cauchy problem for the heat equation on M starting from u_0 has a unique nonnegative bounded solution;*
- *The linear elliptic equation*

$$\Delta V = \alpha V \qquad \text{in } M$$

  does not admit any nonnegative, nontrivial, bounded solution.

Recently, in a joint work with G. Grillo and K. Ishige [JMPA, 2020], we were able to extend the above result to nonlinear equations such as (FDE) and (ELL). More precisely:

Theorem

Let $\alpha > 0$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following assertions are equivalent:

- *M is stochastically complete;*
- *The Cauchy problem* (FDE) *has a unique nonnegative bounded solution;*
- *The semilinear elliptic equation*

  $$\Delta W = \alpha\, W^{\frac{1}{m}} \qquad \text{in } M$$

  does not admit any nonnegative, nontrivial, bounded solution.

Therefore, as an immediate corollary, we infer that on stochastically incomplete manifolds uniqueness for (FDE) fails even within the class of bounded solutions!

Possible future work and open problems

- Weaken the strong-solution assumption in the uniqueness theorem for sign-changing solutions;
- Remove the L^2_{loc} restriction in the proof of the local comparison principle in order to have minimality;
- Extend the results to a nonlinear function more general than $u \mapsto u^m$: note however that a Keller-Osserman-type condition should be required;
- Replace the curvature bound with the stochastic-completeness property (although this seems very ambitious . . .).

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THANKS FOR YOUR ATTENTION!