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Sobolev regularity for nonlocal equations with VMO coefficients

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Classical Calderón-Zygmund-type regularity

Recall the following classical result.

Theorem (Calderón & Zygmund 1952)

Consider a domain $\Omega \subset \mathbb{R}^n$ and some $2 \leq p < \infty$. Then for any (weak) solution of the Poisson equation

$$\Delta u = f \text{ in } \Omega,$$

we have the sharp implication

$$f \in L^p(\Omega) \implies u \in W^{2,p}_{loc}(\Omega).$$

Possible approaches: Singular integrals, Fourier multipliers, Geometric (level set decay)...

Question: What happens if we replace the Laplacian by more complicated operators?

Second-order elliptic equations in divergence form

Next, given $f \in L^{\frac{2n}{n+2}}(\Omega)$, we consider weak solutions $u \in W^{1,2}(\Omega)$ to equations of the form

$$\operatorname{div}(b\nabla u) = f \text{ in } \Omega \subset \mathbb{R}^n, \tag{1}$$

where $b : \Omega \to \mathbb{R}$ is measurable such that

$$\Lambda^{-1} \leq b \leq \Lambda \text{ for some } \Lambda \geq 1.$$

Meyers: If $f \in L^{\frac{np}{n+p}}(\Omega)$ for some $p > 2$, then $u \in W^{1,2+\varepsilon}_{loc}(\Omega)$ for some $\varepsilon = \varepsilon(n, p, \Lambda) > 0$.

To prove more regularity of u, we need to impose more regularity on b.

VMO coefficients

Definition

Define

$$\eta_{b,\Omega}(\rho) := \sup_{\substack{0<r\leq\rho, x\in\Omega \\ B_r(x)\subset\Omega}} \fint_{B_r(x)} \left| b(y) - \overline{b}_{B_r(x)} \right| dy,$$

where $\overline{b}_{B_r(x)} := \fint_{B_r(x)} b dx$. We say that b belongs to $\mathrm{VMO}(\Omega)$ if

$$\lim_{\rho\to 0} \eta_{b,\Omega}(\rho) = 0.$$

Clearly, $b \in C(\overline{\Omega}) \implies b \in \mathrm{VMO}(\Omega)$.

However, not every VMO coefficient is continuous. For example,

$$b(x) = \sin\left(|\log(|x|)|^{\alpha}\right) + 2$$

is VMO for $\alpha \in (0,1)$, but **discontinuous** at the origin.

$W^{1,p}_{loc}$ regularity

Theorem (Di Fazio 1996)

Let $2 < p < \infty$. If $b \in \mathrm{VMO}(\Omega)$, then for any weak solution $u \in W^{1,2}(\Omega)$ of

$$\mathrm{div}(b \nabla u) = f \text{ in } \Omega,$$

we have the implication

$$f \in L^{\frac{np}{n+p}}(\Omega) \implies u \in W^{1,p}_{loc}(\Omega).$$

Many further contributions by Caffarelli, Peral, Iwaniec, Sbodorne, Kinnunen, Zhou, Byun, Wang, Acerbi, Mingione, Duzaar, Krylov, Dong, Kim, Ok, Mengesha, Diening, Balci,....

Higher differentiability?

Higher differentiability

Counterexample: In the one-dimensional case when $n = 1$, $u(x) := \int_0^x \frac{dt}{b(t)}$ solves

$$\left(bu'\right)' = 0.$$

Since $u' = 1/b$, higher differentiability of u' *requires* higher differentiability of b.
$\hookrightarrow$ **No** differentiability gain attainable under VMO or even continuous coefficients!

If the coefficient is Lipschitz, then the classical Calderón-Zygmund regularity remains valid.

Theorem (e.g. Gilbarg & Trudinger, Theorem 8.8 + Theorem 9.11)

If $b \in C^{0,1}(\Omega)$, then for any weak solution $u \in W^{1,2}(\Omega)$ of $\text{div}(b\nabla u) = f$ in Ω and any $2 \leq p < \infty$, we have the implication

$$f \in L^p(\Omega) \implies u \in W^{2,p}_{loc}(\Omega).$$

Fractional Calderón-Zygmund-type regularity

For $s \in (0,1)$, the *fractional* Laplacian of $u : \mathbb{R}^n \to \mathbb{R}$ is formally defined by

$$(-\Delta)^s u(x) := C_{n,s} \, p.v. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy.$$

For example by classical Fourier methods, it is possible to prove the following.

Theorem

Consider a domain $\Omega \subset \mathbb{R}^n$ and $2 \leq p < \infty$. Then for any (weak) solution of

$$(-\Delta)^s u = f \text{ in } \Omega,$$

we have the sharp implication

$$f \in L^p(\Omega) \implies u \in W^{2s,p}_{loc}(\Omega).$$

Question: What about more general fractional/nonlocal operators?

Nonlocal equations with measurable coefficients

Fix $s \in (0,1)$. We consider equations of the form

$$L_A u = f \text{ in } \Omega \subset \mathbb{R}^n,$$

where

$$L_A u(x) := p.v. \int_{\mathbb{R}^n} \frac{A(x,y)}{|x-y|^{n+2s}} (u(x)-u(y)) dy$$

is a **nonlocal** operator.

Here $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a measurable and symmetric coefficient that satisfies

$$\Lambda^{-1} \leq A(x,y) \leq \Lambda \text{ for all } x,y \in \mathbb{R}^n \text{ and some } \Lambda \geq 1.$$

Note that for $A = C_{n,s}$, we recover the fractional Laplacian $(-\Delta)^s$.

Weak solutions

For $s \in (0,1)$ and $p \in [1,\infty)$, define the fractional Sobolev space

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \;\middle|\; \underbrace{\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dydx}_{=:[w]^p_{W^{s,p}(\Omega)}} < \infty \right\}.$$

Given $f \in L^{\frac{2n}{n+2s}}(\Omega)$, $u \in W^{s,2}(\mathbb{R}^n)$ is a **weak solution** of $L_A u = f$ in Ω, if

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{A(x,y)}{|x-y|^{n+2s}} (u(x)-u(y))(\varphi(x)-\varphi(y)) dydx = \int_\Omega f \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Further regularity? Many results on Hölder regularity e.g. by Kassmann, Caffarelli, Chan, Vasseur, Di Castro, Kuusi, Palatucci, Ros-Oton, Serra, Cozzi, Brasco, Lindgren, Schikorra, De Filippis, Fall, Bonforte, Figalli, Vázquez, Chaker, Kim, Weidner,...

What about Sobolev regularity?

Sobolev regularity without further assumptions

Theorem

If $f \in L^{\frac{np}{n+sp}}(\Omega)$ for some $p > 2$, then $u \in W^{s+\varepsilon,2+\varepsilon}_{loc}(\Omega)$ for some $\varepsilon = \varepsilon(n,s,p,\Lambda) > 0$.

T. Kuusi, G. Mingione, Y. Sire, *Nonlocal self-improving properties*, Anal. PDE (2015)

A. Schikorra, *Nonlinear commutators for the fractional p-Laplacian and applications*, Math. Ann. (2016).

The improvement of differentiability under such irregular coefficients is a **purely nonlocal phenomenon**!

Question: If A is more regular, can the integrability gain and more interestingly, can the differentiability gain be improved to larger exponents?

Previous result on higher Sobolev regularity

Theorem (T. Mengesha, A. Schikorra, S. Yeepo, Adv. Math. 2021)

If $A \in C^\alpha$ for *some* $\alpha > 0$, then for any weak solution $u \in W^{s,2}(\mathbb{R}^n)$ of $L_A u = f$ in Ω, any $p \in (2, \infty)$ and any $s \leq t < \min\{2s, 1\}$, we have

$$f \in L^{\frac{np}{n+(2s-t)p}}(\Omega) \implies u \in W^{t,p}_{loc}(\Omega).$$

The proof relies on commutator estimates inspired by

R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. Math. (2) (1976).

Question: Does this result remain valid if $A \in \mathrm{VMO}$?

Main result

Theorem (S. Nowak 2022)

If $A \in \mathrm{VMO}(\Omega \times \Omega)$, then for any weak solution $u \in W^{s,2}(\mathbb{R}^n)$ of $L_A u = f$ in Ω, any $p \in (2,\infty)$ and any $s \leq t < \min\{2s, 1\}$, we have

$$f \in L^{\frac{np}{n+(2s-t)p}}(\Omega) \implies u \in W^{t,p}_{loc}(\Omega). \tag{2}$$

Extensions: (2) remains valid, if A is sufficiently small in BMO, or if $A(x,y) = a(x-y)$ for some measurable $a : \mathbb{R}^n \to \mathbb{R}$ with $\Lambda^{-1} \leq a \leq \Lambda$.
The results also remain valid for nonlinear equations with linear growth.

S. Nowak, *Improved Sobolev regularity for linear nonlocal equations with VMO coefficients*, Math. Ann. (2022).

S. Nowak, *Regularity theory for nonlocal equations* (2022), PhD thesis.

Almost $W^{2s,p}$ regularity for $s \leq 1/2$

For $s \leq 1/2$, we almost match the optimal Calderón-Zygmund-type regularity for the fractional Laplacian, despite the presence of a **discontinuous** coefficient!

Corollary (S. Nowak 2022)

If $A \in \mathrm{VMO}(\Omega \times \Omega)$ and $s \in (0, 1/2]$, then for any weak solution $u \in W^{s,2}(\mathbb{R}^n)$ of $L_A u = f$ in Ω and any $2 \leq p < \infty$, we have the implication

$$f \in L^p(\Omega) \implies u \in W^{2s-\varepsilon,p}_{loc}(\Omega) \text{ for any } \varepsilon > 0.$$

In particular, observe that the case $p = 2$ is included.

$\hookrightarrow$ Pure higher differentiability result under VMO coefficients.

Auxiliary equation

Instead of $L_A u = f$, we focus on equations of the type $L_A u = (-\Delta)^s g$.

If $f \in L^{\frac{np}{n+(2s-t)p}}(\Omega)$, then there exists a weak solution $g \in W^{2s,\frac{np}{n+(2s-t)p}}_{loc}(\Omega) \hookrightarrow W^{t,p}_{loc}(\Omega)$ of

$$(-\Delta)^s g = f \text{ in } \Omega.$$

Therefore, it suffices to prove the implication

$$g \in W^{t,p}_{loc}(\Omega) \implies u \in W^{t,p}_{loc}(\Omega).$$

We do so by adapting and combining techniques introduced in

- L. Caffarelli and I. Peral, *On $W^{1,p}$ estimates for elliptic equations in divergence form*, Comm. Pure Appl. Math. (1998)
- T. Kuusi, G. Mingione, Y. Sire, *Nonlocal self-improving properties*, Anal. PDE (2015)

Dual pairs

For $\theta \in (0, \min\{s, 1-s\})$, we define a locally finite doubling measure μ on $\mathbb{R}^{2n}$:

$$\mu(E) := \int_E \frac{dx\,dy}{|x-y|^{n-2\theta}}.$$

For $u : \mathbb{R}^n \to \mathbb{R}$, we define the *gradient-type* function $U : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$U(x,y) := \frac{|u(x)-u(y)|}{|x-y|^{s+\theta}}.$$

For any $p \geq 2$ and $\widetilde{s} := s + \theta\left(1 - \frac{2}{p}\right) \geq s$,

$$u \in W^{\widetilde{s},p}(\Omega) \iff u \in L^p(\Omega) \text{ and } U \in L^p(\Omega \times \Omega, \mu).$$

Proving higher integrability of U w.r.t. μ implies **higher differentiability** of u!

Thus, as a first step we focus on proving $U \in L^p_{loc}(\Omega \times \Omega, \mu)$.

Freezing coefficients

We approximate our weak solution u of $L_A u = (-\Delta)^s g$ in $B_{2r}(x_0)$ by a solution v of

$$\begin{cases} L_{\widetilde{A}} v = 0 & \text{in } B_{2r}(x_0) \\ v = u & \text{a.e. in } \mathbb{R}^n \setminus B_{2r}(x_0), \end{cases}$$

$$\widetilde{A}(x,y) := \begin{cases} \overline{A}_{x_0,r} & \text{if } (x,y) \in B_r(x_0) \times B_r(x_0) \\ A(x,y) & \text{if } (x,y) \notin B_r(x_0) \times B_r(x_0), \end{cases}$$

Since $s + \theta < \min\{2s, 1\}$, by previous results $v \in C^{s+\theta}_{loc}(B_r(x_0))$. Since A is VMO,

$$\omega(A - \overline{A}_{x_0,r}) := \fint_{B_r(x_0)} \fint_{B_r(x_0)} |A(x,y) - \overline{A}_{x_0,r}| dydx$$

is small whenever r is small.

Testing with $w := u - v \in W^{s,2}_0(B_{2r}(x_0))$ along with applying the Kuusi-Mingione-Sire estimate to u then leads to $[w]_{W^{s,2}(\mathbb{R}^n)}$ being small whenever r is small.

Higher integrability of U via covering the level sets

Idea: Prove sufficiently fast decay of the level sets $E_\lambda := \{\mathcal{M}_\mu(U^2) > N^2\lambda^2\} \subset \mathbb{R}^{2n}$.
We cover E_λ by **dyadic cubes** $\mathcal{K} = K_1 \times K_2$ in $\mathbb{R}^{2n}$ with $\mu(E_\lambda \cap \mathcal{K}) \geq \varepsilon\mu(\mathcal{K})$ and

$$\mu\left(E_\lambda\right) \lesssim \varepsilon \sum \mu(\mathcal{K}), \quad \varepsilon > 0 \text{ to be chosen.}$$

Diagonal case: If $\operatorname{dist}(K_1, K_2)$ is small, then $\mu(\mathcal{K})$ can be controlled by using the $C^{s+\theta}_{loc}$ estimate for the approximate solution.

Off-Diagonal case: If $\operatorname{dist}(K_1, K_2)$ is large, then no useful comparison estimate is available! Nevertheless, combining certain reverse Hölder inequalities with involved combinatorial arguments still allows to control the measures of such cubes.

For any $p > 2$, the level set decay then yields

$$\left(\fint_{\mathcal{B}} U^p d\mu\right)^{\frac{1}{p}} \lesssim \left(\fint_{2\mathcal{B}} G^p d\mu\right)^{\frac{1}{p}} + W^{s,2}\text{-tail terms.}$$

Restricted $W^{t,p}$ regularity

We have arrived at the following intermediate result.

Theorem

Let $p \in (2, \infty)$ and fix some t such that

$$s \leq t < \begin{cases} 2s\left(1 - \frac{1}{p}\right), & \text{if } s \leq 1/2 \\ 1 - \frac{2-2s}{p}, & \text{if } s > 1/2 \end{cases} =: t_{sup}. \tag{3}$$

If $A \in \mathrm{VMO}(\Omega \times \Omega)$, then for any weak solution $u \in W^{s,2}(\mathbb{R}^n)$ of the equation $L_A u = f$ in Ω, we have the implication

$$f \in L_{loc}^{\frac{np}{n+(2s-t)p}}(\Omega) \implies u \in W_{loc}^{t,p}(\Omega).$$

S. Nowak, *Regularity theory for nonlocal equations with VMO coefficients*, Ann. Inst. H. Poincaré Anal. Non Linéaire (2022).

Sharp higher Hölder regularity by embedding

By embedding, the restricted $W^{t,p}_{loc}$ regularity already implies sharp higher Hölder regularity.

Theorem

Let $f \in L^q_{loc}(\Omega)$ for some $q > \frac{n}{2s}$. If $A \in \mathrm{VMO}(\Omega \times \Omega)$, then for any weak solution $u \in W^{s,2}(\mathbb{R}^n)$ of $L_A u = f$ in Ω, we have

$$u \in \begin{cases} C^{2s-\frac{n}{q}}_{loc}(\Omega), & \text{if } 2s - \frac{n}{q} < 1 \\ C^{\alpha}_{loc}(\Omega) \quad \forall \alpha \in (0,1), & \text{if } 2s - \frac{n}{q} \geq 1. \end{cases}$$

For $2s - \frac{n}{q} < 1$, the result is sharp already in the case of the fractional Laplacian.

For $2s - \frac{n}{q} \geq 1$, we also expect the result to be sharp, since already weak solutions of $\mathrm{div}(b \nabla u) = 0$ are in general **not Lipschitz** if b is merely continuous, see

T. Jin, V. Maz'ya, J. Van Schaftingen, *Pathological solutions to elliptic problems in divergence form with continuous coefficients*, C. R. Math. Acad. Sci. Paris (2009).

Higher-order dual pairs and improved differentiability

To further improve the differentiability gain, we consider dual pairs of **higher order**.

For any $s \leq \alpha < s + \theta < \min\{2s, 1\}$ and $\theta_\alpha := s + \theta - \alpha$,

$$\mu_\alpha(E) := \int_E \frac{dx\,dy}{|x-y|^{n-2\theta_\alpha}}, \quad U_\alpha(x,y) := \frac{|u(x)-u(y)|}{|x-y|^{\alpha+\theta_\alpha}}.$$

For any $p > 2$ and $\widetilde{\alpha} := \alpha + \theta_\alpha \left(1 - \frac{2}{p}\right) > \alpha$,

$$u \in W^{\widetilde{\alpha},p}(\Omega) \iff u \in L^p(\Omega) \text{ and } U_\alpha \in L^p(\Omega \times \Omega, \mu_\alpha).$$

Higher-order approximation and iteration

Suppose that $g \in W^{t,p}_{loc}$ for some $p > 2$, $s \leq t < \min\{2s, 1\}$ and that u solves $L_A u = (-\Delta)^s g$ in some ball B, and let v be the solution of

$$\begin{cases} L_A v = 0 & \text{in } B \\ v = u & \text{a.e. in } \mathbb{R}^n \setminus B. \end{cases} \tag{4}$$

$w := u - v$ solves $L_A w = (-\Delta)^s g$, so that by the previous case when $\alpha = s$ we obtain

$$[w]_{W^{\alpha_1,2}} \lesssim [w]_{W^{s,2}} + [g]_{W^{\alpha_1,m}} + \text{ tail terms}$$

for any $m > 2$ and $\alpha_1 := s + \theta \left(1 - \frac{2}{m}\right) > s$.

Since $v \in C^{s+\theta}_{loc} = C^{\alpha_1 + \theta_{\alpha_1}}_{loc}$, adapting the above covering procedure leads to $U_{\alpha_1} \in L^p(\mu_{\alpha_1})$. In particular, u satisfies a $W^{\alpha_2,p}$ estimate for some $\alpha_2 > \alpha_1$, **improving the differentiability gain**.

Iterating this procedure finitely many times leads to $u \in W^{t,p}_{loc}(\Omega)$ as desired.

Summary

For **local** equations $\operatorname{div}(b\nabla u) = f$ with $b \in \mathrm{VMO}$, we have

$$u \in W^{1,2},\ f \in L^{\frac{np}{n+p}}_{loc} \implies u \in W^{1,p}_{loc} \quad \forall 2 < p < \infty,$$

but in general no higher differentiability.

For **nonlocal** equations $\int_{\mathbb{R}^n} \frac{A(x,y)}{|x-y|^{n+2s}}(u(x)-u(y))dy = f$ with $s \in (0,1)$, $A \in \mathrm{VMO}$,

$$u \in W^{s,2},\ f \in L^{\frac{np}{n+(2s-t)p}}_{loc} \implies u \in W^{t,p}_{loc} \quad \forall 2 < p < \infty,\ s \leq t < \min\{2s,1\},$$

gaining also higher differentiability $\rightarrow$ **Purely nonlocal phenomenon**.

Result remains true for **nonlinear** nonlocal equations.