New results on nonlinear aggregation-diffusion equations with Riesz kernels

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Regularity for nonlinear diffusion equations. Green functions and functional inequalities

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The classical Patlak-Keller-Segel model

The classical PKS model is

$$\begin{cases} \rho_t = \Delta\phi(\rho) - \chi\nabla\cdot(\rho\nabla c) & x \in \mathbb{R}^2,\ t > 0 \\ c_t - \Delta c = \rho - \alpha c & x \in \mathbb{R}^2,\ t > 0 \\ \rho(x,0) = \rho_0(x) & x \in \mathbb{R}^2 \end{cases}$$

Patlak (1953), Keller-Segel (1971), Nanjundiah (1973).

The classical Patlak-Keller-Segel model

- Let us consider again the PKS model with $\Phi(\rho) = \rho$ (linear diffusion)

$$\begin{cases} \rho_t = \Delta\rho - \nabla\cdot(\rho\nabla c) & x \in \mathbb{R}^2,\ t > 0 \\ -\Delta c = \rho & x \in \mathbb{R}^2,\ t > 0 \\ \rho(x,0) = \rho_0(x) & x \in \mathbb{R}^2 \end{cases}$$

Now, since we can write $c(x) = \mathcal{N} * \rho$, being $\mathcal{N} = -\frac{1}{2\pi}\log|x|$ the Newtonian kernel in $\mathbb{R}^2$ the equation coming from the parabolic-elliptic KS system is

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$$\rho_t = \Delta\rho - \nabla\cdot(\rho\nabla(\mathcal{N} * \rho)).$$

- There is a critical mass $M_c = 8\pi$ such that, for solutions $\rho \in L^1(\mathbb{R}^2, (1+|x|^2)dx)$
  - if the mass satisfies $M < M_c$, the solution exists globally in time.
  - if $M > M_c$, then we have a blow-up in finite time.
  - if $M = M_c$, then we have a blow-up in infinite time.

  (Jager-Luckhaus '92, Nagai '01, Dolbeault-Perthame '04, Blanchet-Carrillo-Masmoudi '16)

The Keller-Segel model with degenerate diffusion

In $\mathbb{R}^N$ with $N \geq 2$, the model with degenerate diffusion is

$$\rho_t = \Delta\rho^m - \nabla \cdot (\rho\nabla(\mathcal{N} * \rho)),$$

with $m > 1$, being $\mathcal{N}$ the Newtonian kernel in $\mathbb{R}^N$.

- The nonlinear degenerate diffusion term for the 2D Keller-Segel equation avoids the blow-up phenomenon (anti-overcrowding effect). (Boi-Capasso-Morale '00, Topaz-Bertozzi-Lewis '06).
- The behaviour of solutions depends on m and on the so called critical exponent $m_c = 2 - \frac{2}{N}$:
  - for $m > m_c$, for any $\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$, the solution exists globally in time and there is a uniform estimate in time of the L^∞ norm. (Sugiyama '06)
  - for $m < m_c$, there is a blow-up in finite time for an initial data with arbitrarily small mass. (Sugiyama '06)
  - for $m = m_c$ (fair competition) the behaviour of solution depends on the mass, and there is the presence of a critical mass M_c. (Blanchet-Carrillo-Laurençot '09)

The Keller-Segel model with degenerate diffusion

From now on, we will focus on the “subcrictical case” $m > 2 - \frac{2}{N}$, in which solutions exist globally in time.

Question

What about the asymptotic behaviour of solutions?

There is the existence of a free-energy functional $\mathcal{F}$ associated to the model:

$$\mathcal{F}[\rho] = \frac{1}{m-1} \int_{\mathbb{R}^N} \rho^m dx - \frac{1}{2} \int_{\mathbb{R}^N} \rho(\mathcal{N} * \rho) dx;$$

we can write the KS equation as

$$\rho_t = \nabla \cdot \left(\rho \nabla (\frac{m}{m-1} \rho^{m-1} - \mathcal{N} * \rho) \right) =: \nabla \cdot \left(\rho \nabla \left(\frac{\delta \mathcal{F}}{\delta \rho} \right) \right)$$

where $\frac{\delta \mathcal{F}}{\delta \rho} = \frac{m}{m-1} \rho^{m-1} - \mathcal{N} * \rho$.
If ρ is a solution of the KS-equation, then $\mathcal{F}[\rho]$ **decreases** in time, hence it is a Lyapunov functional.

The Keller-Segel model with degenerate diffusion

The following properties are known for the global minimizers of $\mathcal{F}$, among densities with fixed mass M:

- Existence: (Lions '84) for $N \geq 3$ and (Carrillo, Castorina, V. 2014) for $N = 2$;
- Radial symmetry (rearrangement techniques);
- Uniqueness + compact support (Lieb-Yau '87), (Kim-Yao 2012) for $N \geq 3$, (Carrillo, Castorina, V. 2014) for $N = 2$

Let ρ_M be a minimizer of $\mathcal{F}$ with mass M. Then ρ_M must be a stationary solution.

The Keller-Segel model with degenerate diffusion

Question

If $\rho_0 = \rho(0, \cdot)$ has mass M, is it always true that $\rho(\cdot, t)$ converges to (a translation of) ρ_M when $t \to \infty$?

The answer is affirmative only if we have a positive answer to the following questions:

Question

Is ρ_M the unique stationary state of mass M (up to translations)?

We know the uniqueness of stationary solutions with **radial symmetry**, with fixed mass (Lieb-Yau '87), , (Kim-Yao 2014) hence the question above reduces to

Question

Is it true that every steady state is radially symmetric (up to translations)?

Stationary solutions of the Keller-Segel equation

Rewriting the KS-equation in the divergence form

$$\rho_t - \nabla \cdot \left(\rho \nabla (\frac{m}{m-1} \rho^{m-1} - \mathcal{N} * \rho) \right) = 0,$$

then any stationary solution ρ_s satisfies

$$\frac{m}{m-1} \rho_s^{m-1} - \mathcal{N} * \rho_s = C_i$$

in each connected component of $\{\rho_s > 0\}$ (C_i may be get different values in each connected component).

Stationary solutions for the degenerate aggregation-diffusion equation

Now we consider the equation with a general attractive kernel $\mathcal{K}$:

$$\rho_t = \nabla \cdot \left(\rho \nabla (\frac{m}{m-1}\rho^{m-1} + \mathcal{K} * \rho)\right),$$

where $\mathcal{K}$ is radial and strictly increasing in $|x|$. Similarly, each steady state ρ_s verifies

$$\frac{m}{m-1}\rho_s^{m-1} + \mathcal{K} * \rho_s = C_i$$

in each connected component of $\{\rho_s > 0\}$.

Theorem (Carrillo-Hittmeir-Yao, V., Invent. Math., 2019)

Let $\rho_s \in L^1_+(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ a steady state. Then ρ_s must be radially decreasing, up to translastions.

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Main ingredients: Steiner and continuous Steiner symmetrization.

Uniqueness?

In principle, nothing can be said on the uniqueness of the stationary states for a **general kernel** $\mathcal{K}$: if $\mathcal{K} = -\mathcal{N}$, there is a unique radial stationary state with mass *M* (up to translation) (Kim-Yao 2012).

Existence of global minimizers

It is possible to show the existence of a radially decreasing global minimizer of the energy functional

$$\mathcal{F}[\rho] = \frac{1}{m-1}\int_{\mathbb{R}^N} \rho^m dx + \frac{1}{2}\int_{\mathbb{R}^N} \rho(\mathcal{K} * \rho) dx,$$

in the class of admissible densities

$$\mathcal{Y}_M := \left\{\rho \in L^1_+(\mathbb{R}^N) \cap L^m(\mathbb{R}^N) : \ ||\rho||_1 = M, \omega(1+|x|)\,\rho(x) \in L^1(\mathbb{R}^N)\right\},$$

where we assume $\int_{\mathbb{R}^N} x\rho(x)\,dx = 0$, with $\mathcal{K}(x) = \omega(|x|)$. More precise assumptions on $\mathcal{K}$ are

(K1) $\omega'(r) > 0$ for all $r > 0$ with $\omega(1) = 0$.

(K2) $\mathcal{K}$ is not more singular than the Newtonian kernel in $\mathbb{R}^N$ close to the origin,i.e., there exists $C_w > 0$ such that $\omega'(r) \leq C_w r^{1-N}$ per $r \leq 1$.

(K3) There is some $C_w > 0$ such that $\omega'(r) \leq C_w$ for all $r > 1$.

(K4) Condition at infinity: $\lim_{r\to+\infty} \omega_+(r) = +\infty$.

Regularity of minimizers

If ρ_0 is a global minimizer, one has

- ρ_0 satisfies
$$\frac{m}{m-1}\rho_0^{m-1} + \mathcal{K} * \rho_0 = C \quad \text{a.e. in } \{\rho_0 > 0\}$$
hence it is a stationary state;
- From this equation and from the asymptotic bahavior of $\mathcal{K} * \rho_0$ one can show that ρ_0 is of compact support; moreover $\rho_0 \in L^\infty(\mathbb{R}^N)$;
- Using the locally Lipschitz regularity $W^{1,\infty}_{loc}$ of $\mathcal{K} * \rho_0$ one shows that $\rho \in C^{0,\alpha}(\mathbb{R}^N)$, $\alpha = \min\{1, \frac{1}{m-1}\}$.

Remark: uniqueness

For $\mathcal{K} = -\mathcal{N}$, using the uniqueness result for radial steady states, for any mass $M > 0$, the unique radial steady state of mass M (up to translation) is the minimizer of the energy functional $\mathcal{F}$.

What happens when $\mathcal{K}$ is a Riesz kernel?

$$\partial_t \rho = \Delta \rho^m - \chi \nabla \cdot (\rho \, \nabla (W_s * \rho)) \quad \text{in } \mathbb{R}^N \times (0, T),$$

The interaction is given by the the Riesz kernel

$$W_s(x) := c_{N,s} \, |x|^{2s-N} \quad 0 < s < N/2.$$

Free energy:

$$\mathcal{F}[\rho] = \mathcal{H}_m[\rho] + \mathcal{W}_s[\rho]$$

$$\mathcal{H}_m[\rho] = \frac{1}{m-1} \int_{\mathbb{R}^N} \rho^m(x) \, dx \, , \qquad \mathcal{W}_s[\rho] = -\frac{\chi \, c_{N,s}}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^{2s-N} \rho(x)\rho(y) \, dxdy \, .$$

The Riesz kernels case

$\mathcal{H}_m$ and $\mathcal{W}_s$ are homogeneous by taking dilations $\rho^\lambda(x) = \lambda^N \rho(\lambda x)$

$$\mathcal{F}[\rho^\lambda] = \lambda^{N(m-1)}\mathcal{H}_m[\rho] + \lambda^{N-2s}\chi\mathcal{W}_k[\rho]\,.$$

Critical exponent $m_c := 2 - 2s/N$

- $m = m_c$: fair competition regime (critical mass)
- $m > m_c$: diffusion dominated regime ← we focus on this case
- $m < m_c$: attraction dominated regime

Our results have many analogous in fair competition regime
[Blanchet, Carrillo, Laurencot 2009], [Calvez, Carrillo, Hoffmann 2016, 2017]
and in case of Newtonian potential interaction
[Kim, Yao 2012], [Carrillo, Castorina, V. 2015], [Carrillo, Hittmeir, V., Yao 2019]

Stationary states

Basic facts:

Stationary states

Basic facts:if ρ is a stationary state then

$$\rho(x)^{m-1} = \frac{m-1}{m} \left(\chi W_s * \rho(x) - C[\rho](x)\right)_+ , \quad x \in \mathbb{R}^N$$

where $C[\rho](x)$ is constant on each connected component of $\mathrm{supp}(\rho)$.

Radial symmetry of stationary states

Using a suitable variation of the radial symmetry result contained in [Carrillo, Hittmeir, V., Yao 2019]:

Theorem (Carrillo-Hoffmann-Mainini-V., Calc. Var. 2018)

Stationary states are radially symmetric decreasing (up to translations), compactly supported.

Existence of global minimizers

Theorem

Let $s \in (0, N/2)$ and $m > m_c$. There exist a minimizer of $\mathcal{F}$ on $\mathcal{Y}_M := \left\{\rho \in L^1_+(\mathbb{R}^N) \cap L^m(\mathbb{R}^N)\,,\ ||\rho||_1 = M\,,\ \int_{\mathbb{R}^N} x\rho(x)\,dx = 0\right\}$.

- By Lions concentration-compactness, as for instance in [Kim,Yao 2012]

Properties of minimizers

Theorem

Let $s \in (0, N/2)$ and $m > m_c$. If ρ is a global minimizer of the free energy functional $\mathcal{F}$ in $\mathcal{Y}$, then ρ is radially symmetric and non-increasing, bounded, compactly supported, and

$$\rho^{m-1}(x) = \left(\frac{m-1}{m}\right) (\chi W_s * \rho(x) - C[\rho])_+ \quad \text{in } \mathbb{R}^N$$

where

$$C[\rho] := -\frac{2}{M}\mathcal{F}[\rho] - \frac{1}{M}\frac{m-2}{m-1}\int_{\mathbb{R}^N} \rho^m(x)\, dx > 0, \qquad \rho \in \mathcal{Y}_M \,.$$

Regularity of minimizers

There exists a critical exponent

$$m^* := \begin{cases} \dfrac{2-2s}{1-2s} & \text{if} \quad N \geq 1 \quad \text{and} \quad s \in (0, 1/2)\,, \\ +\infty & \text{if} \quad N \geq 2 \quad \text{and} \quad s \in [1/2, N/2)\,. \end{cases}$$

Theorem

Let $s \in (0, N/2)$ and ρ a minimizer of $\mathcal{F}$ on $\mathcal{Y}_M$, with $m > m_c$.

1. *if $s \in (1/2, N/2)$ we have $(-\Delta)^{-s}\rho \in W^{1,\infty}(\mathbb{R}^N)$, $\rho^{m-1} \in W^{1,\infty}(\mathbb{R}^N)$ and $\rho \in C^{0,\alpha}(\mathbb{R}^N)$ with $\alpha = \min\{1, \frac{1}{m-1}\}$.*
2. *if $s \in (0, 1/2]$ we have two subcases:*

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   - (i) *if $m \leq 2$ or $2 < m < m^*$ the same conclusion of case (1) holds;*
   - (ii) *if $m \geq m^*$, then $(-\Delta)^{-s}\rho \in C^{0,\gamma}\left(\mathbb{R}^N\right)$ for any $\gamma < (2s(m-1))/(m-2)$ and $\rho \in C^{0,\alpha}\left(\mathbb{R}^N\right)$ for any $\alpha < 2s/(m-2)$.*

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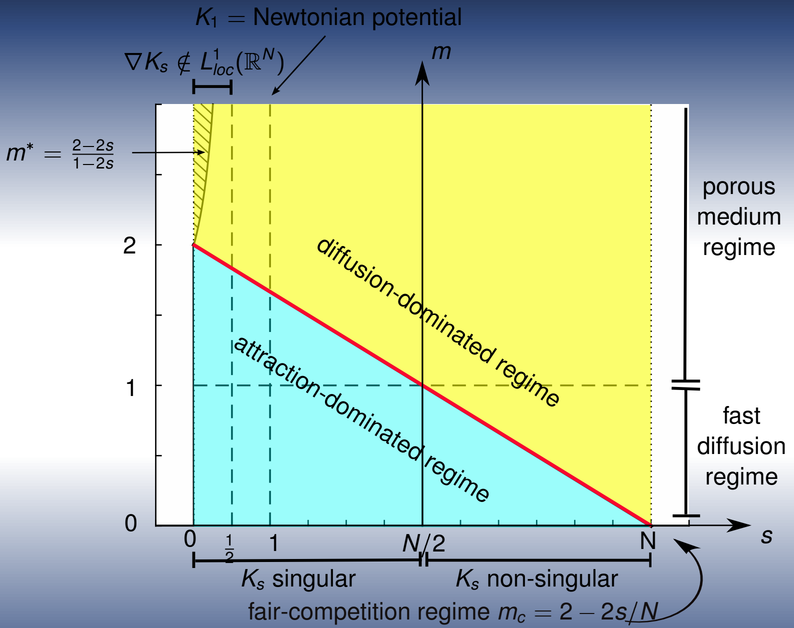
3. *If $m \geq m_c$ and B is the interior of suppρ, then $\rho \in C^\infty(B)$.*


K_1 = Newtonian potential
$\nabla K_s \notin L^1_{loc}(\mathbb{R}^N)$
m
$m^* = \frac{2-2s}{1-2s}$
2
1
0
diffusion-dominated regime
attraction-dominated regime
porous medium regime
fast diffusion regime
0
$\frac{1}{2}$
1
$N/2$
N
s
K_s singular
K_s non-singular
fair-competition regime $m_c = 2 - 2s/N$

Uniqueness of steady states with Riesz aggregation kernels

Uniqueness of radial steady states is well-known with newtonian kernels $\mathcal{N}$.
In the case of Riesz kernels $W_s(x) = c_{N,s}|x|^{2s-N}$, uniqueness was proved for $N = 1$ in [CHMV2018]; for $N > 1$, the situation is much more complicated.
Recall that such special solutions satisfy

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$$\rho(x)^{m-1} = \frac{m-1}{m}\left(\chi W_s * \rho(x) - C\right)_+ , \quad x \in \mathbb{R}^N$$

for some $C > 0$. Some results:

- Calvez-Carrillo-Hoffmann, 2020: case $m > 2 - \frac{2s}{N}$, $s \in (0, 1)$.
- Delgaldino-Yan-Yao,2020: case $m \geq 2$, $s \in (0, N/2)$ (and some other general potentials)

A PDE approach

Putting $u = (-\Delta)^{-s}\rho$, $s \in (0,1)$, $p = 1/(m-1)$, $a = \frac{m-1}{m}$, $\chi = 1$ in

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- local case $s = 1$: Flucher Wei 1988, $N \geq 3$, $1 < p < \frac{N+2}{N-2}$ by an ODE argument;
- Chan-Gonzalez-Huang-Mainini-V., Calc. Var. 2020: case $p \geq 1$, $s \in (0,1)$.

Relation between uniqueness of steady states an uniqueness of solutions to the FPP

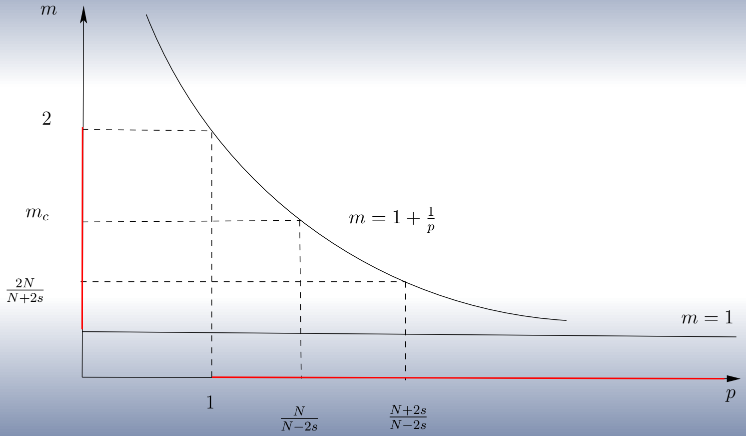


FIGURE 1. Sub and supercritical regimes in terms of m and p

The nonlocal case

The case $s \in (0,1)$ is more challenging: no ODE technique can be used!

Theorem (Subcritical case,CGHMV, Calc. Var. 2020)

Let $1 \leq p < (N+2s)/(N-2s)$ and $C > 0$. There exists a unique non-negative, radially decreasing solution to the problem

$$\begin{cases} (-\Delta)^s u = a(u-C)_+^p & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

Singular limits of the KS equation

Let us consider the Cauchy problem in the whole space $\mathbb{R}^N$, $N \geq 1$, for the aggregation-diffusion equation

$$\begin{cases} \rho_t = \Delta\rho^m + \beta\Delta\rho^2 - \chi\nabla\cdot(\rho\nabla(W_s * \rho)), \\ \rho(0) = \rho^0, \end{cases} \tag{1}$$

where $\beta \geq 0$ and $m > 2$.
The natural free energy associated with the nonlocal PDE (2) is given by

$$\mathcal{F}_s[\rho] = \frac{1}{m-1}\int_{\mathbb{R}^N} \rho^m(x)\,dx + \beta\int_{\mathbb{R}^N} \rho^2(x)\,dx - \frac{\chi}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W_s(x-y)\rho(x)\rho(y)\,dx\,dy.$$

We are interested in the limiting behavior of solutions to (2) and the stationary states as $s \to 0$: Huang-Mainini-Vázquez, V. 2022.

Limiting behavior of the stationary states

The limit functional is formally given by

$$\mathcal{F}_0[\rho] := \frac{1}{m-1}\int_{\mathbb{R}^N} \rho^m(x)\,dx + \left(\beta - \frac{\chi}{2}\right)\int_{\mathbb{R}^N} \rho^2(x)\,dx.$$

It is clear that the minimization problem $\min_{\mathcal{Y}_M} \mathcal{F}_0$,

$$\mathcal{Y}_M := \left\{\rho \in L^1_+(\mathbb{R}^d) \cap L^m(\mathbb{R}^d) : \int_{\mathbb{R}^d} \rho(x)\,dx = M,\ \int_{\mathbb{R}^d} x\rho(x)\,dx = 0\right\},$$

is strongly influenced by the sign of the coefficient $\beta - \chi/2$. Indeed, it can be proven that [HMVV, 2022] for $0 \leq \beta < \chi/2$, $\mathcal{F}_0$ admits a unique radially decreasing minimizer over $\mathcal{Y}_M$, given by

$$\rho_0(x) := \left(\frac{\chi - 2\beta}{2}\right)^{1/(m-2)} \mathbb{1}_{B_{R_0}}(x), \qquad \text{where} \quad R_0 = \left(\frac{NM}{\sigma_N}\right)^{1/d} \left(\frac{\chi - 2\beta}{2}\right)^{-\frac{1}{N(m-2)}}.$$

Else if $\beta \geq \chi/2$, functional $\mathcal{F}_0$ does not admit a minimizer over $\mathcal{Y}_M$ and $\inf_{\mathcal{Y}_M} \mathcal{F}_0 = 0$.

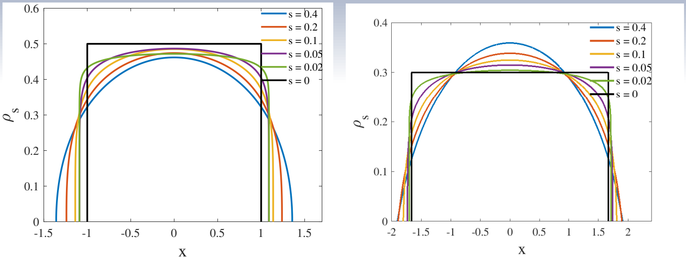
Limiting behavior of the stationary states

We have the following result

Theorem (HMVV, 2022)

For any $s \in (0, 1/2)$, let $\rho_s \in \mathcal{Y}_M$ be the unique minimizer of $\mathcal{F}_s$ over $\mathcal{Y}_M$. If $0 \le \beta < \chi/2$, there exists $\rho \in \mathcal{Y}_M$ such that $\rho_s \to \rho$ strongly in $L^m(\mathbb{R}^N)$ as $s \downarrow 0$, and moreover ρ is the unique radially decreasing minimizer of the functional $\mathcal{F}_0$ over $\mathcal{Y}_M$. Else if $\beta \ge \chi/2$, we have $\lim_{s\downarrow 0} \mathcal{F}_s[\rho_s] = 0$ and $\rho_s \to 0$ uniformly on $\mathbb{R}^N$.

Limiting behavior of the stationary states



The steady states for different $s > 0$ with $m = 3$ and $\chi = 1$ (Left figure: $\beta = 0$ and Right figure: $\beta = 0.2$). The expected limiting steady state with $s = 0$, which is a characteristic function with height $\left(\frac{\chi - 2\beta}{2}\right)^{1/(m-2)}$ is also plotted for reference.

Limiting behavior of the stationary states: main ingredients for the case $\beta < \chi/2$

Lemma

Fix any $s_0 \in (0, 1/2)$. For any $s \in (0, s_0)$, let $\rho_s \in \mathcal{Y}_M$ be the unique minimizer of $\mathcal{F}_s$ over $\mathcal{Y}_M$. Then $\sup_{s \in (0, s_0)} ||\rho_s||_{L^\infty(\mathbb{R}^N)} < +\infty$.

Lemma

For any $s \in (0, 1/2)$, let $\rho_s \in \mathcal{Y}_M$ be the unique minimizer of $\mathcal{F}_s$ over $\mathcal{Y}_M$. Then

$$\liminf_{s \downarrow 0} \mathcal{C}_s \geq \frac{m-2}{m-1} \left(\frac{\chi - 2\beta}{2} \right)^{\frac{m-1}{m-2}}$$

where $\mathcal{C}_s$ is the Lagrange multiplier of ρ_s.

Lemma

Let $0 \leq \beta < \chi/2$. For any $s \in (0, 1/2)$, let $\rho_s \in \mathcal{Y}_M$ be the unique minimizer of $\mathcal{F}_s$ over $\mathcal{Y}_M$. Then there exists $R \in (0, +\infty)$ and $s_0 \in (0, 1/2)$ such that $\mathrm{supp}(\rho_s) \subset B_R$ for any $s \in (0, s_0)$.

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Lemma

For any $s \in (0, 1/2)$, let $\rho_s \in \mathcal{Y}_M$ be the unique minimizer of $\mathcal{F}_s$ over $\mathcal{Y}_M$. For any vanishing sequence $(s_n) \subset (0, 1/2)$, the sequence (ρ_{s_n}) admits limit points in the strong $L^p(\mathbb{R}^N)$ topology as $n \to +\infty$ for any $p \in [1, +\infty)$.

Lemma

Suppose that $\rho_s \in \mathcal{Y}_M$ for any $s > 0$ and that $\rho \in \mathcal{Y}_M$. If $\rho_s \to \rho$ strongly in $L^2(\mathbb{R}^d)$ as $s \downarrow 0$, then

$$\lim_{s\downarrow 0} \int_{\mathbb{R}^{2N}} c_{d,s}|x-y|^{2s-d}\rho_s(x)\rho_s(y)\,dx\,dy = \int_{\mathbb{R}^N} \rho^2(x)\,dx.$$

Limiting behavior of the solutions to the KS equation

The formal limiting equation as $s \to 0$ to the KS equation

$$\rho_t = \Delta\rho^m + \beta\Delta\rho^2 - \chi\nabla\cdot(\rho\nabla(W_s * \rho)) \tag{2}$$

reads

$$\rho_t = \Delta\rho^m + (\beta - \chi/2)\ \Delta\rho^2, \tag{3}$$

and its behavior is again crucially depending on the sign of the coefficient $\beta - \chi/2$. We only treat the case $\beta \geq \chi/2$, for which the limiting equation becomes a purely diffusive equation. We have the following result

Theorem (HMVV, 2022)

Let $\beta \geq \chi/2$. Let $\rho^0 \in \mathcal{Y}_{M,2}$. Let $(s_n)_{\{n\in\mathbb{N}\}} \subset (0, 1/2)$ be a vanishing sequence, and for every $n \in \mathbb{N}$ let ρ_n be a gradient flow solution to (3) *with $s = s_n$. Then the sequence $(\rho_n)_{n\in\mathbb{N}}$ admits strong $L^2_{loc}((0,+\infty); L^2(\mathbb{R}^N))$ limit points. If ρ is one of such limit points, then $[0,+\infty) \ni t \mapsto \rho(t,\cdot)$ is narrowly continuous with values in $\mathcal{Y}_{M,2}$, $\rho(0,\cdot) = \rho^0$ and ρ is a distributional solution to the nonlinear diffusion equation* (4).

First step: existence of gradient flow solutions

We construct weak solutions to problem

$$\begin{cases} \rho_t = \Delta\rho^m + \beta\Delta\rho^2 - \chi\nabla\cdot(\rho\nabla(W_s * \rho)), \\ \\ \rho(0) = \rho^0, \end{cases}$$

by applying the Jordan-Kinderlehrer-Otto scheme. Therefore, denoting by W_2 the Wasserstein distance of order 2, for a discrete time step $\tau > 0$, we solve the recursive minimization problems

$$\rho_\tau^0 = \rho^0, \qquad \rho_\tau^k \in \operatorname*{argmin}_{\rho\in\mathcal{Y}_M} \left(\mathcal{F}_s[\rho] + \frac{1}{2\tau} W_2^2(\rho, \rho_\tau^{k-1})\right), \quad k \in \mathbb{N},$$

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and we prove that piecewise constant in time interpolations ρ_τ of minimizers do converge to a weak solution to (2) as $\tau \to 0$ along a suitable vanishing sequence $(\tau_n)_{n\in\mathbb{N}}$. A weak solution that is constructed in this way, that is, as a limit of the JKO scheme applied to $\mathcal{F}_s$, will be called a gradient flow solution.

First step: existence of gradient flow solutions

A crucial step for the existence part is the derivation of energy estimates:

- if $\beta > 0$, for every $T > 0$ there holds

$$\frac{4}{m}\int_0^T \int_{\mathbb{R}^N} |\nabla(\rho_\tau(t,x))^{m/2}|^2\,dx\,dt \leq C_1^* + C_2^*(T+\tau) + C_3^*(T+\tau)\,\chi s\left(\frac{\chi(1-s)}{2\beta}\right)^{\frac{1-s}{s}},$$

  where C_i^*, $i = 1,2,3$, are a suitable explicit constants, only depending on χ, M, m, s, d, β, and on ρ^0.

- if $\beta = 0$, let $N \geq 2$, $s \in [1/2, 1)$. Let $T > 0$. Then

$$\int_0^T \int_{\mathbb{R}^N} |\nabla(\rho_\tau(t,x))^{m-1}|^2\,dx\,dt \leq C_1^{**} + (T+\tau)C_2^{**}$$

  where C_1^{**}, C_2^{**} are a suitable explicit constants, only depending on χ, M, m, s, d and the initial datum ρ^0.

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Remark: the red constant is bounded if and only if $\beta \ge \chi/2$.

Second step: properties of gradient flow solutions

Let $\beta \geq 0$. Let $\rho^0 \in \mathcal{Y}_{M,2}$. Let ρ_τ be the piecewise constant interpolation, and let ρ be a limit function obtained as a limit of the JKO scheme. Then

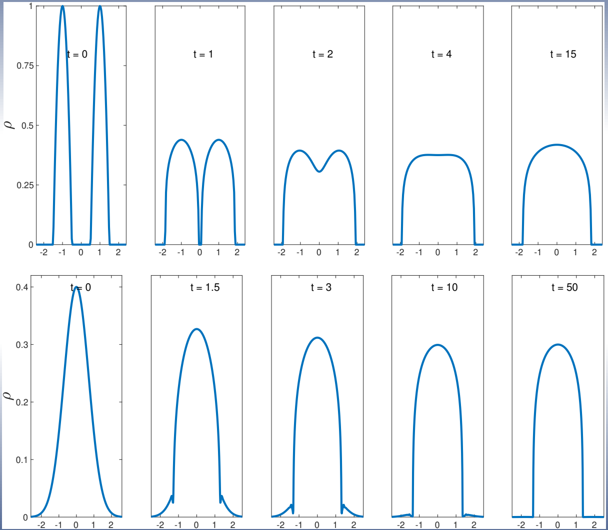
(i) The function $[0,+\infty) \ni t \mapsto \rho(t,\cdot) \in \mathcal{Y}_{M,2}$ is absolutely continuous with respect to the Wasserstein distance W_2.

(ii)
$$\frac{1}{2(m-1)} \int_{\mathbb{R}^d} (\rho(t,x))^m \, dx \leq \frac{1}{m-1} \int_{\mathbb{R}^d} (\rho^0(x))^m \, dx + \beta \int_{\mathbb{R}^d} (\rho^0(x))^2 \, dx + \bar{C},$$

(iii) if $\beta > 0$, then $\rho^{m/2} \in L^2((0,T); H^1(\mathbb{R}^d))$ for every $T > 0$ along with the estimate
$$\frac{4}{m} \int_0^T \int_{\mathbb{R}^d} |\nabla(\rho(t,x))^{m/2}|^2 \, dx \, dt \leq C_1^* + TC_2^* + TC_3^* \chi s \left(\frac{\chi(1-s)}{2\beta} \right)^{\frac{1-s}{s}},$$

(iv) for every $T > 0$ there holds
$$\int_{\mathbb{R}^d} |x|^2 \rho(t,x) \, dx \leq 4T \left(\frac{1}{m-1} \int_{\mathbb{R}^d} (\rho^0(x))^m \, dx + \beta \int_{\mathbb{R}^d} (\rho^0(x))^2 \, dx \right) + 2 \int_{\mathbb{R}^d} |x|^2 \rho^0(x) \, dx.$$

Some simulations



Open problems

- A rigorous proof that every solution to the Cauchy problem associated to the KS equation does converge to the unique stationary state. We mention that a similar result is available in the two dimensional setting, in the case of aggregation with the Newtonian potential instead of the Riesz potential, with $\beta = 0$ and $m > 1$ (i.e., diffusion-dominated regime), see CHVY, 2019:
- show that the family of solutions ρ_s to the Cauchy problem associated to the KS equation converges as $s \to 0$ to a solution (in an appropriate sense) to the equation

$$\rho_t = \Delta \rho^m + (\beta - \chi/2)\ \Delta \rho^2 = \Delta \varphi(\rho),$$

where if $\beta < \chi/2$ the nonlinearity φ is nonmonotone and the equation (39) is of **forward-backward** type.

Thank you for your attention!