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Regularity estimates for nonlocal operators related to nonsymmetric forms

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joint work with Moritz Kassmann

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4. Hölder estimates
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Introduction: Setup

Study weak solutions $u : I \times \mathbb{R}^d \to \mathbb{R}$ to

$$\partial_t u - Lu = 0 \quad \text{in } I \times \Omega =: Q. \tag{PDE}$$

- $I \subset \mathbb{R}$ bounded open interval, $\Omega \subset \mathbb{R}^d$ bounded domain.
- L is a linear, **nonsymmetric**, nonlocal operator of the form

$$-Lu(t,x) := 2\ p.v. \int_{\mathbb{R}^d} (u(t,x) - u(t,y))K(x,y)\mathrm{d}y. \tag{L}$$

  - jumping kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ is measurable,
  - K does NOT necessarily satisfy $K(x,y) = K(y,x)$,
- Example: $K(x,y) = c(x,y)|x-y|^{-d-\alpha}$, $\alpha \in (0,2)$, $c : \mathbb{R}^d \times \mathbb{R}^d \to [\lambda, \Lambda]$, $0 \leq \lambda \leq \Lambda < \infty$.

Goal: Derive Hölder estimates for weak solutions u to (PDE) under certain assumptions on K.

Introduction: Bilinear form

Study weak solutions $u : I \times \mathbb{R}^d \to \mathbb{R}$ to

$$\partial_t u - Lu = 0 \quad \text{in } I \times \Omega =: Q, \tag{PDE}$$

$$Lu(t,x) := 2\ p.v. \int_{\mathbb{R}^d} (u(t,y) - u(t,x))K(x,y)\mathrm{d}y. \tag{L}$$

Via the relation $\mathcal{E}^K(u(t),\phi) := -(Lu(t),\phi)_{L^2(\mathbb{R}^d)}$, associate to L a bilinear form.
Then the **weak formulation** of (PDE) reads:

$$(\partial_t u(t),\phi)_{L^2(\mathbb{R}^d)} + \mathcal{E}^K(u(t),\phi) = 0, \quad \forall t \in I,$$

for every ϕ in a suitable test function space with $\mathrm{supp}(\phi) \subset\subset \Omega$.

$$\mathcal{E}^K(f,g) = 2 \iint_{\mathbb{R}^d\mathbb{R}^d} (f(x) - f(y))g(x)K(x,y)\mathrm{d}y\mathrm{d}x$$
$$= \iint_{\mathbb{R}^d\mathbb{R}^d} (f(x) - f(y))(g(x) - g(y))K_s(x,y)\mathrm{d}y\mathrm{d}x + \iint_{\mathbb{R}^d\mathbb{R}^d} (f(x) - f(y))(g(x) + g(y))K_a(x,y)\mathrm{d}y\mathrm{d}x$$

Introduction: Goal

Study weak solutions $u : I \times \mathbb{R}^d \to \mathbb{R}$ to

$$\partial_t u - Lu = 0 \quad \text{in } I \times \Omega =: Q. \tag{PDE}$$

Goal: Derive Hölder estimates for weak solutions u to (PDE) under certain assumptions on K.

Definition (Hölder regularity **PHR**(α))

We say that **PHR**(α) is satisfied by L (or by K) if there exists $C > 0$ such that for every weak solution u to (PDE) in Q, and every $0 < R \leq 1$, $t_0 > 0$, $x_0 \in \mathbb{R}^d$ with $Q_R^{(\alpha)} := I_R^{(\alpha)}(t_0) \times B_{2R}(x_0) \subset Q$:

$$\sup_{(t,x),(s,y) \in Q_{R/2}} \frac{|u(t,x) - u(s,y)|}{\left(|x-y| + |t-s|^{1/\alpha}\right)^{\gamma}} \leq \frac{\|u\|_{L^\infty(I_R^{(\alpha)}(t_0) \times \mathbb{R}^d)}}{CR^{\gamma}}, \tag{PHR}$$

where $I_R^{(\alpha)}(t_0) = (t_0 - R^\alpha, t_0 + R^\alpha)$ and $\gamma \in (0,1)$ is the Hölder exponent.

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2nd order divergence form operators (symmetric)

Let $A = (A_{j,k})_{j,k=1}^d : \Omega \to \mathbb{R}^{d\times d}$ measurable, s.t. $A(x)$ is a symmetric matrix for every $x \in \Omega$.
Study

$$\partial_t u(t,x) - \operatorname{div}(A(x)\nabla u(t,x)) = 0 \quad \text{in } Q. \tag{PDE$_A$}$$

Assume:

- There is $\Lambda \geq 1$ s.t. for a.e. $x \in \Omega$, every $\xi \in \mathbb{R}^d$:

$$\Lambda^{-1}|\xi|^2 \leq \textstyle\sum_{j,k=1}^d A_{j,k}(x)\xi_j\xi_k \leq \Lambda|\xi|^2. \tag{A_{ell}}$$

Theorem (Hölder regularity (Nash 1957; De Giorgi 1957; Moser 1964))

Let A be as above. Then there is $\gamma \in (0,1)$, $C > 0$ s.t. for every weak solution u to (PDE$_A$) in Q and every $Q_R^{(2)} \subset Q$, $0 < R \leq 1$:

$$\sup_{(t,x),(s,y)\in Q_{R/2}^{(2)}} \frac{|u(t,x) - u(s,y)|}{\left(|x-y| + |t-s|^{1/2}\right)^{\gamma}} \leq \frac{\|u\|_{L^\infty(Q)}}{CR^\gamma}. \tag{PHR}$$

$\Rightarrow$ **PHR**(2) is satisfied for $L = \operatorname{div}(A\cdot\nabla)$.

2nd order divergence form operators (symmetric)

Let $A = (A_{j,k})_{j,k=1}^d : \Omega \to \mathbb{R}^{d \times d}$ measurable, s.t. $A(x)$ is a symmetric matrix for every $x \in \Omega$. Study

$$\partial_t u(t,x) - \mathrm{div}(A(x)\nabla u(t,x)) = 0 \quad \text{in } Q. \qquad (\mathrm{PDE}_A)$$

Assume:

- There is $\Lambda \geq 1$ s.t. for a.e. $x \in \Omega$, every $\xi \in \mathbb{R}^d$:

$$\Lambda^{-1}|\xi|^2 \leq \sum_{j,k=1}^d A_{j,k}(x)\xi_j\xi_k \leq \Lambda|\xi|^2. \qquad (A_{ell})$$

Theorem (Harnack inequality (Moser 1964))

Let A be as above. Then there is $C > 0$ such that for every weak solution u to (PDE_A) in Q, with $u \geq 0$ in Q, and every $Q_R^{(2)} \subset Q$, $0 < R \leq 1$:

$$\sup_{Q_R^{\ominus}} u \leq C \inf_{Q_R^{\oplus}} u \qquad (\mathrm{PHI})$$

$\Rightarrow$ **PHI**(2) is satisfied for $L = \mathrm{div}(A \cdot \nabla)$.

2nd order divergence form operators (symmetric)

Let $A = (A_{j,k})_{j,k=1}^{d} : \Omega \to \mathbb{R}^{d\times d}$ measurable, s.t. $A(x)$ is a symmetric matrix for every $x \in \Omega$. Study

$$\partial_t u(t,x) - \mathrm{div}(A(x)\nabla u(t,x)) = 0 \quad \text{in } Q. \qquad (\mathrm{PDE}_A)$$

- Note that parabolic Harnack (**PHI**(2)) $\Rightarrow$ parabolic Hölder (**PHR**(2)).
- Main ingredients in Moser's proof: Sobolev- and Poincaré inequality

$$\|f^2\|_{L^{\frac{d}{d-2}}(B_R)} \le C\Big(\|\nabla f\|_{L^2(B_R)}^2 + R^{-2}\|f\|_{L^2(B_R)}^2\Big) \qquad (\mathrm{Sob})$$

$$\int_{B_R} (f(x) - [f]_{B_R})^2 \mathrm{d}x \le CR^2 \|\nabla f\|_{L^2(B_R)}^2 \qquad (\mathrm{Poinc})$$

2nd order divergence form operators

Let $A = (A_{j,k})_{j,k=1}^d : \Omega \to \mathbb{R}^{d\times d}$, $b = (b_j)_{j=1}^d, d = (d_j)_{j=1}^d : \Omega \to \mathbb{R}^d$ measurable. Study

$$\partial_t u(t,x) - \operatorname{div}(A(x)\nabla u(t,x)) - \operatorname{div}(b(x)u(t,x)) + d(x)\nabla u(t,x) = 0 \quad \text{in } Q. \qquad (\text{PDE}_{A,b,d})$$

Assume that A satisfies (A_{ell}).

- $|b|^2, |d|^2 \in L^\theta(\Omega)$, $\theta \in (d/2, \infty] \Rightarrow$ elliptic Hölder **EHR**(2) (Morrey 1959)
- $|b|^2, |d|^2 \in L^\theta(\Omega)$, $\theta \in (d/2, \infty] \Rightarrow$ parabolic Harnack **PHI**(2) (Trudinger 1967)
- $|b|^2 \in L^\theta(\Omega), |d|^2 \in L^{d/2}(\Omega) \Rightarrow$ elliptic Harnack **EHI**(2) (Stampacchia 1965)
- $b = 0$, $|d|^2 \in \mathcal{K}^{d,2}(\Omega) \Rightarrow$ elliptic Harnack **EHI**(2) (Kurata 1994)
- $b = 0$, $|d| \in \mathcal{K}^{d,1}(\Omega) \Rightarrow$ parabolic Harnack **PHI**(2) (Zhang 1995)
- ...

2nd order divergence form operators

Let $A = (A_{j,k})_{j,k=1}^{d} : I \times \Omega \to \mathbb{R}^{d\times d}$, $b = (b_j)_{j=1}^{d}, d = (d_j)_{j=1}^{d} : I \times \Omega \to \mathbb{R}^{d}$ measurable. Study

$$\partial_t u(t,x) - \mathrm{div}(A(t,x)\nabla u(t,x)) - \mathrm{div}(b(t,x)u(t,x)) + d(t,x)\nabla u(t,x) = 0 \quad \text{in } Q. \quad (\mathrm{PDE}^{t}_{A,b,d})$$

Assume that A satisfies (A_{ell}).

- $|b|^2, |d|^2 \in L^{\mu,\theta}_{t,x}(\Omega)$, $\frac{d}{2\theta} + \frac{1}{\mu} < 1 \Rightarrow$ parabolic Harnack **PHI**(2) (Aronson-Serrin 1967, Ladyzhenskaya-Solonnikov-Ural'tseva 1968)
- ...

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The fractional Laplacian

Recall: We are interested in Hölder regularity of weak solutions u to

$$\partial_t u - Lu = 0 \quad \text{in} \quad I \times \Omega =: Q, \tag{PDE}$$

where L is of the form

$$Lu(t,x) := 2 \; p.v. \int_{\mathbb{R}^d} (u(t,y) - u(t,x))K(x,y)\mathrm{d}y. \tag{L}$$

Example (Fractional Laplacian)

Consider $L = -(-\Delta)^{\alpha/2}$ for some $\alpha \in (0,2)$,

$$Lf(x) = c_{d,\alpha} \; p.v. \int_{\mathbb{R}^d} (f(y) - f(x))|x-y|^{-d-\alpha}\mathrm{d}y.$$

This corresponds to

$$K(x,y) = K(y,x) := c_{d,\alpha}/2|x-y|^{-d-\alpha}, \quad c_{d,\alpha} \asymp (2-\alpha)\alpha.$$

Example 1: K-comparability (symmetric)

Consider more general symmetric kernels K such that

- There is $\Lambda \geq 1$ such that for a.e. $x, y \in \mathbb{R}^d$:

$$\Lambda^{-1}|x-y|^{-d-\alpha} \leq K(x,y) \leq \Lambda |x-y|^{-d-\alpha}. \qquad (K_{comp})$$

Theorem (Chen, Kumagai 2003)

Let L and K be as above. Then there is $\gamma = \gamma(d, \Lambda, \alpha) \in (0,1)$, $C > 0$ such that for every weak solution u to (PDE) in Q and every $Q_R^{(\alpha)} \subset Q$, $0 < R \leq 1$:

$$\sup_{(t,x),(s,y) \in Q_{R/2}^{(\alpha)}} \frac{|u(t,x) - u(s,y)|}{\left(|x-y| + |t-s|^{1/\alpha}\right)^{\gamma}} \leq \frac{\|u\|_{L^\infty(\mathbb{R}^d)}}{CR^{\gamma}}. \qquad \text{(PHR)}$$

$\Rightarrow$ **PHR**(α) satisfied for $K(x,y) \asymp |x-y|^{-d-\alpha}$.

- They also prove a Harnack inequality and heat kernel bounds (using stochastic process)

Example 2: $\mathcal{E}$-comparability (symmetric)

- **Recall**: We associate to L given by

$$-Lf(x) := 2\ p.v. \int_{\mathbb{R}^d} (f(x) - f(y))K(x,y)\mathrm{d}y \tag{L}$$

the following bilinear form

$$\mathcal{E}^K(f,g) = 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x) - f(y))g(x)K(x,y)\mathrm{d}y\mathrm{d}x.$$

Let K be symmetric. Then one can write

$$\mathcal{E}^K(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x) - f(y))(g(x) - g(y))K(x,y)\mathrm{d}y\mathrm{d}x.$$

For $0 < R \leq 1$, $x_0 \in \mathbb{R}^d$, define:

$$\mathcal{E}^K_{B_R(x_0)}(f,f) = \int_{B_R(x_0)}\int_{B_R(x_0)} (f(x) - f(y))^2 K(x,y)\mathrm{d}y\mathrm{d}x.$$

Example 2: $\mathcal{E}$-comparability (symmetric)

For $0 < R \leq 1$, $x_0 \in \mathbb{R}^d$, define:

$$\mathcal{E}^K_{B_R(x_0)}(f,f) = \int_{B_R(x_0)} \int_{B_R(x_0)} (f(x) - f(y))^2 K(x,y) \mathrm{d}y \mathrm{d}x.$$

Consider symmetric K such that

- There is $\Lambda \geq 1$ such that for every ball $B_R \subset \Omega$, $0 < R \leq 1$, $f \in L^2(B_R)$:

$$\Lambda^{-1}[f]^2_{H^{\alpha/2}(B_R)} \leq \mathcal{E}^K_{B_R}(f,f) \leq \Lambda [f]^2_{H^{\alpha/2}(B_R)} \qquad (\mathcal{E}_{comp})$$

Recall that

$$c_{d,\alpha} \int_{B_R} \int_{B_R} (f(x) - f(y))^2 |x-y|^{-d-\alpha} \mathrm{d}y \mathrm{d}x = [v]^2_{H^{\alpha/2}(B_R)}.$$

- $K(x,y) \asymp |x-y|^{-d-\alpha}$ satisfies $(\mathcal{E}_{comp})$
- $(\mathcal{E}_{comp})$ allows for a much larger class of kernels K. See
  - (Chaker, Silvestre 2020)
  - (Dyda, Kassmann 2020)
  - (Bux, Schulze, Kassmann 2017)
  - Extension to $\mu(x, \mathrm{d}y)$

Example 2: $\mathcal{E}$-comparability (symmetric)

Theorem (Felsinger, Kassmann 2013)

Let K be symmetric and such that for some $\alpha \in (0,2)$, $\Lambda \geq 1$:

- $(\mathcal{E}_{comp})$ *For every ball $B_R \subset \Omega$, $0 < R \leq 1$, $f \in L^2(B_R)$:*

$$\Lambda^{-1}[f]^2_{H^{\alpha/2}(B_R)} \leq \mathcal{E}^K_{B_R}(f,f) \leq \Lambda[f]^2_{H^{\alpha/2}(B_R)} \qquad (\mathcal{E}_{comp})$$

- *(generalized upper bound) For every $\rho > 0$:*

$$\sup_{x\in\Omega} \int_{\mathbb{R}^d \setminus B_\rho(x)} K(x,y)\mathrm{d}y \leq \Lambda\rho^{-\alpha}. \qquad \text{(tail-est)}$$

Then L satisfies $\mathbf{PHR}(\alpha)$*, i.e., weak solutions to $\partial_t u - Lu = 0$ satisfy the Hölder estimate.*

- Note that $\int_{\mathbb{R}^d\setminus B_\rho(x)} |x-y|^{-d-\alpha}\mathrm{d}y \leq c\rho^{-\alpha}$.

Example 2: $\mathcal{E}$-comparability (symmetric)

(Felsinger, Kassmann 2013) relies on nonlocal Moser iteration:

- $(\mathcal{E}_{comp}) + (\text{tail-est}) \Rightarrow$ **weak parabolic Harnack inequality**

Definition (Weak parabolic Harnack inequality (**wPHI**(α)))

There exists $C > 0$ such that for every globally nonnegative weak supersolution to (PDE) in Q and every $Q_R^{(\alpha)} \subset Q$, $0 < R \leq 1$:

$$\frac{1}{|Q_R^{\ominus}|} \int_{Q_R^{\ominus}} u \leq C \inf_{Q_R^{\oplus}} u. \qquad \text{(wPHI)}$$

- **wPHI**(α) + (tail-est) $\Rightarrow$ **PHR**(α).
- symmetry of K: $(f(x) - f(y))(g(x) - g(y))$-shape required for nonlocal "chain rules".

Note that by $(\mathcal{E}_{comp})$:

$$\|f^2\|_{L^{\frac{d}{d-\alpha}}(B_R)} \leq C\left(\mathcal{E}^K_{B_R}(f,f) + R^{-\alpha}\|f\|^2_{L^2(B_R)}\right), \qquad \text{(Sob)}$$

$$\int_{B_R} (f(x) - [f]_{B_R})^2 \mathrm{d}x \leq C R^{\alpha} \mathcal{E}^K_{B_R}(f,f). \qquad \text{(Poinc)}$$

Regularity for (symmetric) nonlocal operators

Further research works based on energy methods for symmetric nonlocal operators:
Hölder estimates and Harnack inequalities (Moser / De Giorgi technique)

- (Caffarelli, Chan, Vasseur 2011), (Di Castro, Kuusi, Palatucci 2014, 2016), (Cozzi 2017), (Strömqvist 2019), (Chaker, Kassmann, W. 2019), (Chen, Kumagai, Wang 2020), (Ding, Zhang, Zhou 2021), (Liao 2022), (Adimurthi, Prasad, Tewary 2022)...

Higher regularity / integrability:

- (Kuusi, Mingione, Sire 2015), (Cozzi 2017), (Brasco, Lindgren, Schikorra 2018), (Mengesha, Schikorra, Yeepo 2020), (Nowak 2021, 2022), ...

Regularity theory via "non energy approaches":

- Abatangelo, Abels, Bonforte, Caffarelli, Chang Lara, De Filippis, Dávila, Dipierro, Endal, Fernández Real, Figalli, Grubb, Hoh, Jacob, Ros-Oton, Schwab, Serra, Silvestre, Valdinoci, Vázquez, ...

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Nonsymmetric bilinear forms

- $K \geq 0$ nonsymmetric, i.e., NOT necessarily $K(x,y) = K(y,x)$.

We decompose $K = K_s + K_a$, where

$$K_s(x,y) = \frac{K(x,y) + K(y,x)}{2}, \quad K_a(x,y) = \frac{K(x,y) - K(y,x)}{2}$$

Note that: $K_s(x,y) = K_s(y,x) \geq 0$, $K_a(x,y) = -K_a(y,x)$.

- **Recall**: We associate to L the following bilinear form

$$\mathcal{E}^K(f,g) = 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x) - f(y))g(x)K(x,y)\mathrm{d}y\mathrm{d}x.$$

Rewrite $\mathcal{E}^K = \mathcal{E}^{K_s} + \mathcal{E}^{K_a}$, where

$$\mathcal{E}^{K_s}(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x) - f(y))(g(x) - g(y))K_s(x,y)\mathrm{d}y\mathrm{d}x,$$

$$\mathcal{E}^{K_a}(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x) - f(y))(g(x) + g(y))K_a(x,y)\mathrm{d}y\mathrm{d}x.$$

Nonsymmetric bilinear forms

Rewrite $\mathcal{E}^K = \mathcal{E}^{K_s} + \mathcal{E}^{K_a}$, where

$$\mathcal{E}^{K_s}(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x)-f(y))(g(x)-g(y))K_s(x,y)\mathrm{d}y\mathrm{d}x,$$

$$\mathcal{E}^{K_a}(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x)-f(y))(g(x)+g(y))K_a(x,y)\mathrm{d}y\mathrm{d}x.$$

Assume that K_a, K_s satisfy

- There exists $\Lambda \geq 1$ such that:

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|K_a(x,y)|^2}{K_s(x,y)}\mathrm{d}y \leq \Lambda. \tag{K1}$$

Then $\mathcal{E}^K$ satisfies a **sector-type condition** and **Gårding's inequality**:

$$\mathcal{E}^K(f,g) \leq \left(2\Lambda\mathcal{E}^{K_s}(f,f)\right)^{1/2}\left(\mathcal{E}^{K_s}(g,g) + \|g\|_{L^2}^2\right)^{1/2},$$

$$\mathcal{E}^K(f,f) \geq \frac{1}{2}\mathcal{E}^{K_s}(f,f) - \frac{\Lambda}{2}\|f\|_{L^2}^2.$$

$\Rightarrow$ Lévy int. on K_s: $(\mathcal{E}^K, \mathcal{F})$ is a reg. lower bounded **semi-Dirichlet form** (Schilling, Wang 2015).

Nonsymmetric bilinear forms

Rewrite $\mathcal{E}^K = \mathcal{E}^{K_s} + \mathcal{E}^{K_a}$, where

$$\mathcal{E}^{K_s}(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x)-f(y))(g(x)-g(y))K_s(x,y)\mathrm{d}y\mathrm{d}x,$$
$$\mathcal{E}^{K_a}(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x)-f(y))(g(x)+g(y))K_a(x,y)\mathrm{d}y\mathrm{d}x.$$

Assume that K_a, K_s satisfy

- There exists $\Lambda \geq 1$ such that:

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|K_a(x,y)|^2}{K_s(x,y)}\mathrm{d}y \leq \Lambda. \tag{K1}$$

$\Rightarrow \mathcal{E}^K(u,\phi)$ is **well-defined** for

- $u \in V^{K_s}(B_R(x_0)|\mathbb{R}^d) = \left\{ u \in L^2(B_R(x_0)) : (u(x)-u(y))K_s^{1/2} \in L^2(B_R(x_0)\times\mathbb{R}^d)\right\}$
- $\phi \in H^{K_s}_{B_R(x_0)}(\mathbb{R}^d) = \{\phi \in L^2(\mathbb{R}^d) : \mathcal{E}^{K_s}(\phi,\phi) < \infty,\ \phi \equiv 0 \text{ in } \mathbb{R}^d \setminus B_R(x_0)\}$

see (Felsinger, Kassmann, Voigt 2015)

Example: nonlocal drift

- There exists $\Lambda \geq 1$ such that:

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|K_a(x,y)|^2}{K_s(x,y)} \mathrm{d}y \leq \Lambda. \tag{K1}$$

Example

Consider $K(x,y) = c(x,y)|x-y|^{-d-\alpha}$. Choose $c(x,y) = c_{d,\alpha}/2[1 + (V(x) - V(y))]$, where

- $V : \mathbb{R}^d \to \mathbb{R}$ with $|V(x) - V(y)| \leq 1$.

$$K_s(x,y) = c_{d,\alpha}/2|x-y|^{-d-\alpha},$$
$$K_a(x,y) = c_{d,\alpha}/2(V(x) - V(y))|x-y|^{-d-\alpha},$$

Sufficient for $(K1)$ to hold: $V \in C^{0,\gamma}(\mathbb{R}^d)$ for some $\gamma > \alpha/2$.

$$-Lf(x) = \int_{\mathbb{R}^d} (f(x) - f(y))|x-y|^{-d-\alpha} \mathrm{d}y + \int_{\mathbb{R}^d} (f(x) - f(y))(V(x) - V(y))|x-y|^{-d-\alpha} \mathrm{d}y$$
$$= (-\Delta)^{\alpha/2} f(x) + 2\Gamma^{(\alpha)}(f,V)(x) \to (-\Delta) f(x) + 2\nabla V(x) \nabla f(x), \quad \text{as } \alpha \to 2.$$

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$\mathcal{E}^{K_s}$-comparability

Assume that K_s satisfies ($\mathcal{E}_{comp}$):

- There is $\Lambda \geq 1$ such that for every ball $B_R \subset \Omega$, $0 < R \leq 1$, $f \in L^2(B_R)$:

$$\Lambda^{-1}[f]^2_{H^{\alpha/2}(B_R)} \leq \mathcal{E}^{K_s}_{B_R}(f,f) \leq \Lambda[f]^2_{H^{\alpha/2}(B_R)} \qquad (\mathcal{E}_{comp})$$

Then Sobolev- and Poincaré inequality hold true for $\mathcal{E}^{K_s}$:

$$\|f^2\|_{L^{\frac{d}{d-\alpha}}(B_R)} \leq C\left(\mathcal{E}^{K_s}_{B_R}(f,f) + R^{-\alpha}\|f\|^2_{L^2(B_R)}\right), \qquad \text{(Sob)}$$

$$\int_{B_R}(f(x) - [f]_{B_R})^2 \mathrm{d}x \leq CR^{\alpha}\mathcal{E}^{K_s}_{B_R}(f,f). \qquad \text{(Poinc)}$$

Moreover, assume that K_s satisfies (tail-est):

- There is $\Lambda \geq 1$ such that for every $\rho > 0$:

$$\sup_{x\in\Omega}\int_{\mathbb{R}^d\setminus B_\rho(x)} K_s(x,y)\mathrm{d}y \leq \Lambda\rho^{-\alpha}. \qquad \text{(tail-est)}$$

generalized (K1)

Recall: There exists $\Lambda \geq 1$ such that:

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|K_a(x,y)|^2}{K_s(x,y)} \mathrm{d}y \leq \Lambda. \tag{K1}$$

Generalized assumption: Let $\theta \in [\frac{d}{\alpha}, \infty]$.

- There is a (symm.) J s.t. for every ball $B_R \subset \Omega$ with $R \leq 1$:

$$\left\| \int_{B_R} \frac{|K_a(\cdot,y)|^2}{J(\cdot,y)} \mathrm{d}y \right\|_{L^\theta(B_R)} \leq \Lambda, \qquad \mathcal{E}^J_{B_R}(v,v) \leq \Lambda [v]^2_{H^{\alpha/2}(B_R)}, \quad \forall v \in L^2(B_R), \tag{K1$_{loc}$}$$

$$\left\| \int_{\mathbb{R}^d} \frac{|K_a(\cdot,y)|^2}{J(\cdot,y)} \mathrm{d}y \right\|_{L^\theta(\mathbb{R}^d)} \leq \Lambda, \qquad \mathcal{E}^J_{B_R}(v,v) \leq \Lambda [v]^2_{H^{\alpha/2}(B_R)}, \quad \forall v \in L^2(B_R). \tag{K1$_{glob}$}$$

Think: $J(x,y) = |x-y|^{-d-\alpha}$ or $J(x,y) = K_s(x,y)$.

Main result I: (PDE)

Study weak solutions $u : I \times \mathbb{R}^d \to \mathbb{R}$ to

$$\partial_t u - Lu = 0 \quad \text{in } I \times \Omega =: Q, \tag{PDE}$$

Theorem (Kassmann, W. 2022)

Let K be such that for some $\alpha \in (0,2)$, $\Lambda \geq 1$, $D < 1$ and $\theta \in [\frac{d}{\alpha}, \infty]$:

- $(K1_{loc})$: *There is a (symm.) J s.t. for every ball $B_R \subset \Omega$ with $R \leq 1$:*

$$\left\| \int_{B_R} \frac{|K_a(\cdot,y)|^2}{J(\cdot,y)} \mathrm{d}y \right\|_{L^\theta(B_R)} \leq \Lambda, \qquad \mathcal{E}^J_{B_R}(v,v) \leq \Lambda [v]^2_{H^{\alpha/2}(B_R)}, \quad \forall v \in L^2(B_R). \tag{$K1_{loc}$}$$

- $(K2)$: *There is a (symm.) j s.t. for every ball $B_R \subset \Omega$ with $R \leq 1$:*

$$K(x,y) \geq (1-D) j(x,y), \quad \forall x,y \in B_R, \qquad [v]^2_{H^{\alpha/2}(B_R)} \leq \Lambda \mathcal{E}^j_{B_R}(v,v), \quad \forall v \in L^2(B_R). \tag{K2}$$

- K_s *satisfies* $(\mathcal{E}_{comp})$, (tail-est).

Then, L satisfies **wPHI**(α), **PHR**(α), *i.e., weak solutions to* (PDE) *satisfy the Hölder estimate.*

$(K1_{loc})$ in the proof of Main result I

$(K1_{loc})$: Let $\theta \in [\frac{d}{\alpha}, \infty]$. For every ball $B_R \subset \Omega$ with $R \leq 1$, every $v \in L^2(B_R)$:

$$\|W\|_{L^\theta(B_R)} := \left\| \int_{B_R} \frac{|K_a(\cdot,y)|^2}{J(\cdot,y)} \mathrm{d}y \right\|_{L^\theta(B_R)} \leq \Lambda, \qquad \mathcal{E}^J_{B_R}(v,v) \leq \Lambda [v]^2_{H^{\alpha/2}(B_R)}. \tag{K1$_{loc}$}$$

- For every $\delta > 0$ there is $C(\delta) > 0$ such that:

$$\int_{B_R} v^2(x) W(x) \mathrm{d}x \leq \delta \mathcal{E}^{K_s}_{B_R}(v,v) + c_1 (C(\delta) + \delta R^{-\alpha}) \|v^2\|_{L^1(B_R)}, \quad \forall v \in L^2(B_R).$$

- If $\theta \in (\frac{d}{\alpha}, \infty]$:

$$C(\delta) = \begin{cases} \|W\|_{L^\infty(B_R)} & , \quad \theta = \infty, \\ \delta^{\frac{d}{d-\theta\alpha}} \|W\|^{\frac{\theta\alpha}{\theta\alpha - d}}_{L^\theta(B_R)} & , \quad \theta \in (\frac{d}{\alpha}, \infty). \end{cases}$$

Think: $v^2(x) = \tau^2 (u+\varepsilon)^{-p+1}(x)$ for some $p \in (0,\infty)$, $\varepsilon > 0$.

Main result II: $(\widehat{\text{PDE}})$

- **Recall**: Weak formulation of (PDE) reads:

$$(\partial_t u(t), \phi)_{L^2(\mathbb{R}^d)} + \mathcal{E}^K(u(t), \phi) = 0, \quad \forall t \in I. \tag{PDE}$$

- Consider the dual problem, i.e.,

$$(\partial_t u(t), \phi)_{L^2(\mathbb{R}^d)} + \widehat{\mathcal{E}^K}(u(t), \phi) = 0, \quad \forall t \in I, \tag{$\widehat{\text{PDE}}$}$$

- **Here**: $\widehat{\mathcal{E}^K}(f,g) = \mathcal{E}^K(g,f)$. Associate to $\widehat{\mathcal{E}^K}$ the operator $\widehat{L}$ via $-(\widehat{L}f, g)_{L^2(\mathbb{R}^d)} = \widehat{\mathcal{E}^K}(f,g)$.

Theorem (Kassmann, W. 2022)

Let K be such that for some $\alpha \in (0,2)$, $\Lambda \geq 1$, $D < 1$ and $\theta \in (\frac{d}{\alpha}, \infty]$:
$(K1_{glob})$, (K2) hold true, and K_s satisfies (tail-est), $(\mathcal{E}_{comp})$.
*Then, $\widehat{L}$ satisfies **wPHI**(α), **PHR**(α), i.e., weak solutions to $\partial_t u - \widehat{L}u = 0$ satisfy Hölder estimate.*

Local analogy: $d\nabla u \leftrightarrow \operatorname{div}(bu)$.

Main result III: time-dependent jumping kernels

Study weak solutions $u : I \times \mathbb{R}^d \to \mathbb{R}$ to

$$\partial_t u - L_t u = 0 \quad \text{in } I \times \Omega =: Q, \tag{PDE^t}$$

$$L_t u(t,x) := 2 \text{ p.v.} \int_{\mathbb{R}^d} (u(t,y) - u(t,x)) k(t;x,y) \mathrm{d}y. \tag{L_t}$$

Assume there is a symmetric $K_s : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$:

$$\Lambda^{-1} K_s(x,y) \le k_s(t;x,y) \le \Lambda K_s(x,y), \quad \forall t \in I, \ x,y \in \mathbb{R}^d.$$

Main result III: time-dependent jumping kernels

Theorem (Kassmann, W. 2022)

Let k be s.t. for some $\alpha \in (0,2)$, $\Lambda \geq 1$, there is a symmetric $K_s : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$:

$$\Lambda^{-1} K_s(x,y) \leq k_s(t;x,y) \leq \Lambda K_s(x,y), \quad \forall t \in I, \ x,y \in \mathbb{R}^d.$$

Assume that for some $D < 1$ and (μ,θ) with $\frac{d}{\alpha\theta} + \frac{1}{\mu} < 1$:

- $(K1^t_{loc})$: *For every interval $I_R \subset I$ and every ball $B_R \subset \Omega$ with $R \leq 1$:*

$$\left\| \int_{B_R} \frac{|k_a(\cdot;\cdot,y)|^2}{J(\cdot,y)} \mathrm{d}y \right\|_{L^{\mu,\theta}_{t,x}(I_R \times B_R)} \leq \Lambda, \qquad \mathcal{E}^J_{B_R}(v,v) \leq \Lambda [v]^2_{H^{\alpha/2}(B_R)}, \quad \forall v \in L^2(B_R). \quad (\mathrm{K}1^t_{loc})$$

- $(K2^t)$: *$k(t)$ satisfies (K2) for every $t \in I_R$.*
- *K_s satisfies $(\mathcal{E}_{comp})$, (tail-est).*

Then, L_t satisfies **wPHI**(α) *and* **PHR**(α).
Moreover, if $(K1^t_{glob})$ holds true, then $\widehat{L}_t$ satisfies **wPHI**(α) *and* **PHR**(α).

Main result IV: Full Harnack inequality

Theorem (Kassmann, W. 2022)

Let K be s.t. for some $\alpha \in (0,2)$, $\Lambda \geq 1$, $D < 1$, and $\theta \in [\frac{d}{\alpha}, \infty]$: $(K1_{loc})$, (K2) *hold true and K_s satisfies* (tail-est), $(\mathcal{E}_{comp})$. *Moreover,*

$$K(x,y) \leq \Lambda |x-y|^{-d-\alpha}, \quad \forall x,y \in \mathbb{R}^d : |x-y| \leq 2. \qquad (K_{\leq})$$

Then L satisfies the **full Harnack inequality**, *i.e., for every nonnegative weak solution to* (PDE) *in Q and every $Q_R^{(\alpha)} \subset Q$, $0 < R \leq 1$:*

$$\sup_{Q_R^{\ominus}} u \leq C \inf_{Q_R^{\oplus}} u + C \sup_{t \in I_R^{\ominus}} \mathrm{Tail}(u(t), R).$$

Main ingredient: $\mathbf{L}^{\infty} - \mathbf{L}^{\mathbf{p}}$**-estimate** for L (where $0 < p < 2$):

$$\sup_{Q_{R/2}^{\ominus}} u \leq c \left(\frac{1}{|Q_R^{\ominus}|} \int_{Q_R^{\ominus}} u^p \right)^{1/p} + \sup_{t \in I_{R/2}^{\ominus}} \mathrm{Tail}(u(t), R) \qquad (L^{\infty} - L^p)$$

Analogous results hold true for $\widehat{L}$, and also for time-dependent jumping kernels.

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1. Introduction
2. Classical examples
   - Local operators
   - Nonlocal operators
3. Nonsymmetric nonlocal operators
4. Hölder estimates
5. Heat kernel estimates

Cauchy problem

- Recall

$$-Lu(t,x) := 2\ p.v. \int_{\mathbb{R}^d} (u(t,x) - u(t,y))K(x,y)\mathrm{d}y. \tag{L}$$

Let $u : (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be the solution to the Cauchy problem

$$\begin{cases} \partial_t u - Lu &= 0, \quad \text{in } (0,\infty) \times \mathbb{R}^d, \\ u(0) &= f, \quad \text{in } \mathbb{R}^d, \quad f \in L^2(\mathbb{R}^d). \end{cases} \tag{CP}$$

If the fundamental solution (heat kernel) $(t,x) \mapsto p(t,x,y)$ to (CP) exists:

$$u(t,x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)\mathrm{d}y =: P_t f(x).$$

Lemma (Blumental, Getoor 1960)

If $L = -(-\Delta)^{\alpha/2}$, then $p(t,x,y) = \left(e^{-t|\cdot|^{\alpha}}\right)^{\vee}(x-y) \asymp t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}$.

Symmetric case

Theorem (Chen, Kumagai 2003)

Let K be a symmetric *jumping kernel such that for some $\alpha \in (0,2)$, $\Lambda \geq 1$:*

$$\Lambda^{-1}|x-y|^{-d-\alpha} \leq K(x,y) \leq \Lambda|x-y|^{-d-\alpha}. \qquad (K_{comp})$$

Then, there are $c_1, c_2 > 0$ s.t. for the fundamental solution to (CP) it holds:

$$c_1 \left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \leq p(t,x,y) \leq c_2 \left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}} \right), \quad \forall t > 0, \; x,y \in \mathbb{R}^d.$$

Further results:

- (Chen, Kumagai 2008), (Chen, Kumagai, Wang 2021), (Chen, Kim, Kumagai, Wang 2021)
- (Barlow, Grigoryan, Kumagai 2009), (Grigoryan, Hu, Hu 2017, 2018), (Hu, Li 2018), ...
- (Kassmann, W. 2021)
- Bae, Bass, Bogdan, Cho, Grzywny, Jin, Kang, Kim, Knopova, Kochubei, Kulczycki, Kulik, Lee, Levin, Minecki, Ryznar, Schilling, Song, Szczypkowski, Sztonyk, Vondraček, Zhang, ...

Nonsymmetric case

Assume: For some $\Lambda \geq 1$ and $\theta \in (\frac{d}{\alpha}, \infty]$:

- There is a (symm.) J s.t. for every ball $B \subset \mathbb{R}^d$:

$$\left\| \int_{\mathbb{R}^d} \frac{|K_a(\cdot,y)|^2}{J(\cdot,y)} \mathrm{d}y \right\|_{L^\theta(\mathbb{R}^d)} \leq \Lambda, \qquad \mathcal{E}_B^J(v,v) \leq \Lambda \mathcal{E}_B^{K_s}(v,v), \quad \forall v \in L^2(B). \qquad (\mathrm{K1}_{glob})$$

- For every ball $B \subset \mathbb{R}^d$:

$$\Lambda^{-1}[v]^2_{H^{\alpha/2}(B)} \leq \mathcal{E}_B^{K_s}(v,v), \quad \forall v \in L^2(B). \qquad (\mathcal{E}_{\geq})$$

$\Rightarrow$ $(\mathcal{E}^K, V^{K_s}(\mathbb{R}^d|\mathbb{R}^d))$ is a regular lower bounded semi-Dirichlet form.
$\Rightarrow$ heat semigroup (P_t) exists.

- (P_t) is **not** symmetric. $(P_t f, g)_{L^2(\mathbb{R}^d)} = (\widehat{P}_t g, f)_{L^2(\mathbb{R}^d)}$, where $(\widehat{P}_t)$ is the dual semigroup.

Nonsymmetric case

$\Rightarrow (\mathcal{E}^K, V^{K_s}(\mathbb{R}^d|\mathbb{R}^d))$ is a regular lower bounded semi-Dirichlet form.
$\Rightarrow$ heat semigroup (P_t) exists.

- (P_t) is **not** symmetric. $(P_t f, g)_{L^2(\mathbb{R}^d)} = (\widehat{P}_t g, f)_{L^2(\mathbb{R}^d)}$, where $(\widehat{P}_t)$ is the dual semigroup.

- $(t,x) \mapsto P_t f(x)$ solves (CP)

$$\begin{cases} \partial_t u - Lu &= 0, \\ u(0) &= f. \end{cases} \qquad \text{(CP)}$$

- $P_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ bounded,
- $P_{t+s} f = P_t P_s f$,
- $f \geq 0 \Rightarrow P_t f \geq 0$,
- $0 \leq f \leq 1 \Rightarrow 0 \leq P_t f \leq 1$

- $(t,x) \mapsto \widehat{P}_t f(x)$ solves $(\widehat{\text{CP}})$

$$\begin{cases} \partial_t u - \widehat{L}u &= 0, \\ u(0) &= f. \end{cases} \qquad (\widehat{\text{CP}})$$

- $\widehat{P}_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ bounded,
- $\widehat{P}_{t+s} f = \widehat{P}_t \widehat{P}_s f$,
- $f \geq 0 \Rightarrow \widehat{P}_t f \geq 0$,
- $0 \leq f \leq 1 \not\Rightarrow 0 \leq \widehat{P}_t f \leq 1$

Nonsymmetric case: Lower heat kernel estimate

Theorem (W. 2022+)

Let $T > 0$. Let K be such that for some $\alpha \in (0,2)$, $\Lambda \geq 1$, and $\theta \in (\frac{d}{\alpha}, \infty]$:

- $(K1_{glob})$*: There is a (symm.) J s.t. for every ball $B \subset \mathbb{R}^d$:*

$$\left\| \int_{\mathbb{R}^d} \frac{|K_a(\cdot,y)|^2}{J(\cdot,y)} \mathrm{d}y \right\|_{L^{\theta}(\mathbb{R}^d)} \leq \Lambda, \qquad \mathcal{E}_B^J(v,v) \leq \Lambda \mathcal{E}_B^{K_s}(v,v), \quad \forall v \in L^2(B). \qquad (K1_{glob})$$

- $(K_{\asymp})$*: For every $x, y \in \mathbb{R}^d$:*

$$\Lambda^{-1}|x-y|^{-d-\alpha} \leq K(x,y) \leq \Lambda |x-y|^{-d-\alpha}. \qquad (K_{\asymp})$$

Then, there exists $c > 0$ s.t. for the fundamental solution to (CP) *it holds*

$$p(t,x,y) \geq c \left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}} \right), \quad \forall t \in (0,T), \quad x,y \in \mathbb{R}^d.$$

Nonsymmetric case: Lower heat kernel estimate

Step 1: Near-diagonal lower bound

$$p(t,x,y) \geq ct^{-\frac{d}{\alpha}}, \quad t \in (0,T), \quad |x-y| \leq ct^{\frac{1}{\alpha}}.$$

- Main ingredient: (**wPHI**(α)) for L and $\widehat{L}$. Apply to

$$u(s,x) = \begin{cases} 1, & s \leq c_1 t, \\ P_{s-c_1 t} \mathbb{1}_{B_{c_2 t^{1/\alpha}}}, & s \geq c_1 t. \end{cases}$$

Step 2: Off-diagonal lower bound

$$p(t,x,y) \geq c \frac{t}{|x-y|^{d+\alpha}}, \quad t \in (0,T), \quad |x-y| \geq ct^{\frac{1}{\alpha}}.$$

- Adapt method developed in (Grigoryan, Hu, Hu 2017, 2018) to nonsymmetric operators.
- Main ingredients: (**wPHI**(α)) and parabolic maximum principle for L and $\widehat{L}$.

Nonsymmetric case: Upper heat kernel estimate

Theorem (W. 2022+)

Let $T > 0$. Let K be such that for some $\alpha \in (0,2)$, $\Lambda \geq 1$, and $\theta \in (\frac{d}{\alpha}, \infty]$:

- $(K1_{glob})$: *There is a (symm.) J s.t. for every ball $B \subset \mathbb{R}^d$:*

$$\left\| \int_{\mathbb{R}^d} \frac{|K_a(\cdot,y)|^2}{J(\cdot,y)} \mathrm{d}y \right\|_{L^\theta(\mathbb{R}^d)} \leq \Lambda, \qquad \mathcal{E}_B^J(v,v) \leq \Lambda \mathcal{E}_B^{K_s}(v,v), \quad \forall v \in L^2(B). \tag{K1$_{glob}$}$$

- $(K_\leq)$: *For every $x, y \in \mathbb{R}^d$:*

$$K(x,y) \leq \Lambda |x-y|^{-d-\alpha}. \tag{$K_\leq$}$$

- *For every ball $B \subset \mathbb{R}^d$:*

$$\Lambda^{-1}[v]^2_{H^{\alpha/2}(B)} \leq \mathcal{E}_B^{K_s}(v,v), \quad \forall v \in L^2(B) \tag{$\mathcal{E}_\geq$}$$

Then, there exists $c > 0$ s.t. for the fundamental solution to (CP) it holds

$$p(t,x,y) \leq c \left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}} \right), \quad \forall t \in (0,T), \quad |x-y| \leq cT^{1/\alpha}.$$

Nonsymmetric case: Upper heat kernel estimate

Step 1: On-diagonal upper bound

$$p(t,x,y) \le ct^{-\frac{d}{\alpha}}, \quad t \in (0,T), \quad x,y \in \mathbb{R}^d.$$

- Main ingredient: $\mathbf{L}^\infty - \mathbf{L}^2$**-estimate** for L and $\widehat{L}$ (Kassmann, W. 2022):

$$\sup_{Q^{\ominus}_{R/2}} u \le c \left(\frac{1}{|Q^{\ominus}_R|} \int_{Q^{\ominus}_R} u^2 \right)^{1/2} + \sup_{t \in I^{\ominus}_{R/2}} \mathrm{Tail}(u(t),R) \qquad (L^\infty - L^2)$$

  for every weak subsolution to (PDE), respectively $(\widehat{\text{PDE}})$ in Q, where $Q_R \subset Q$.
- Prove that for every $f \in L^2(\mathbb{R}^d)$ and $t \in (0,T)$:

$$\|P_t f\|_{L^\infty(\mathbb{R}^d)} \le ct^{-\frac{d}{2\alpha}} \|f\|_{L^2(\mathbb{R}^d)}, \quad \|\widehat{P}_t f\|_{L^\infty(\mathbb{R}^d)} \le ct^{-\frac{d}{2\alpha}} \|f\|_{L^2(\mathbb{R}^d)}.$$

$$\Rightarrow p(t,x,y) = \int_{\mathbb{R}^d} p(t/2,x,z)p(t/2,z,y)\mathrm{d}x \le \|p(t/2,x,\cdot)\|_{L^2(\mathbb{R}^d)} \|p(t/2,\cdot,y)\|_{L^2(\mathbb{R}^d)} \le ct^{-\frac{d}{\alpha}}$$

Nonsymmetric case: Upper heat kernel estimate

Step 2: Off-diagonal upper bound

$$p(t,x,y) \leq c\left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\right), \quad t \in (0,T), \quad ct^{1/\alpha} \leq |x-y| \leq cT^{1/\alpha}.$$

- Adapt **nonlocal Aronson method** from (Kassmann, W. 2021) to nonsymmetric operators.
- Key ingredient: $(L^\infty - L^2)$ for L^ρ and $\widehat{L}^\rho$, where

$$-L^\rho u(t,x) := 2 \; p.v. \int_{B_\rho(x)} (u(t,x) - u(t,y))K(x,y)\mathrm{d}y.$$

- Difficulty: Lack of Markovianity for dual semigroups $(\widehat{P}_t)$, $(\widehat{P}_t^\rho)$. Can prove: For every $\rho > 0$

$$\widehat{P}_t^\rho \mathbb{1}(x) = \int_{\mathbb{R}^d} \widehat{p}^\rho(t,x,y)\mathrm{d}y = \int_{\mathbb{R}^d} p^\rho(t,y,x)\mathrm{d}y \leq C\rho^d, \qquad 0 < t \leq c\rho^\alpha, \quad x \in \mathbb{R}^d,$$

where $C = C(d,\alpha,T) > 0$.

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Approximation: nonlocal to local

Lemma

Let $(K^{(\alpha)})_\alpha$ *be such that* $(K1)$ *holds true and* $(K_s^{(\alpha)})_\alpha$ *satisfies*

- *There exists* $\Lambda \geq 1$ *such that for every* $\alpha \in (0,2)$, *a.e.* $x, y \in \mathbb{R}^d$:

$$\Lambda^{-1}(2-\alpha)|x-y|^{-d-\alpha} \leq K_s^{(\alpha)}(x,y) \leq \Lambda(2-\alpha)|x-y|^{-d-\alpha}.$$

Then for $f, g \in H^1(\mathbb{R}^d)$ *it holds as* $\alpha \nearrow 2$:

$$\mathcal{E}^{K_s^{(\alpha)}}(f,g) \to \int_{\mathbb{R}^d} (\nabla f(x), A(x)\nabla g(x))\mathrm{d}x \qquad \mathcal{E}^{K_a^{(\alpha)}}(f,g) \to 2\int_{\mathbb{R}^d} (d(x), \nabla f(x))g(x)\mathrm{d}x,$$

with $A = (A_{j,k})_{j,k=1}^d : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $d = (d_j)_{j=1}^d : \mathbb{R}^d \to \mathbb{R}^d$ *given as*

$$A_{j,k}(x) = \lim_{\alpha \nearrow 2} \int_{B_\delta(0)} (-h_j)(-h_k)K_s^{(\alpha)}(x, x+h)\mathrm{d}h, \qquad d_j(x) = \lim_{\alpha \nearrow 2} \int_{B_\delta(0)} (-h_j)K_a^{(\alpha)}(x, x+h)\mathrm{d}h.$$

Further examples

Let $0 < \lambda \leq \Lambda < \infty$, $0 < \beta < \alpha/2 < \alpha < 2$.

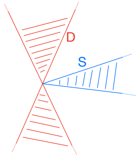
- $V, W \in W^{1,2\theta}(\mathbb{R}^d)$, where $\theta \in [\frac{d}{\alpha}, \infty]$,
- $c : \mathbb{R}^d \times \mathbb{R}^d \to [\lambda, \Lambda]$ symmetric, i.e., $c(x, y) = c(y, x)$,
- $L \in [0, \infty]$ (suitably chosen),
- $D \subset \mathbb{R}^d$ double cone centered at $0 \in \mathbb{R}^d$, $S \subset \mathbb{R}^d$ single cone centered at $0 \in \mathbb{R}^d$.

Consider

(1) $K(x, y) = c(x, y)|x - y|^{-d-\alpha} + (V(x) - V(y))\mathbb{1}_{\{|x-y|\leq L\}}|x - y|^{-d-\alpha}$,

(2) $K(x, y) = V(x)W(y)c(x, y)|x - y|^{-d-\alpha}$ if $V(x)W(y) \in [\lambda, \Lambda]$,

(3) $K(x, y) = \mathbb{1}_D(x - y)|x - y|^{-d-\alpha} + \mathbb{1}_S(x - y)|x - y|^{-d-\beta}$.



Examples (1), (2), (3) all satisfy the Hölder estimate **PHR**(α).