Nonlocal phenomena in local problems

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Regularity for nonlinear diffusion equations.
Green functions and functional inequalities

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Problem 1: maximal regularity in nonlinear mixed local and nonlocal problems

$$\mathcal{L}(w) := \int_{\Omega} [|Dw|^p - fw] \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^\gamma}{|x - y|^{n+s\gamma}} \, dx \, dy$$

$$p, \gamma \in (1, \infty), \qquad s \in (0, 1)$$

- What is the maximal regularity expected for minima of $\mathcal{L}(\cdot)$?
- Best conditions on f and on $(p; s, \gamma)$?.

$\rightsquigarrow$ D. & Mingione *Preprint* (2022). arXiv:2204.06590

Mixed problems: the linear case

$$-\Delta u + (-\Delta)^s u = f$$

- Foondun *Electron. J. Probab.* (2009). $\leftsquigarrow$ Heat kernel estimates and Harnack inequality.
- Chen & Kim & Song & Vondraček *Trans. AMS* (2012); *Illinois J. Math.* (2010). $\leftsquigarrow$ Boundary Harnack principle and Green functions.
- Biagi & Dipierro & Valdinoci & Vecchi *Comm. PDE* (2021); *Proc. Royal Soc. Edinburgh A: Mathematics* (2021); *Preprint* (2022). $\leftsquigarrow$ Regularity, maximum principle and Faber-Krahn inequality.
- Dipierro & Proietti Lippi & Valdinoci *Preprint* (2021). $\leftsquigarrow$ Logistic equations.
- Biswas & Modasya & Sen *Preprint* (2022). $\leftsquigarrow$ Global regularity for viscosity inequalities and overdetermined mixed problems.
- dos Santos & Oliva & Rossi *Computational and Applied Mathematics* (2022). $\leftsquigarrow$ Existence, uniqueness and numerical approximation for mixed local and nonlocal diffusion problems.

Mixed problems: the semilinear case

$$-\Delta u + \mathcal{I}[u] = f$$

- Caffarelli & Patrizi & Quitalo *JEMS* (2017). $\leftsquigarrow$ Long range segregation models.
- Soave & Tavares & Terracini & Zilio *ARMA* (2018). $\leftsquigarrow$ Problems with long range interaction in variational form, employs approximating functionals with mixed local and nonlocal structure to overcome infinite dimensional constraint.
- Abatangelo & Cozzi *SIAM J. Math. Anal.* (2021). $\leftsquigarrow$ Existence and uniqueneness for solutions to problems governed by the sum of a Laplacian and a power of the s-Laplacian.

Mixed problems: the quasilinear case

$$-\Delta_p u + (-\Delta_p)^s u = f$$

- Del Pezzo & Ferreira & Rossi *Fractional Calculus and Applied Analysis* (2019). $\looparrowleft$ Regularity and spectral analysis for nonlinear mixed local and nonlocal problems with compactly supported kernels.
- Da Silva & Salort *Z. Ang. Math. Physik* (2020). $\looparrowleft$ Asymptotic analysis for the limiting profile as $p \to \infty$.
- Garain & Kinnunen *Trans. AMS* to appear; *Preprint* (2021); *Preprint* (2021). $\looparrowleft$ Hölder continuity for weak solutions in both elliptic and parabolic cases. - De Giorgi-Nash-Moser Theory.
- Salort & Vecchi *Preprint* (2021). $\looparrowleft$ Mixed local and nonlocal Hénon equation.

Mixed problems: the quasilinear case

- D. & Mingione *Anal. Math. Phys.* (2021). ↜ Interpolative results for (p,q)-functionals via mixed local and nonlocal approximation.
- Fang & Shang & Zhang *J. Geom. Anal.* (2022). ↜ Hölder continuity for weak solutions to parabolic equations.
- Biagi & Dipierro & Valdinoci & Vecchi *Mathematics in Engineering* (2022). ↜ Hong-Krahn-Szegö inequality.
- D. & Mingione *Preprint* (2022). ↜ Optimal $C^{1,\beta}$-regularity for solutions. Covers also the case $-\Delta_p + (-\Delta_\gamma)^s$ with $\gamma \neq p$. - Maximal regularity.

Mixed problems: the fully nonlinear case

$$-\lambda u + H(x, u, Du) - \Delta u - \mathcal{I}[u] = f$$

- Lenhart, *Appl. Math. Optim.* (1982). $\leftsquigarrow$ Diffusion processes with jumps.
- Caffarelli & Silvestre *CPAM* (2009). $\leftsquigarrow$ Fully nonlinear integro-differential equations.
- Barles & Imbert *Ann. IHP-AN* (2008). $\leftsquigarrow$ Viscosity setting for integro-differential equations.
- Barles & Chasseigne & Imbert *JEMS* (2011). $\leftsquigarrow$ Elliptic mixed integro-differential equations.
- Bjorland & Caffarelli & Figalli *Adv. in Math.* (2012). $\leftsquigarrow$ Nonlocal gradient dependent operators.
- dos Prazeres & Topp *J. Diff. Equations* (2021). $\leftsquigarrow$ Fractional elliptic equations that degenerate with the gradient
- Jakobsen & Rutkowski *Forthcoming*. $\leftsquigarrow$ Mixed mean field games.

Regularity for linear mixed equations

Theorem (Biagi & Dipierro & Valdinoci & Vecchi *Comm. PDE* (2021))

Let $\Omega \subset \mathbb{R}^n$ be a strictly convex domain and $f \in C^\infty(\Omega) \cap L^\infty(\Omega)$ be a function. Then there exists a unique solution $u \in C^2(\Omega) \cap C(\mathbb{R}^n)$ of problem

$$\begin{cases} -\Delta u + (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Nonlinear mixed equations: first regularity results

Elliptic mixed PDE

Theorem (Garain & Kinunnen *Trans. AMS* to appear)

Let u be a weak solution to $-\Delta_p u + (-\Delta_p)^s u = 0$ in Ω. Then $u \in C^{0,\beta}_{\mathrm{loc}}(\Omega)$.

- Di Castro & Kuusi & Palatucci *Ann. IHP-AN* (2016).

Parabolic mixed PDE

Theorem (Fang & Shang & Zhang *J. Geom. Anal.* (2022); Garain & Kinnunen *Preprint* (2021); *Preprint* (2021))

Let u be a weak solution to $\partial_t u - \Delta_p u + (-\Delta_p)^s u = 0$ in $\Omega \times (0, T)$. Then $u \in C^{0,\beta}_{\mathrm{loc}}(\Omega \times (0, T))$.

- Brasco & Lindgren & Strömqvist *J. Evol. Equ.* (2021).

That's all?

The two aforementioned papers adapt the techniques of **Di Castro & Kuusi & Palatucci** *Ann. IHP-AN (2016) (elliptic PDE) and of* **Brasco & Lindgren & Strömqvist** *J. Evol. Equ. (2021) (parabolic PDE) to the mixed setting.*

Let's try to scale the operator. For $B_\varrho(x_0) \Subset \Omega$, set $u_\varrho(x) := u(x_0 + \varrho x)$. If u is a local minimizer of

$$w \mapsto \int_{B_\varrho(x_0)} |Dw|^p \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^\gamma}{|x - y|^{n+s\gamma}} \, dx \, dy,$$

then u_ϱ is a local minimizer of

$$w \mapsto \int_{B_1(0)} |Dw|^p \, dx + \varrho^{p-s\gamma} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^\gamma}{|x - y|^{n+s\gamma}} \, dx \, dy.$$

If $p > s\gamma$, this scaling argument suggests that the (s, γ)-Laplacian behaves as bulky **lower order term**.

Optimal gradient $C^{1,\beta}$-regularity

Theorem (D. & Mingione *Preprint* (2022))

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain, and let u be a weak solution to

$$-\Delta_p u + (-\Delta_\gamma)^s u = f \qquad \text{in } \ \Omega,$$

with

$$p \in (1, \infty), \qquad s \in (0,1), \qquad p > s\gamma$$

and $f \in L^d_{\mathrm{loc}}(\Omega)$ for some $d > n$. Assume that

$$u \in L^\infty_{\mathrm{loc}}(\Omega) \qquad \text{only if } \ \gamma > p.$$

Then $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega)$ for some $\beta \in (0,1)$.

- Ural'tseva *Semin. in Mathematics, V. A. Steklov Math. Inst., Leningrad* (1968).

Global Hölder continuity

Theorem (D. & Mingione *Preprint* (2022))

Let $\Omega \subset \mathbb{R}^n$ *be an open, bounded domain with* C^{1,α_0}*-regular boundary and* u *be the solution of Dirichlet problem*

$$\begin{cases} -\Delta_p u + (-\Delta_\gamma)^s u = f & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

with

$$p \in (1, \infty), \qquad s \in (0,1), \qquad p > s\gamma,$$

and

$$f \in L^n(\Omega), \qquad g \in W^{1,q}(\Omega) \cap W^{a,\chi}(\mathbb{R}^n),$$

for some

$$q > p, \qquad a > s, \qquad \chi > \gamma, \qquad \min\{1 - n/q, a - n/\chi\} > 0.$$

Then $u \in C^{0,\beta}(\Omega)$ *for all* $\beta \in (0, \min\{1 - n/q, a - n/\chi\})$*.*

Evolution of the tail

$$\mathtt{tail}(\varrho) := \left(\varrho^{s\gamma} \int_{\mathbb{R}^n \setminus B_\varrho(x_0)} \frac{|u(x)|^{\gamma-1}}{|x_0 - y|^{n+s\gamma}}\, dy\right)^{1/(\gamma-1)}$$

- Di Castro & Kuusi & Palatucci *Ann. IHP-AN* (2016).

$$\downarrow$$

$$\mathtt{tail}(\varrho) := \left(\varrho^{s\gamma} \int_{\mathbb{R}^n \setminus B_\varrho(x_0)} \frac{|u(x) - (u)_{B_\varrho(x_0)}|^{\gamma-1}}{|x_0 - y|^{n+s\gamma}}\, dy\right)^{1/(\gamma-1)}$$

- Kuusi & Mingione & Sire *Comm. Math. Phys.* (2015).

$$\downarrow$$

$$\mathtt{snail}(\varrho) := \left(\varrho^{s\gamma} \int_{\mathbb{R}^n \setminus B_\varrho(x_0)} \frac{|u(x)|^{\gamma}}{|x_0 - y|^{n+s\gamma}}\, dy\right)^{1/\gamma}$$

- Brasco & Lindgren *Adv. Math.* (2017).

The δ-snail

$$\mathtt{snail}_\delta(\varrho) := \left(\varrho^\delta \int_{\mathbb{R}^n \setminus B_\varrho(x_0)} \frac{|u(y) - (u)_{B_\varrho(x_0)}|^\gamma}{|x - x_0|^{n+s\gamma}}\,dy\right)^{1/\gamma}$$

$$s\gamma \le \delta < p$$

Lemma (D. & Mingione *Preprint* (2022); Kuusi & Mingione & Sire *Comm. Math. Phys.* (2015))

Let $B_t(x_0) \subset B_\varrho(x_0) \subseteq B_1(x_0)$, $\gamma \ge 1$, $\delta \ge s\gamma$ and $u \in W^{s,\gamma}(\mathbb{R}^n)$. Then,

$$\begin{aligned}\mathtt{snail}_\delta(t) \lesssim \left(\frac{t}{\varrho}\right)^{\delta/\gamma} \mathtt{snail}_\delta(\varrho) \\ &+ t^{\delta/\gamma - s} \int_t^\varrho \left(\frac{t}{\nu}\right)^s \left(\fint_{B_\nu(x_0)} |u - (u)_{B_\nu(x_0)}|^\gamma\,dx\right)^{1/\gamma} \frac{d\nu}{\nu} \\ &+ t^{\delta/\gamma - s}\left(\frac{t}{\varrho}\right)^s \left(\fint_{B_\varrho(x_0)} |u - (u)_{B_\varrho(x_0)}|^\gamma\,dx\right)^{1/\gamma}.\end{aligned}$$

- Brasco & Lindgren *Adv. Math.* (2017).

Fractional embedding theorem

Lemma

Let $1 \leq \gamma \leq p < \infty$, $s \in (0,1)$ *and* $B_\varrho(x_0) \subset \mathbb{R}^n$ *be a ball. If* $w \in W_0^{1,p}(B_\varrho(x_0))$, *then* $w \in W^{s,\gamma}(B_\varrho(x_0))$ *and*

$$\left(\int_{B_\varrho(x_0)} \fint_{B_\varrho(x_0)} \frac{|w(x) - w(y)|^\gamma}{|x-y|^{n+s\gamma}}\, dx\, dy \right)^{1/\gamma} \lesssim \varrho^{1-s} \left(\fint_{B_\varrho(x_0)} |Dw|^p\, dx \right)^{1/p}.$$

- Adams & Fournier *Pure and Applied Mathematics series, Academic Press, Elsevier Science* (2003).
- Di Nezza & Palatucci & Valdinoci *Bull. Sci. Math.* (2012).

Interpolation inequality

Lemma

Let $1 < p < \gamma \le p/s$ *and* $s \in (0,1)$, $B_\varrho(x_0) \subset \mathbb{R}^n$. *If* $w \in W^{1,p}_0(B_\varrho(x_0)) \cap L^\infty(B_\varrho(x_0))$, *then* $w \in W^{s,\gamma}(B_\varrho(x_0))$ *and*

$$\left(\int_{B_\varrho(x_0)} \fint_{B_\varrho(x_0)} \frac{|w(x) - w(y)|^\gamma}{|x-y|^{n+s\gamma}}\, dx\, dy \right)^{1/\gamma} \lesssim \|w\|^{1-s}_{L^\infty(B_\varrho(x_0))} \left(\fint_{B_\varrho(x_0)} |Dw|^p\, dx \right)^{s/p}.$$

- Brasco & Parini & Squassina *Discr. Contin. Dyn. Syst.* (2016).
- Brezis & Mironescu *J. Evol. Equ.* (2001).

Main steps of the proof

Step 1: Caccioppoli inequality.

$$\begin{aligned}
&\fint_{B_{\varrho/2}(x_0)} |Du|^p\,dx + \int_{B_{\varrho/2}(x_0)}\fint_{B_{\varrho/2}(x_0)} \frac{|u(x)-u(y)|^\gamma}{|x-y|^{n+s\gamma}}\,dx\,dy \\
&\lesssim \varrho^{-p}\fint_{B_\varrho(x_0)} |u-(u)_{B_\varrho(x_0)}|^p\,dx + \|f\|_{L^n(B_\varrho(x_0))}^{\frac{p}{p-1}} \\
&\quad + \varrho^{-s\gamma}\fint_{B_\varrho(x_0)} |u-(u)_{B_\varrho(x_0)}|^\gamma\,dx + \varrho^{-\delta}\,\mathtt{snail}_\delta(\varrho)^\gamma
\end{aligned}$$

Step 2: Comparison.

$$h \mapsto \min_{w\in u+W_0^{1,p}(B_{\varrho/4}(x_0))} \int_{B_{\varrho/4}(x_0)} |Dw|^p\,dx$$

By standard energy estimates and maximum principle it is

$$\fint_{B_{\varrho/4}(x_0)} |Dh|^p\,dx \lesssim \fint_{B_{\varrho/4}(x_0)} |Du|^p\,dx \quad \text{and} \quad \|h\|_{L^\infty(B_{\varrho/4}(x_0))} \le \|u\|_{L^\infty(B_\varrho(x_0))}.$$

The L^∞-information will be used only when $\gamma > p$.

Main steps of the proof

It holds that

$$\fint_{B_{\varrho/4}(x_0)} |u-h|^p\,dx \lesssim \varrho^{\sigma\theta}\left(\fint_{B_\varrho(x_0)} |u-(u)_{B_\varrho(x_0)}|^p\,dx + \mathtt{snail}_\delta(\varrho)^\gamma\right) + \varrho^{p-\theta}\left(\|f\|^{p/(p-1)}_{L^n(B_\varrho(x_0))}+1\right),$$

for all $\theta \in (0,1)$.

Step 3: Almost Lipschitz continuity. Recall the Lipschitz reference estimates for p-harmonic maps: whenever $B_{\sigma_1}(x_0) \subset B_{\sigma_2}(x_0) \subset B_{\varrho/4}(x_0)$ it is

$$\fint_{B_{\sigma_1}(x_0)} |Dw|^p\,dx \lesssim \fint_{B_{\sigma_2}(x_0)} |Dw|^p\,dx.$$

Comparison estimates and Lipschitz estimates can be matched and iterated to conclude that for all $\alpha \in (0,1)$ is holds:

$$\fint_{B_\sigma(x_0)} |u-(u)_{B_\sigma(x_0)}|^p\,dx \lesssim \left(\frac{\sigma}{\varrho}\right)^{\alpha p}\left[\fint_{B_\varrho(x_0)} |u-(u)_{B_\varrho(x_0)}|^p\,dx + r^{\alpha p}\|f\|^{p/(p-1)}_{L^n(B_\varrho(x_0))}\right] + \left(\frac{\sigma}{\varrho}\right)^{\alpha p}\int_{\mathbb{R}^n\setminus B_\varrho(x_0)} \frac{|u-(u)_{B_\varrho(x_0)}|^\gamma}{|x_0-x|^{n+s\gamma}}\,dx.$$

Step 4: Gradient Hölder continuity. With the local Hölder continuity at all exponents, the Morrey type decay estimates

$$\int_{B_{\varrho/2}(x_0)} \fint_{B_{\varrho/2}(x_0)} \frac{|u(x) - u(y)|^{\gamma}}{|x-y|^{n+s\gamma}}\, dx\, dy \lesssim \varrho^{(\beta - s)\gamma} \qquad s < \beta < 1,$$

and

$$\fint_{B_{\varrho/2}(x_0)} |Du|^p\, dx \lesssim \varrho^{-p\lambda} \qquad \text{for all} \quad \lambda > 0$$

easily follow. In particular, it is

$$t^{-\delta}[\mathtt{snail}_{\delta}(t)]^{\gamma} \equiv t^{-\delta}[\mathtt{snail}_{\delta}(u, B_t(x_0))]^{\gamma} \lesssim 1$$

for all $0 < t \leq \varrho$.

Main steps of the proof

As a direct consequence of the above three bounds, we have

$$\fint_{B_{\varrho/4}(x_0)} |Du - Dh|^p \, dx \lesssim \varrho^{\sigma_2 p},$$

and the Hölder continuity of Du follows by standard means - recall the oscillation decay for p-harmonic maps:

$$\operatorname{osc}_{B_t(x_0)} Dh \lesssim \left(\frac{t}{\varrho}\right)^{\alpha_0} \left(\fint_{B_{\varrho/4}(x_0)} |Dh|^p \, dx\right)^{1/p}.$$

Problem 2: interpolative gap bounds for nonuniformly elliptic integrals

$$\mathcal{F}(w) := \int_\Omega F(x, Dw)\,dx$$
$$|z|^p \lesssim F(x,z) \lesssim 1 + |z|^q$$

It is well-known - **Marcellini** *Preprint Ist. U. Dini* (1987); *ARMA* (1989); *JDE* (1991), **Giaquinta** *manuscripta math.* (1987) - that

$$\frac{q}{p} < 1 + \mathrm{o}(n)$$

is necessary and sufficient condition for regularity of minima of $\mathcal{F}(\cdot)$.

- By coercivity, minima belong to $W^{1,p}(\Omega)$. Imposing a priori extra regularity on minima is it possible to relax the bound on q/p?
- What about a priori **Hölder continuous** minimizers?

$\rightsquigarrow$ D. & Mingione *Anal. Math. Phys.* (2021). arXiv:2103.00743

Autonomous integrands.

$$W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega) \ni u \mapsto \min_{W^{1,p}_{\mathrm{loc}}(\Omega)} \int_{\Omega} F(Dw)\,dx$$

$$1 < p < q < p + 2 \implies u \in W^{1,q}_{\mathrm{loc}}(\Omega)$$

- Choe *Nonlinear Anal.* (1992).
- Esposito & Leonetti & Mingione *NoDEA* (1999).
- Bildhauer *Lect. Notes Math. Springer, Berlin* (2003).
- Carozza & Kristensen & Passarelli Di Napoli *Ann. IHP-AN* (2011).

Baroni & Colombo & Mingione *Calc. Var. & PDE* (2018) proved that if $u \in W^{1,p}(\Omega)$ is a local minimizer of the **Double Phase energy**

$$w \mapsto \int_{\Omega} [|Dw|^p + a(x)|Dw|^q] \, dx,$$

it holds that

- $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ and $q - p \leq \alpha \implies u \in C^{1,\beta_0}_{\mathrm{loc}}(\Omega)$;
- $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C^{0,\gamma}_{\mathrm{loc}}(\Omega)$ and $q - p < \frac{\alpha}{1-\gamma} \implies u \in C^{1,\beta_0}_{\mathrm{loc}}(\Omega)$.

Functionals with (p,q)-growth

Let $F\colon \Omega\times\mathbb{R}^n\to\mathbb{R}$ be a Carathéodory integrand such that

$$\begin{cases} |z|^p \lesssim F(x,z) \lesssim (1+|z|^q) \\ \left(\mu^2+|z|^2\right)^{\frac{p-2}{2}}|\xi|^2 \lesssim \langle \partial^2 F(x,z)\xi,\xi\rangle \\ |\partial^2 F(x,z)| \lesssim \left(\mu^2+|z|^2\right)^{\frac{p-2}{2}} + \left(\mu^2+|z|^2\right)^{\frac{q-2}{2}} \\ |\partial F(x,z)-\partial F(y,z)| \lesssim |x-y|^{\alpha}(1+|z|^{q-1}), \end{cases}$$

with $\mu\in[0,1]$ and $\alpha\in(0,1]$ and consider functional

$$W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\ni w\mapsto \mathcal{F}(w,\Omega):=\int_\Omega F(x,Dw)\,dx.$$

- Esposito & Leonetti & Mingione *JDE* (2004).

Measuring ellipticity

Pointwise ellipticity ratio.

$$\mathcal{R}(z) := \sup_{x\in B} \frac{\text{highest eigenvalue of } \partial^2 F(x,z)}{\text{lowest eigenvalue of } \partial^2 F(x,z)}$$

Nonlocal ellipticity ratio.

$$\bar{\mathcal{R}}(z) := \frac{\sup_{x\in B} \text{highest eigenvalue of } \partial^2 F(x,z)}{\inf_{x\in B} \text{lowest eigenvalue of } \partial^2 F(x,z)}$$

- Uniform ellipticity $\rightsquigarrow$ $\mathcal{R}(z) \lesssim 1$ and $\bar{\mathcal{R}}(z) \lesssim 1$ for all $z \in \mathbb{R}^n$.
- Soft nonuniform ellipticity $\rightsquigarrow$ $\mathcal{R}(z) \lesssim 1$ and $\bar{\mathcal{R}}(z) \to \infty$ as $|z| \to \infty$.
- Strong nonuniform ellipticity $\rightsquigarrow$ $\mathcal{R}(z) \to \infty \implies \bar{\mathcal{R}}(z) \to \infty$ as $|z| \to \infty$.

When proving higher integrability for local minimizers of variational integrals, **double phase energy** *and* **general functionals with (p,q)-growth** *are expected to behave in a similar fashion.*

Bounded minimizers of nonautonomous functionals

Nonautonomous integrands.

$$W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega) \ni u \mapsto \min_{W^{1,p}(\Omega)} \int_{\Omega} F(x, Dw)\, dx$$

$$1 < p < q < p + \alpha \min\left\{1, \frac{p}{2}\right\}$$

- D. & Mingione *J. Geom. Anal.* (2020).

Lavrentiev Phenomenon

$$\inf_{w\in W^{1,p}_{\mathrm{loc}}(\Omega)} \mathcal{F}(w,\Omega) < \inf_{w\in W^{1,q}_{\mathrm{loc}}(\Omega)} \mathcal{F}(w,\Omega)$$

$\mathbf{p} = \mathbf{q}$ $\rightsquigarrow$ Lavrentiev Phenomenon does not occur.

- Buttazzo & Mizel *J. Funct. Anal.* (1992).

When $p < q$, $\mathcal{F}(\cdot)$ is well defined and finite in $W^{1,q}_{\mathrm{loc}}(\Omega)$, but is is coercive only on $W^{1,p}_{\mathrm{loc}}(\Omega)$ so it can be extended by relaxation to $W^{1,p}_{\mathrm{loc}}(\Omega)$.

- Marcellini *Ann. IHP-AN* (1986).

Extension by relaxation

Let $B \subset 2B \Subset \Omega$ be any ball and $w \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C^{0,\gamma}_{\mathrm{loc}}(\Omega)$. The extension by relaxation of functional $\mathcal{F}(\cdot)$ is defined as

$$\bar{\mathcal{F}}(w,B) := \inf_{\{w_j\} \in \mathcal{C}_*(w)} \left\{ \liminf_{j \to \infty} \int_B F(x, Dw_j)\, dx \right\},$$

where, class $\mathcal{C}_*(w)$ is defined as

$$\mathcal{C}_*(w) := \Big\{ \{w_j\} \subset W^{1,\infty}(B) \colon w_j \rightharpoonup w \text{ in } W^{1,p}(B), \\ \sup_{j \in \mathbb{N}} [w_j]_{0,\gamma;B} \le \|w\|_{C^{0,\gamma}(2B)} \Big\}$$

The Lavrentiev gap functional

The Lavrentiev gap functional is defined as:

$$\mathcal{L}(w,B) := \begin{cases} \bar{\mathcal{F}}(w,B) - \mathcal{F}(w,B) & \text{if } \mathcal{F}(w,B) < \infty \\ 0 & \text{if } \mathcal{F}(w,B) = \infty \end{cases}$$

and in general it is $\mathcal{L}(\cdot) \geq 0$.

Proposition

Let $w \in W^{1,p}(2B) \cap C^{0,\gamma}(2B)$ be any map such that $\mathcal{F}(w,B) < \infty$. Then

$$\mathcal{L}(w,B) = 0 \iff \exists \{\tilde{w}_j\}_{j\in\mathbb{N}} \in \mathcal{C}_*(w) \colon \mathcal{F}(\tilde{w}_j, B) \to \mathcal{F}(w,B).$$

- Buttazzo & Mizel *J. Funct. Anal.* (1992).
- Marcellini *Annales IHP-AN* (1986); *ARMA* (1989).

Higher integrability for a priori $C^{0,\gamma}$-minimizers

Theorem (D. & Mingione *Anal. Math. Phys.* (2021))

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C^{0,\gamma}_{\mathrm{loc}}(\Omega)$ *be a minimizer of functional* $\mathcal{F}(\cdot)$ *with exponents* (p, q) *verifying*

$$q < p + \frac{\min\{\alpha, 2\gamma\}}{(1-\gamma)} \min\left\{1, \frac{p}{2}\right\}, \qquad \gamma \in (0,1).$$

Assume also that $\mathcal{L}(u, B_r) = 0$ *holds for* $B_r \Subset \Omega$ *with* $r \le 1$*. If* $\tilde{q}$ *is a number such that*

$$q \le \tilde{q} < p + \frac{\min\{\alpha, 2\gamma\}}{(1-\gamma)} \min\left\{1, \frac{p}{2}\right\}$$

and $B_\varrho \Subset B_r \subset B_{2r} \Subset \Omega$ *is a ball concentric to* B_r*, then*

$$\|Du\|_{L^{\tilde{q}}(B_\varrho)} \lesssim \frac{1}{(r-\varrho)^{\kappa_1}} \left[\mathcal{F}(u, B_r) + \|u\|_{C^{0,\gamma}(B_{2r})} + 1\right]^{\kappa_2}.$$

Lemma (Brezis & Mironescu *J. Evol. Equations* (2001))

Let $B_\varrho \Subset B_r \subset \mathbb{R}^n$ be concentric balls with $r \le 1$. If $0 \le s_1 < 1 < s_2 < 2$, $1 < a, t < \infty$, $\tilde{p} > 1$ and $\theta \in (0,1)$ are so that

$$1 = \theta s_1 + (1-\theta) s_2, \qquad \frac{1}{\tilde{p}} = \frac{\theta}{a} + \frac{1-\theta}{t},$$

then $w \in W^{s_1,a}(B_r) \cap W^{s_2,t}(B_r)$ belongs to $W^{1,\tilde{p}}(B_\varrho)$ with

$$\|Dw\|_{L^{\tilde{p}}(B_\varrho)} \lesssim \frac{1}{(r-\varrho)^\kappa} [w]^{\theta}_{s_1,a;B_r} \|Dw\|^{1-\theta}_{W^{s_2-1,t}(B_r)}.$$

Technical obstructions

- Need to correct the growth of $\mathcal{F}(\cdot)$ with more regular functionals with balanced polynomial growth $\mathcal{F}_{\varepsilon}(\cdot)$.
- Need to transfer the a priori $C^{0,\gamma}$-regularity from the original minimizer to the approximating minimizers of $\mathcal{F}_{\varepsilon}(\cdot)$.
- Given the poor structure of $F(\cdot)$ (even if regularized) no maximum principles are available - no way for controlling the $C^{0,\gamma}$-norm of approximating minimizers via the one of the boundary data.

Add to the approximating integrals certain penalization terms controlling suitable Lebesgue or fractional Sobolev norms.

- Carozza & Kristensen & Passarelli Di Napoli *Ann. IHP-AN* (2011).

Plugging maximum principles in approximating functionals

$M_0 \approx \|u\|_{L^\infty(B_{2r})}$, $M \approx \|u\|_{C^{0,\gamma}(B_{2r})}$, $d > 1$ large enough.

$$\mathcal{F}_\varepsilon(w, B_r) := \mathcal{F}(w, B_r) + \varepsilon \int_{B_r} |Dw|^{2d}\, dx + \int_{B_r} (|w|^2 - M_0^2)_+^d\, dx$$
$$+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left(|w(x) - w(y)|^2 - M^2|x-y|^{2\gamma}\right)_+^d}{|x-y|^{n+2sd}}\, dx\, dy.$$

Sketch of the proof

Step 1: Approximating problems and convergence. Let u_ε be the solution of Dirichlet problem

$$\tilde{u}_\varepsilon + W_0^{1,2d}(B_r) \cap W^{s,2d}(\mathbb{R}^n) \ni w \mapsto \min \mathcal{F}_\varepsilon(w, B_r).$$

Recall that

$$\mathcal{L}(u, B_r) = 0 \implies \tilde{u}_\varepsilon \rightharpoonup u \;\text{ in }\; W^{1,p}(B_r) \;\text{ and }\; \mathcal{F}(\tilde{u}_\varepsilon, B_r) \to \mathcal{F}(u, B_r).$$

Since

$$\|\tilde{u}_\varepsilon\|_{L^\infty(B_r)} \le M_0 \quad \text{and} \quad |\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)| \le M|x-y|^\gamma$$

it follows that, for $\varepsilon << \|D\tilde{u}_\varepsilon\|_{L^{2d}(B_r)}^{4d}$,

$$\int_{B_r} (|\tilde{u}_\varepsilon|^2 - M_0^2)_+^d \, dx = 0, \qquad \varepsilon \int_{B_r} |D\tilde{u}_\varepsilon|^{2d} \, dx \equiv o(\varepsilon)$$

and

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left(|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)|^2 - M^2|x-y|^{2\gamma}\right)_+^d}{|x-y|^{n+2sd}} \, dx \, dy = 0.$$

Sketch of the proof

The minimality of u_ε in class $\tilde{u}_\varepsilon + W^{1,2d}_0(B_r) \cap W^{s,2d}(\mathbb{R}^n)$ yields that

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, B_r) \le \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, B_r) \le \mathcal{F}(u, B_r).$$

Then $u_\varepsilon \rightharpoonup v$ in $W^{1,p}(\Omega)$ with $v \in u + W^{1,p}_0(\Omega)$, so weak lower semicontinuity, strict convexity and minimality yield that

$$\mathcal{F}(u, B_r) \le \mathcal{F}(v, B_r) \le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, B_r) \le \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, B_r) \le \mathcal{F}(u, B_r),$$

therefore $v = u$, $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(B_r)$ and

$$\|u_\varepsilon\|_{L^{2d}(B_r)} \lesssim 1, \qquad [u_\varepsilon]_{s,2d;\mathbb{R}^n} \lesssim 1.$$

Sketch of the proof

Step 2: Local/nonlocal Caccioppoli type inequality. Recall that we are dealing with the functional

$$\mathcal{F}_\varepsilon(w, B_r) := \mathcal{F}(w, B_r) + \varepsilon \int_{B_r} |Dw|^{2d}\, dx + \int_{B_r} (|w|^2 - M_0^2)_+^d\, dx$$
$$+ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left(|w(x) - w(y)|^2 - M^2|x-y|^{2\gamma}\right)_+^d}{|x-y|^{n+2sd}}\, dx\, dy.$$

Via difference quotient argument on the Euler-Lagrange equation for functional $\mathcal{F}_\varepsilon(\cdot)$ derive

$$\int_{B_{(\tau_1+\tau_2)/2}} |\tau_h V_p(Du_\varepsilon)|^2\, dx \lesssim \frac{|h|^{\alpha_0}\left(\mathcal{F}(u, B_r) + \|Du_\varepsilon\|^q_{L^q(B_{\tau_2})} + \|u\|^{2d}_{C^{0,\gamma}(B_{2r})} + 1\right)}{(\tau_2 - \tau_1)^2}.$$

Sketch of the proof

Step 3: Fractional Gagliardo-Nirneberg inequality and conclusions. Standard manipulations and fractional Gagliardo-Nirneberg inequality yield

$$\begin{aligned}\|Du_\varepsilon\|_{L^{\tilde p}(B_{\tau_1})} &\lesssim \frac{1}{(\tau_2-\tau_1)^\kappa}[u_\varepsilon]^{\theta}_{s,2d;B_{\tau_2}}\|Du_\varepsilon\|^{1-\theta}_{W^{\beta/p,p}(B_{(\tau_2+\tau_1)/2})}\\ &\lesssim \frac{1}{(\tau_2-\tau_1)^\kappa}\left(\mathcal{F}(u,B_r)+\|Du_\varepsilon\|^q_{L^q(B_{\tau_2})}+\|u\|^{2d}_{C^{0,\gamma}(B_{2r})}+1\right),\end{aligned}$$

which implies via interpolation and Young inequality that

$$\|Du_\varepsilon\|_{L^{\tilde q}(B_\varrho)} \le \frac{c}{(r-\varrho)^\kappa}\left(\mathcal{F}(u,B_r)+\|u\|^{2d}_{C^{0,\gamma}(B_{2r})}+1\right)^{\vartheta},$$

for all

$$\tilde q \in \left[q, p+\frac{\min\{\alpha,2\gamma\}}{(1-\gamma)}\min\left\{1,\frac{p}{2}\right\}\right).$$

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