A convergent discretization of the porous medium equation with fractional pressure

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Workshop: On nonlocal and nonlinear PDEs

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Based on a joint work with:

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Porous Medium Equation / Fast Diffusion Equation

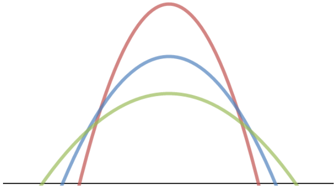
$$u_t - \Delta u^m = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,T) \qquad \text{(PME/FDE)}$$

Self Similar solutions: $\mathscr{U}(x,t) = t^{-\alpha} F(|x| t^{-\beta})$

Finite/Infinite speed of propagation with $m = 1$ as borderline cases

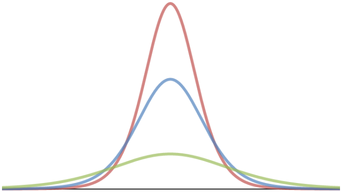
Slow Diffusion: $m > 1$

$$F(y) \sim (R^2 - |y|^2)_+^{1/(m-1)}$$



Fast Diffusion : $m < 1$

$$F(y) \sim (R^2 + |y|^2)^{-1/(1-m)}$$



Reference: Vázquez's book 2007 on the PME

Fractional porous medium models

$$u_t - \Delta u^m = 0$$

$$\downarrow$$

$$u_t + (-\Delta)^s u^m = 0$$

de Pablo, Quirós, Rodriguez and Vázquez

Infinite speed of propagation for all $m > 0$.

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$$u_t = \nabla \cdot \left(u^{m-1} \nabla u \right)$$

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Caffarelli and Vázquez ($m = 2$)
& Stan, del Teso, and Vázquez
$m \in (1, \infty)$

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Biler, Imbert, Karch and Monneau

Finite speed of propagation for all $m > 1$.

$$u_t = \nabla \cdot \left(u^{m-1} \nabla (-\Delta)^{-\sigma} u \right)$$

Continuity equation: $u_t = -\nabla \cdot (u\mathbf{v})$

$u \equiv$ density (mass per unit volume), $\mathbf{v} \equiv$ flow velocity field.

In this case

$$\mathbf{v} = u^{m-2} \nabla (-\Delta)^{-\sigma} u$$

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- The velocity field is nonlocal.

$$\nabla(-\Delta)^{-\sigma}f(x) = \nabla^{1-2\sigma}f(x) = \int (f(x)-f(y))\frac{x-y}{|x-y|^{N+2-2\sigma}}\,\mathrm{d}y$$

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Consequences:

- Finite speed of propagation if $m > 2$ (also $m = 2$, Caffarelli and Vázquez 11')
- Infinite speed of propagation if $m \in (1, 2)$.

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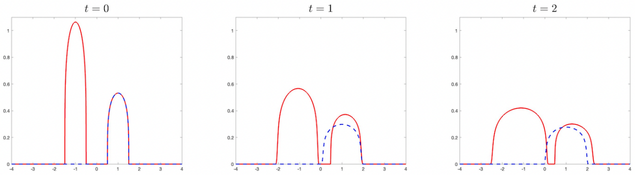
Consequences:

- Finite speed of propagation if $m>2$ (also $m=2$, Caffarelli and Vázquez 11')
- Infinite speed of propagation if $m\in(1,2)$.
- No comparison principle (Caffarelli and Vázquez 11')



$$\partial_t u - \nabla\cdot(u^{m-1}\nabla(-\Delta)^{-\sigma}u) = 0,$$
$$u(\cdot,0) = \mu_0.$$

Weak solution

$$\int_0^T\int_{\mathbb{R}^d} u\phi_t - \int_0^T\int_{\mathbb{R}^d} u^{m-1}\nabla(-\Delta)^{-\sigma}u\nabla\phi + \int_{\mathbb{R}^d}\phi(x,0)\,d\mu_0(x) = 0,$$

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Properties (Stan, dT & Vázquez – 19')

- Existence of nonnegative weak solutions for $\mu_0 \in \mathcal{M}_+(\mathbb{R}^d)$.
- Conservation of mass.
- L^∞-decay

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- *(dT, Jakobsen)* Tightness: For every $\varepsilon, T > 0$ there exists $R = R(\varepsilon, \mu_0, T) > 0$ s.t.

$$\sup_{t \in [0,T]} \int_{|x|>R} u(x,t) \mathrm{d}x < \varepsilon.$$

Cumulative density in dimension $d = 1$

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Integrating the equation

$$\partial_t v + (\partial_x v)^{m-1} (-\Delta)^s v = 0,$$

with $s = 1 - \sigma$.

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• A general class of nonlocal quasilinear equations including this one was studied in (Chasseigne and Jakobsen 17'):

$$L[\phi](x) = \int_{|z|>0} (\phi(x + f(\nabla\phi(x))z) - \phi(x)) \frac{\mathrm{d}z}{|z|^{d+2s}}$$
$$= -f(\nabla\phi(x))^{2s}(-\Delta)^s\phi(x)$$

$$L[\phi] = (\partial_x \phi)^{m-1}(-\Delta)^s \phi$$

- Let us recall the definition of the p-Laplacian

$$\Delta_p \phi = \nabla \cdot (|\nabla \phi|^{p-2} \nabla \phi)$$

that in dimension $d = 1$ can we expressed as

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- (Chasseigne and Jakobsen 17') The operator L is a nonlocal version of the p-Laplacian. It does not coincide with other ones in the literature

(Variational) $\quad (-\Delta)_p^s \phi(x) = \int \frac{|\phi(x) - \phi(x+z)|^{p-2} (\phi(x) - \phi(x+z))}{|z|^{d+sp}} \mathrm{d}z$

(Game th.) $\quad -\Delta_p^s \phi(x) = \sup_{|y|=1} \int_{\mathscr{C}(x,y)} \frac{\phi(x) - \phi(x+z)}{|z|^{d+sp}} \mathrm{d}z + \inf_{|\tilde{y}|=1} \int_{\mathscr{C}(x,\tilde{y})} \frac{\phi(x) - \phi(x+z)}{|z|^{d+sp}} \mathrm{d}z$

$$u_t - \partial_x \left(u^{m-1} \partial_x (-\partial_{xx})^{-\sigma} u \right) = 0 \quad \text{(FPE)}$$

$$v(x,t) = \int_{-\infty}^{x} u(y,t)\,\mathrm{d}y \quad \Downarrow \qquad\qquad \Uparrow \quad u = \partial_x v$$

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- $u \geq 0$ (nonnegative weak sol.) $\iff$ $\partial_x v \geq 0$ (nondecreasing viscosity sol.)

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By (Chasseigne and Jakobsen 17'):

- (IP) has existence and uniqueness of viscosity solutions.

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By (Chasseigne and Jakobsen 17'):

- (IP) has existence and uniqueness of viscosity solutions.
- (IP) has comparison principle for sub and super solutions.

This seems to be a right context for numerical schemes!

How to discretize

$$L[\phi] = (\partial_x \phi)^{m-1}(-\Delta)^s \phi$$

so that it preserves the monotonicity property:

If $\psi(x) = \phi(x)$ and $\psi \geq \phi$ then $L[\psi](x) \geq L[\phi](x)$.

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- *Discretization of the fractional Laplacian:*

$$-(-\Delta)^s \phi(x) = \text{P.V} \int_{|z|>0} (\phi(x+z) - \phi(x)) \frac{\mathrm{d}z}{|z|^{1+2s}}$$

$$-(-\Delta)^s_h \phi(x) \sim \sum_{k \neq 0} (\phi(x+z_k) - \phi(x)) \omega_k \qquad \text{with} \quad \omega_k = \omega_{-k} \geq 0.$$

such that for $\phi \in C^2 \cap C_b$

$$(-\Delta)^s_h \phi \xrightarrow{h \to 0} (-\Delta)^s \phi \quad \text{in} \quad L^\infty_{\text{loc}}.$$

- *Discretization of ∂_x:* We think of $L[\phi] = (\partial_x \phi)^{m-1}(-\Delta)^s \phi$ as

$$L[\phi] = (\partial_x \phi)^{m-1} b(x,t),$$

with b changing signs.

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with b changing signs. Thus, we choose an UPWIND-type discretization:

$$\partial_x \phi(x) \sim D_h \phi(x) := \begin{cases} \dfrac{\phi(x+h) - \phi(x)}{h} & \text{if} \quad (-\Delta)_h^s \phi(x) \leq 0, \\ \dfrac{\phi(x) - \phi(x-h)}{h} & \text{if} \quad (-\Delta)_h^s \phi(x) > 0. \end{cases}$$

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- *Complete discretization*

$$L_h[\phi](x) = (D_h\phi(x))^{m-1}(-\Delta)_h^s\phi(x)$$

- (Consistent) For $\phi \in C^2 \cap C_b$, $L_h[\phi] \xrightarrow{h\to 0} L[\phi]$ in L^∞_{loc}
- (Monotone) If $\psi(x) = \phi(x)$ and $\psi \geq \phi$ then $L_h[\psi](x) \geq L_h[\phi](x)$

but only for nondecreasing functions ϕ and ψ!

Numerical scheme

Given discretization parameters $h, \tau > 0$, a uniform grids $x_k = kh$ and $t_j = j\tau$:

$$
\begin{aligned}
V_k^{j+1} &= V_k^j - \tau L_h V_k^j, \\
V_k^0 &= \int_{-\infty}^{x_k} \mathrm{d}\mu_0(x).
\end{aligned}
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Key result – Comparison for nondecreasing solutions: For V^j, W^j nondecreasing,

$$V^j \leq W^j \Longrightarrow V^{j+1} \leq W^{j+1}.$$

provided CFL-type conditions:

$$\tau \leq h^{2s+m-1} \quad (V, W \text{ bounded}) \qquad \text{or} \qquad \tau \leq h^{\max\{1,2s\}} \quad (V, W \text{ Lipschitz})$$

Theorem

Let $\{V_i^0\}_{i\in\mathbb{Z}}$ *and* $\{W_i^0\}_{i\in\mathbb{Z}}$ *are nondecreasing and satisfying one of the following:*

$$|V_i^0|,|W_i^0|\leq M,\ i\in\mathbb{Z}, \qquad \text{and} \qquad \tau\leq C_1 h^{2s+m-1},$$

$$|V_i^0|,|W_i^0|\leq M,\ \frac{|V_{i+1}^0-V_i^0|}{h},\frac{|W_{i+1}^0-W_i^0|}{h}\leq L,\ i\in\mathbb{Z}, \quad \text{and} \qquad \tau\leq C_2 h^{\max\{2s,1\}},$$

Then

- ⓐ *(Nondecreasing)* $V_i^j\leq V_{i+1}^j$.
- ⓑ *(ℓ^∞-stability)* $|V_i^j|\leq M$.
- ⓒ *(Nonegative) If* $V_i^0\geq 0$ *then* $V_i^j\geq 0$.
- ⓓ *(ℓ^∞-contraction)* $\sup_{i\in\mathbb{Z},j\in\mathbb{N}}|W_i^j-V_i^j|\leq\sup_{i\in\mathbb{Z}}|W_i^0-V_i^0|$.
- ⓔ *(Lipschitz-stability)* $\sup_{i\in\mathbb{Z},j\in\mathbb{N}}\frac{|V_{i+1}^j-V_i^j|}{h}\leq\sup_{i\in\mathbb{Z}}\frac{|V_{i+1}^0-V_i^0|}{h}$.

Let $\overline{V}$ be an interpolation of V (p.w. linear in space and p.w. constant in time).

Theorem

(a) (Continuous case) *If* $v_0 \in BUC(\mathbb{R}^d)$ *then:*

(i) *(Convergence)* $\overline{V}_h \xrightarrow{h \to 0} v$ locally uniformly in $\overline{Q_T}$.

(ii) *(Limit)* $v \in BUC(\mathbb{R}^d)$ *is a viscosity solution of (IP).*

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(ii) *(Limit)* $v \in BUC(\mathbb{R}^d)$ *is a viscosity solution of (IP).*

(b) (Jump discontinuity) *If*

$$v_0(x) = \begin{cases} 0 & \text{if} \quad x < a, \\ M & \text{if} \quad x \geq a. \end{cases}$$

(i) *(Convergence)* $\overline{V}_h \xrightarrow{h \to 0} v$ locally uniformly in $\overline{Q_T} \setminus \{(a,0)\}$.

(ii) *(Limit)* $v \in C_b(\overline{Q}_T \setminus \{(a,0)\})$ *is a discontinuous viscosity solution of (IP).*

Convergence by *half-relaxed limit method (Barles, Perthame, Souganidis)*.

[1] Uniform-in-h L^∞ bounds. Then

$$\underline{v}(x,t) := \liminf_{(y,\rho,h)\to(x,t,0)} \overline{V}_h(y,s) \qquad \overline{v}(x,t) := \limsup_{(y,\rho,h)\to(x,t,0)} \overline{V}_h(y,s)$$

and $\underline{v} \leq \overline{v}$.

[2] Monotonicity and consistency. Then

$\underline{v}$ discontinuous viscosity supersolution

$\overline{v}$ discontinuous viscosity subsolution

[3] Comparison principle for (IP). Then

$$\overline{v} \leq \underline{v}$$

[4] All together:

$$\overline{v} = \underline{v} =: v = \lim_{h\to 0} \overline{V}_h$$

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Once a numerical method for (IP) is provided, the approximation for (FPE) is naturally given by numerical differentiation

$$U_i^j = \frac{V_{i+1}^j - V_i^k}{h}$$

and $\overline{U}$ the corresponding p.w. constant interpolation.

Rubinstein-Kantorovich metric

$$d_0(f_1, f_2) = \sup_{\varphi \in \mathrm{Lip}_{1,1}(\mathbb{R})} \int_{\mathbb{R}} (f_1(x) - f_2(x))\varphi(x)\mathrm{d}x,$$

where $\mathrm{Lip}_{1,1}(\mathbb{R}) = \{\varphi \in C(\mathbb{R}) : \|\varphi\|_{L^\infty} \leq 1,\ \|D\varphi\|_{L^\infty} \leq 1\}$ and D is the weak derivative.

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Theorem

$\mu_0\in\mathcal{M}_+(\mathbb{R})$ and u the weak solution of (FPE).

a. *(L^1 case) Assume $\mu_0\in L^1(\mathbb{R})$ (resp. $\mu_0\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$), then we have*

$$\sup_{t\in[0,T]} d_0(\overline{U}_h(\cdot,t),u(\cdot,t))\xrightarrow{h\to 0}0.$$

Rubinstein-Kantorovich metric

$$d_0(f_1,f_2)=\sup_{\varphi\in \mathrm{Lip}_{1,1}(\mathbb{R})}\int_{\mathbb{R}}(f_1(x)-f_2(x))\varphi(x)\mathrm{d}x,$$

where $\mathrm{Lip}_{1,1}(\mathbb{R})=\{\varphi\in C(\mathbb{R}):\|\varphi\|_{L^\infty}\leq 1,\ \|D\varphi\|_{L^\infty}\leq 1\}$ and D is the weak derivative.

Theorem

$\mu_0\in\mathscr{M}_+(\mathbb{R})$ and u the weak solution of (FPE).

a. *(L^1 case) Assume $\mu_0\in L^1(\mathbb{R})$ (resp. $\mu_0\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$), then we have*

$$\sup_{t\in[0,T]} d_0(\overline{U}_h(\cdot,t),u(\cdot,t))\xrightarrow{h\to 0}0.$$

b. *(Fundamental solution) Assume $\mu_0=M\delta_a$ then we have*

$$\sup_{t\in[t_0,T]} d_0(\overline{U}_h(\cdot,t),u(\cdot,t))\xrightarrow{h\to 0}0.$$

Sketch of the proof:

- [Previous theorems] $\overline{V}(x,t) \to v(x,t) = \int_{-\infty}^{x} u(y,t)\mathrm{d}y$ locally uniformly.
- [Interpolation choice] $\overline{V}(x,t) = \int_{-\infty}^{x} \overline{U}(y,t)\mathrm{d}y$
- [Convergence I] For $\varphi \in \mathrm{Lip}_{1,1}$ and $\partial_x\varphi$ with compact support:

$$\int \varphi(\overline{U}-u)\mathrm{d}x = \int \varphi_x \int_{-\infty}^{x} (\overline{U}-u)\mathrm{d}y\mathrm{d}x = \int \varphi_x(\overline{V}-v)\mathrm{d}x \stackrel{h\to 0}{\longrightarrow} 0.$$

- [Equi-Tightness]

$$\int \overline{U}(1-\chi_R)\mathrm{d}x < \varepsilon, \qquad \text{and} \qquad \int \overline{u}(1-\chi_R)\mathrm{d}x < \varepsilon.$$

- [Convergence II] For $\varphi \in \mathrm{Lip}_{1,1}$:

$$\int \varphi(\overline{U}-u)\mathrm{d}x = \int \varphi\chi_R(\overline{U}-u)\mathrm{d}x + \int \varphi(1-\chi_R)(\overline{U}-u)\mathrm{d}x \stackrel{h\to 0}{\longrightarrow} 0.$$

Final comments

- The numerical method works in this specific framework just because:
  - $u \geq 0 \quad \Longrightarrow v$ nondecreasing.
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- Numerical methods in dimensions $d > 1$?
- Numerical methods for sign-changing solutions?

Thank you

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