Lévy processes, controlled time rate and mean field games

Miłosz Krupski

University of Wrocław

Joint work with Indranil Chowdhury and Espen Jakobsen

NTNU, May 2023

Part I: Classical mean field games and their heuristic derivation

“Classical” mean field games

$$\begin{cases} -\partial_t u = \Delta u + H(\nabla u) + \mathfrak{f}(m) & \text{on } [0,T] \times \mathbb{R}^d, \\ u(T) = \mathfrak{g}\big(m(T)\big) & \text{on } \mathbb{R}^d, \\ \partial_t m = \Delta m + \operatorname{div}(H'(\nabla u)\, m) & \text{on } [0,T] \times \mathbb{R}^d, \\ m(0) = m_0 & \text{on } \mathbb{R}^d. \end{cases}$$

- Agents control (individually, but interchangeably) **the drift** of a Wiener process describing their positions.

Controlled Wiener process

- Controlled Wiener process $Y_s^{t,x,\gamma} = x + W_s^{t,x} + \gamma(s,\cdot)(s-t)$ — at each point $(s,\cdot)$ we choose a direction γ, i.e. $\gamma : (s,\cdot) \mapsto \mathcal{A} \subset \mathbb{R}^d$.
- Y^γ is a Markov process associated with the families of operators P^γ and transition probabilities $p^\gamma(t,x,s,A) = \mathsf{P}(Y_s^{t,x,\gamma} \in A)$,

$$P_{t,s}^\gamma \phi(x) = \int_{\mathbb{R}^d} \phi(y)\, p^\gamma(t,x,s,dy) = E\phi\big(Y_s^{t,x,\gamma}\big), \quad \phi \in C_b(\mathbb{R}^d).$$

- We may compute the "generator"

$$\lim_{h\to 0} \frac{P_{t+h,t}^\gamma \phi(x) - \phi(x)}{h} = \Delta u + \gamma(t,x)\cdot\nabla u.$$

Dynamic programming

- Total gain functional

$$J(t,x,\gamma) = E\Big(\int_t^T \ell\big(s, Y_s^{t,x,\gamma}, \gamma\big)\, ds + g\big(Y_T^{t,x,\gamma}\big) \Big).$$

- Value function u (the optimal value of J) is given by

$$u(t,x) = \sup_\gamma J(t,x,\gamma).$$

- Dynamic programming principle — assume the "tail" is already optimized

$$u(t,x) = \sup_\gamma E\Big(\int_t^{t+h} \ell\big(s, Y_s^{t,x,\gamma}, \gamma\big)\, ds + u\big(t+h, Y_{t+h}^{t,x,\gamma}\big) \Big).$$

- In the limit we get the Bellman equation

$$-\partial_t u = \Delta u + \sup_{\gamma(t,x)\in\mathcal{A}} \Big(\gamma \cdot \nabla u + \ell(t,x,\gamma) \Big), \tag{1}$$

Hamilton–Jacobi–Bellman

- We now assume that

$$\ell(t,x,\gamma) = -L(\gamma) + f(t,x), \tag{2}$$

  where $L : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is a convex, lower-semicontinuous function.

- Legendre–Fenchel transform H of L (γ disappears, but we will need it later)

$$H(z) = \sup_{\zeta \in \mathbb{R}^d} \big(\zeta z - L(\zeta)\big),$$

- The Bellman equation becomes

$$\begin{cases} -\partial_t u = \Delta u + H\big(\nabla u\big) + f(t,x), \\ u(T,x) = Eg\big(Y_T^{T,x}\big) = g(x). \end{cases}$$

- Backward-in-time evolution equation. Because of Δ it has unique, smooth solutions.

Fokker–Planck–Kolmogorov

- Under reasonable assumptions on L, by the properties of LF transform, we have (the optimal control) $\gamma^* = \nabla H(\nabla u)$ for every $(t,x) \in [0,\infty) \times \mathbb{R}^d$
- For initial condition $m(0) = m_0 \in \mathcal{P}(\mathbb{R}^d)$, input distribution m of Y satisfies

$$\int_{\mathbb{R}^d} \varphi(x)\, m(t+h, dx) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\, p^{\gamma^*}(t, x, t+h, dy)\, m(t, dx),$$

- This leads to

$$\partial_t \int_{\mathbb{R}^d} \varphi(t,x)\, m(t,dx) = \int_{\mathbb{R}^d} \Big(\Delta\varphi + \nabla H(\nabla u)\nabla\varphi + \partial_t\varphi(t,x)\Big)\, m(t,dx).$$

- By duality m is a very weak solution of

$$\partial_t m = \Delta u + \operatorname{div}\big(\nabla H(\nabla u)\, m\big), \quad m(0) = m_0,$$

- m describes the joint distribution of all players, each of whom moves according to their own copy of Y – this leads to the **mean field game**.

Part II: Fully nonlinear, nonlocal mean field games

Fully nonlinear (parabolic, local/nonlocal) MFG

$$\begin{cases} -\partial_t u = F(\mathcal{L}u) + \mathfrak{f}(m) & \text{on } [0,T]\times\mathbb{R}^d, \\ u(T) = \mathfrak{g}\big(m(T)\big) & \text{on } \mathbb{R}^d, \\ \partial_t m = \mathcal{L}^*\big(F'(\mathcal{L}u)\, m\big) & \text{on } [0,T]\times\mathbb{R}^d, \\ m(0) = m_0 & \text{on } \mathbb{R}^d. \end{cases}$$

- Agents control **the time rate** θ of any Lévy process ($\mathcal{L}$)
- θ is a stochastic process such that $\theta(t)$ is a stopping time
- "Local-in-time generator" $\theta'(t)\mathcal{L}$ — not Lévy, but Markov (inhomog.)
- Same for any number of Lévy processes
- To get the classical model: Δ, $dx_1, \dots dx_d$, $-dx_1, \dots - dx_d$

Lévy processes

Definition

A stochastic process $X = \{X_t : t \geq 0\}$ with law P on $\mathbb{R}^{[0,\infty)}$ is a Lévy process if

- X has P-almost surely right-continuous paths with left-limits.
- $\mathsf{P}(X_0 = 0) = 1$
- $X_t - X_s = X_{t-s}$ in distribution
- $X_t - X_s$ is independent of $\{X_r \colon r \leq s\}$

- A Lévy process is a Wiener process if it has P-almost surely continuous paths.
- Lévy–Khintchine–Courrège formula for the generator

$$\mathcal{L}\phi(x) = c\cdot\nabla\phi(x) + \mathrm{tr}\left(aa^T D^2\phi(x)\right) + \int_{\mathbb{R}^d} \Big(\phi(x+z) - \phi(x) - \mathbb{1}_{B_1}(z)\, z\cdot\nabla\phi(x)\Big)\, \nu(dz).$$

- Order $2\sigma \Leftrightarrow \mathcal{L} : C^{2\sigma+\alpha} \to C^{\alpha}$
- Non-degenerate $\Leftrightarrow \nu \asymp |z|^{-d-2\sigma}\, dz$
- Degenerate $\Leftrightarrow \nu \leq |z|^{-d-2\sigma}\, dz$ (or analogue if singular)

Controlled Lévy process

- Controlled Lévy process $Y_s^{t,x,\gamma} = x + X_{\theta(s)}^{t,x}$
- $\theta(s)$ is an **random time change**, i.e. a stochastic process which is almost surely non-negative, non-decreasing, and is a finite stopping time for each fixed s.
- We assume θ is **absolutely continuous**, i.e. there exists an $\mathcal{F}_s$-adapted process θ' such that $\theta(s) - \theta(0) = \int_0^s \theta'(\tau)\, d\tau$.
- Then (with technical assumptions on θ), Y^γ is Markov
- Operators P^γ and transition probabilities $p^\gamma(t,x,s,A) = \mathsf{P}(Y_s^{t,x,\gamma} \in A)$

$$P_{t,s}^\gamma \phi(x) = \int_{\mathbb{R}^d} \phi(y)\, p^\gamma(t,x,s,dy) = E\phi\big(Y_s^{t,x,\gamma}\big), \quad \phi \in C_b(\mathbb{R}^d).$$

- We may compute the "generator" using Dynkin's formula

$$\frac{P_{t+h,t}^\gamma \phi(x) - \phi(x)}{h} = \frac{E\phi\big(Y_{t+h}^{t,x,\gamma}\big) - \phi(x)}{h} = E\left(\frac{1}{h}\int_t^{\theta_s} \mathcal{L}\phi(X_\tau^{t,x})\, d\tau\right)$$

$$= E\left(\frac{1}{h}\int_t^{t+h} \mathcal{L}\phi\big(X_\tau^{t,x}\big)\theta'(\tau)\, d\tau\right) \to \theta'(t)\mathcal{L}\phi(x)$$

Mean field game

- In the same way as before we obtain the pairs of equations

$$\begin{cases} -\partial_t u = F\big(\mathcal{L}u\big) + f(t,x), \\ u(T,x) = g(x). \end{cases} \qquad \begin{cases} \partial_t m = \mathcal{L}^*\big(F'(\mathcal{L}u)\, m\big), \\ m(0) = m_0, \end{cases}$$

- $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$
- Since the process is one-dimensional, ∇H is replaced by F' (Legendre–Fenchel transform of L). F is **convex**.
- Since the time control has non-negative values, F is also **non-decreasing**.
- Mean field game: the cost functions f and g depend on m – individual players move according to the joint distribution m of all players.
- Each player may percieve the distribution as $\widehat{m}$, but in the equilibrium for all of them it should overlap with m.
- We put $f = \mathfrak{f}(m)$ and $g = \mathfrak{g}(m(T))$ and we require $\mathfrak{f}$, $\mathfrak{g}$ to be **continuous, monotone** operators with values in continuous functions.

Part III: Well-posedness

MFG – uniqueness

- Take (m_1, u_1), (m_2, u_2) and test m's against u's

$$\big(m_1(T) - m_2(T)\big)\big[u_1(T) - u_2(T)\big] - \big(m_1(0) - m_2(0)\big)\big[u_1(0) - u_2(0)\big]$$
$$= \int_0^T \Big(m_1\big[\partial_t u + F'(\mathcal{L}u_1)\mathcal{L}u\big] - m_2\big[\partial_t u + F'(\mathcal{L}u_2)\mathcal{L}u\big]\Big)(\tau)\,d\tau = \ldots = 0$$

- F —convex, non-decreasing, $C^{1+\gamma}(\mathbb{R})$, $\mathfrak{f}$, $\mathfrak{g}$ — monotone
- Then

$$m_1 = \mathcal{L}^*(b\,m_1) \text{ and } m_2 = \mathcal{L}^*(b\,m_2), \quad m_1(0) = m_2(0) = m_0,$$

where

$$b(t,x) = \begin{cases} \dfrac{F\big(\mathcal{L}u_1(t,x)\big) - F\big(\mathcal{L}u_2(t,x)\big)}{\mathcal{L}u_1(t,x) - \mathcal{L}u_2(t,x)}, & \text{if } \mathcal{L}u_1(t,x) \neq \mathcal{L}u_2(t,x), \\ F'\big(\mathcal{L}u_1(t,x)\big), & \text{if } \mathcal{L}u_1(t,x) = \mathcal{L}u_2(t,x) \end{cases}$$

- We need: uniqueness of FPK, regularity of HJB.

Fokker–Planck–Kolmogorov

$$\begin{cases} \partial_t m = \mathcal{L}^*(bm) & \text{on } [0,T]\times\mathbb{R}^d, \\ m(0) = m_0 & \text{on } \mathbb{R}^d. \end{cases} \tag{FPK}$$

$b = F'(\mathcal{L}u)$

- $b \in C\big([0,T]\times\mathbb{R}^d\big)$ and $b \geq 0$
- Natural space to look for solutions: $m \in C\big([0,T], \mathcal{P}(\mathbb{R}^d)\big)$:

$$m(t)[\phi(t)] = m_0[\phi(0)] + \int_0^t m(\tau)\big[\partial_t\phi(\tau) + b(\tau)(\mathcal{L}\phi)(\tau)\big]\,d\tau.$$

- Existence: "easy" – set of solutions is convex, **compact** and non-empty.
- Uniqueness by Holmgren: existence of classical solutions to the dual equation

$$\partial_t w = -b\,\mathcal{L}w, \quad w(t) = \psi \in C_c^\infty(\mathbb{R}^d)$$

- Non-deg: $b \in C^\alpha$, $b \geq \kappa > 0$, Mikulevičius & Pragarauskas PotAn14
- Deg: $b \in C^\alpha$, $b \geq 0$, $\mathcal{L}$ of order at most $2\sigma < \frac{7-\sqrt{33}}{4}$
- If $b_n \to b$ locally uniformly, then $\mathcal{M}_n \to \mathcal{M}$ as closed sets ("$K - \limsup$")

Hamilton–Jacobi–Bellman

$$\begin{cases} -\partial_t u = F(\mathcal{L}u) + f(t,x) & \text{on } [0,T]\times\mathbb{R}^d, \\ u(T,x) = g(x) & \text{on } \mathbb{R}^d. \end{cases} \tag{HJB}$$

$$f = \mathfrak{f}(m), \quad g = \mathfrak{g}\big(m(T)\big)$$

- Fully nonlinear equation $\to$ viscosity solutions.
- Comparison principle (VS uniquely exist): Chasseigne & Jakobsen JDE17
- But we need classical solutions and **a bit more**
- Deg: for $2\sigma < 1$ the comparison principle is enough; no regularization

$$f, g \in C^{2\sigma+\alpha} \quad \Rightarrow \quad \partial_t u, \mathcal{L}u \in C^\alpha$$

- Non-deg, local: Schauder–Caccioppoli estimates (interior regularity)

$$f \in C^{\alpha/2,\alpha}(\mathbb{R}^d) \quad \Rightarrow \quad \partial_t u, D^2 u \in C^\alpha(B_1) \quad \text{(Wang CPAM92)}$$

- Non-deg, non-local: **Conjecture**: Schauder estimates as above (works under additional assumptions).
- (we end up assuming $f \in C^{1.\alpha}$ to get global boundedness, but this is bad)

MFG – existence

- We use Kakutani–Glicksberg–Fan fixed point theorem (i.e. Schauder, but for set-valued maps; solutions to FPK are **compact, convex, non-empty sets**)
- Take $\mu \in C\big([0,T], \mathcal{P}(\mathbb{R}^d)\big)$, solve HJB: $\mathcal{K}_1(\mu) = u$.
- Take u and solve FPK: $\mathcal{K}_2(u) = m$
- Look for a fixed point of $\mathcal{K}(\mu) = \mathcal{K}_2(\mathcal{K}_1(\mu))$.
- Compactness of the map is easy (Prohorov theorem)
- For (semi-)continuity:

$$
\begin{array}{ccccccccc}
\mu_n & \xmapsto{\mathcal{K}_1} & \mathfrak{f}(\mu_n), \mathfrak{g}(\mu_n) & \longmapsto & \mathcal{L}u_n & \longmapsto & b_n = F'(\mathcal{L}u_n) & \xmapsto{\mathcal{K}_2} & \mathcal{M}_n \\
\big\downarrow{\scriptstyle weak} & & \big\downarrow{\scriptstyle uniform} & & \big\downarrow{\scriptstyle loc\ unif} & & \big\downarrow{\scriptstyle loc\ unif} & & \big\downarrow{\scriptstyle K-\limsup} \\
\mu & \xmapsto{\mathcal{K}_1} & \mathfrak{f}(\mu), \mathfrak{g}(\mu) & \longmapsto & \mathcal{L}u & \longmapsto & b = F'(\mathcal{L}u) & \xmapsto{\mathcal{K}_2} & \mathcal{M}
\end{array}
$$

Thank you!