The master equation for the mean field games with Lévy diffusions.

Joint work with Espen R. Jakobsen

Artur Rutkowski

NTNU (Trondheim)/WUST (Wrocław)

Workshop: On Nonlinear and Nonlocal Equations
Trondheim 24–26.05.2023

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where $B \in \mathbb{R}^d$, $A \geqslant 0$, $\int (1 \wedge |x|^2)\, \nu(dx) < \infty$ and $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$.

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- $F(x, m(t))$, $G(x, m(T))$ – nonlocal/smoothing coupling.

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- Many other models exist, e.g., common noise, no noise at all, games with a major player.

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$\mathcal{L}$ – more general Lévy operator

- Graber–Ignazio–Neufeld (2021), Δ + nonlocal perturbation on $(0,\infty)$.
- **Ersland–Jakobsen** (2021), time-dependent on $\mathbb{R}^d$, order $\alpha \in (1,2)$.

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Formally, it is easy to show that U is the unique solution of the **master equation**:

$$\begin{cases} \partial_t U(t,x,m) = & -\mathcal{L}_x U(t,x,m) + H(x, D_x U(t,x,m)) - F(x,m) \\ & + \int_{\mathbb{R}^d} D_y \frac{\delta U}{\delta m}(t,x,m,y) H_p(y, D_y U(t,y,m))\, m(dy) \\ & - \int_{\mathbb{R}^d} \mathcal{L}_y \frac{\delta U}{\delta m}(t,x,m,y)\, m(dy) \quad \text{in } (0,T) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d), \\ U(T,x,m) = & G(x,m) \quad \text{in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d). \end{cases} \tag{ME}$$

The space of probability measures

$\mathcal{P}(\mathbb{R}^d)$ is the space of all probability measures on $\mathbb{R}^d$. Let $m, m' \in \mathcal{P}(\mathbb{R}^d)$.

Kantorovich–Rubinstein distance

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- d_0 is a metric for the narrow convergence of measures (tested with $C_b(\mathbb{R}^d)$).
- Most of the works on MFGs in the whole space use 1-Wasserstein or 2-Wasserstein distances, which are equivalent to weak convergence + convergence of 1, resp. 2, moments. The metric d_0 does not require any moments.

Derivative in the space of probability measures

Derivative in $\mathcal{P}(\mathbb{R}^d)$

We say that $V\colon \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is C^1 if there exists a mapping $\frac{\delta V}{\delta m}\colon \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$, bounded and continuous in both variables, such that for all $m, m' \in \mathcal{P}(\mathbb{R}^d)$,

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- Similar to the Gateaux derivative, but the space is not linear.
- The above definition does not give uniqueness of $\frac{\delta V}{\delta m}$.

Most relevant references on the master equation

- Cardaliaguet–Delarue–Lasry–Lions, chapter 3. **Torus/periodic boundary conditions**.
- M. Ricciardi. The master equation in a **bounded domain with Neumann conditions**. *Comm. PDE* (2022).
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All the results above are for local diffusions.

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Our contribution to the well-posedness of the master equation:

- Nonlocal, local and mixed diffusions.
- Handling the whole space for probability measures without moment conditions, using analytic methods (new even for $\mathcal{L} = \Delta$).

Assumptions on the heat kernel

We adopt the following order condition for $\mathcal{L}$ from Ersland and Jakobsen:

There is $\mathcal{K} > 0$ and $\alpha \in (1, 2]$, such that the **heat kernels** K and K^* of $\mathcal{L}$ and $\mathcal{L}^*$ respectively are smooth densities of probability measures, and for $\tilde{K} \in \{K, K^*\}$ and $\beta \geqslant 0$ we have

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<u>Examples:</u>

- $\mathcal{L} = (-\Delta)^{\alpha/2}$ for $\alpha \in (1, 2]$,
- $\nu(z) \approx |z|^{-d-\alpha}$ for $|z| \leqslant 1$, $\alpha \in (1, 2)$, (Grzywny–Szczypkowski, *Forum Math.* 2020)
- $\mathcal{L} = (\partial^2_{x_1 x_1})^{\alpha_1/2} + (\partial^2_{x_2 x_2})^{\alpha_2/2} + \ldots + (\partial^2_{x_d x_d})^{\alpha_d/2}$ for $\alpha_1, \alpha_2, \ldots, \alpha_d > 1$,
- $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, where $\mathcal{L}_1$ satisfies (**K**) and $\mathcal{L}_2$ is any Lévy operator.

Assumptions on H

(H1) $H\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is smooth and for every $l \in \mathbb{N}^{d+1}$ with $|l| \leqslant 4$, $\sup_{x\in\mathbb{R}^d} |D^l H(x,\cdot)|$ is locally bounded.

(H2) For every $R > 0$ there exists $C_R > 0$ such that for $x, y \in \mathbb{R}^d$ and $p \in \mathbb{R}^d$,

$$|H(x,p) - H(y,p)| \leqslant C_R(1 + |p|)|x - y|.$$

Assumptions on F, G

Note: d_0 – Rubinstein–Kantorovich distance, $\alpha \in (1,2]$ $\sim$ order of $\mathcal{L}$. $\exists \sigma \in (0, \alpha-1)$:

(F1) $F\colon \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ satisfies

$$\sup_{m\in\mathcal{P}(\mathbb{R}^d)} \|F(\cdot,m)\|_{C_b^2(\mathbb{R}^d)} < \infty,$$

$$\sup_{x\in\mathbb{R}^d,\, m\neq m'} \frac{|F(x,m)-F(x,m')|}{d_0(m,m')} < \infty.$$

(F2) There exists $C>0$ such that for all $m, m' \in \mathcal{P}(\mathbb{R}^d)$,

$$\left\| \frac{\delta F}{\delta m}(\cdot, m, \cdot) \right\|_{C_b^{2+\sigma}(\mathbb{R}^d, C_b^{2+\sigma}(\mathbb{R}^d))} \leqslant C,$$

$$\left\| \frac{\delta F}{\delta m}(\cdot, m, \cdot) - \frac{\delta F}{\delta m}(\cdot, m', \cdot) \right\|_{C_b^{2+\sigma}(\mathbb{R}^d, C_b^{2+\sigma}(\mathbb{R}^d))} \leqslant C d_0(m,m').$$

(G1) $G\colon \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ satisfies

$$\sup_{m\in\mathcal{P}(\mathbb{R}^d)} \|G(\cdot,m)\|_{C_b^{3+\sigma}(\mathbb{R}^d)} < \infty,$$

$$\sup_{x\in\mathbb{R}^d,\, m\neq m'} \frac{|G(x,m)-G(x,m')|}{d_0(m,m')} < \infty.$$

(G2) There exists $C>0$ such that for all $m, m' \in \mathcal{P}(\mathbb{R}^d)$,

$$\left\| \frac{\delta G}{\delta m}(\cdot, m, \cdot) \right\|_{C_b^{3+\sigma}(\mathbb{R}^d, C_b^{3+\sigma}(\mathbb{R}^d))} \leqslant C,$$

$$\left\| \frac{\delta G}{\delta m}(\cdot, m, \cdot) - \frac{\delta G}{\delta m}(\cdot, m', \cdot) \right\|_{C_b^{3+\sigma}(\mathbb{R}^d, C_b^{2+\sigma}(\mathbb{R}^d))} \leqslant C d_0(m,m').$$

Monotonicity conditions

(M1) The Lasry–Lions monotonicity condition holds for F and G, that is, for all $m, m' \in \mathcal{P}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} (F(x, m') - F(x, m))(m' - m)(dx) \geqslant 0,$$

$$\int_{\mathbb{R}^d} (G(x, m') - G(x, m))(m' - m)(dx) \geqslant 0.$$

(M2) (F2) and (G2) hold and for every $\rho \in C_b^{-2-\sigma}(\mathbb{R}^d) := (C_b^{2+\sigma}(\mathbb{R}^d))^*$ and $m \in \mathcal{P}(\mathbb{R}^d)$ we have

$$\left\langle \left\langle \frac{\delta F(\cdot, m, \cdot)}{\delta m}, \rho \right\rangle_y, \rho \right\rangle_x \geqslant 0,$$
$$\left\langle \left\langle \frac{\delta G(\cdot, m, \cdot)}{\delta m}, \rho \right\rangle_y, \rho \right\rangle_x \geqslant 0,$$

where $\langle \cdot, \cdot \rangle_x, \langle \cdot, \cdot \rangle_y$ are the pairings between $C_b^{2+\sigma}(\mathbb{R}^d)$ and $C_b^{-2-\sigma}(\mathbb{R}^d)$ in x and y respectively.

(M3) There exists $c_1 \geqslant 1$ such that for all $x \in \mathbb{R}^d$

$$\frac{1}{c_1} I_d \leqslant D_{pp}^2 H(x, \cdot) \leqslant c_1 I_d.$$

Digression: monotonicity conditions vs normalization of $\frac{\delta U}{\delta m}$

$$\int_{\mathbb{R}^d} (F(x,m') - F(x,m))(m'-m)(dx) \geqslant 0, \quad m, m' \in \mathcal{P}(\mathbb{R}^d), \tag{M1}$$

$$\left\langle \left\langle \frac{\delta F(\cdot, m, \cdot)}{\delta m}, \rho \right\rangle_y, \rho \right\rangle_x \geqslant 0, \quad \rho \in C_b^{-2-\sigma}(\mathbb{R}^d). \tag{M2}$$

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The following condition is often used in the literature to ensure uniqueness of $\frac{\delta U}{\delta m}$:

$$\int \frac{\delta U}{\delta m}(m,y)\, m(dx) = 0, \quad m \in \mathcal{P}(\mathbb{R}^d). \tag{1}$$

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Example

If $\rho \in C_c^\infty(\mathbb{R}^d)$ and $F(x,m) = \rho * m(x)$, then under (1),

$$\frac{\delta F}{\delta m}(x,m,y) = \rho(x-y) - \rho * m(x).$$

For nontrivial odd ϕ (M1) is always satisfied, but (M2) is never satisfied.

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- We do not adopt condition (1).

Main results — well-posedness for the MFG system

Theorem (Well-posedness of the MFG system)

Assume that (H1), (H2), (K), (F1), *and* (G1) *hold. Then,*

- *for any* $m_0 \in \mathcal{P}(\mathbb{R}^d)$ *the system* (MFG) *has a solution* (u, m) *such that*

$$\|\partial_t u\|_{L^\infty(\mathbb{R}^d)} + \sup_{t\in[t_0,T]} \|u(t,\cdot)\|_{C_b^{3+\sigma}(\mathbb{R}^d)} \leqslant C(d, T, F, G, H, \mathcal{L}, \sigma),$$

$$d_0(m(t), m(s)) \leqslant C(d, T, F, G, H, \mathcal{L})|t-s|^{\frac{1}{2}}, \quad t, s \in [t_0, T].$$

- *If in addition* (M1) *and* (M3) *are true, then the solution is unique.*

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We allow $H = H(x, u, p)$ here under appropriate additional assumptions. Uniqueness follows from a modified monotonicity argument, but it seems too weak to obtain stability needed for the master equation.

Main results — well-posedness for the master equation

Theorem (Well-posedness for the master equation)

Assume that (H1), (H2), (K), (F1), (F2), (G1), (G2), (M1), (M2), *and* (M3) *hold and let* (u, m) *be the solution to the MFG system on* (t_0, T) *with initial measure* $m_0 \in \mathcal{P}(\mathbb{R}^d)$*. Then* U *defined as*

$$U(t_0, x, m_0) = u(t_0, x)$$

is the unique classical solution of the master equation

$$\begin{cases} \partial_t U(t,x,m) = & -\mathcal{L}_x U(t,x,m) + H(x, D_x U(t,x,m)) - F(x,m) \\ & + \int_{\mathbb{R}^d} D_y \frac{\delta U}{\delta m}(t,x,m,y) H_p(y, D_y U(t,y,m))\, m(dy) \\ & - \int_{\mathbb{R}^d} \mathcal{L}_y \frac{\delta U}{\delta m}(t,x,m,y)\, m(dy) \quad \text{in } (0,T) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d), \\ U(T,x,m) = & G(x,m) \quad \text{in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d). \end{cases}$$

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In the remainder of the talk we will discuss the main ingredients of the proof of the above theorem.

Auxiliary results

We use (and prove) several results for single equations.

- Schauder estimates for linear equations and Hamilton–Jacobi equations. We gain $\alpha - \varepsilon$ derivatives over f, but it seems that (**K**) might be too weak to gain α.
  Linear: Mikulevičius–Pragarauskas (1992), supercritical case: Chaudru de Raynal–Menozzi –Priola (2020), nonlinear case: Dong–Jin–Zhang (2018).

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Auxiliary results: well-posedness in L^1

Lemma

Assume (K) *and let* $V_1 \in C_b([0,T]\times\mathbb{R}^d)$, $V_2 \in L^\infty([0,T],L^1(\mathbb{R}^d))$, *and* $\rho_0 \in L^1(\mathbb{R}^d)$. *Then there exists a unique mild solution (satisfying Duhamel)* $\rho \in C([0,T],L^1(\mathbb{R}^d))$ *to*

$$\begin{cases} \partial_t \rho - \mathcal{L}\rho - div(V_1\rho) - div(V_2) = 0, & \text{in } (0,T)\times\mathbb{R}^d, \\ \rho(0) = \rho_0, & \text{in } \mathbb{R}^d. \end{cases}$$

The mild solution is also a distributional solution.

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The mild solution is also a distributional solution.

In addition to that we get (Kolmogorov–Riesz) compactness properties:

- uniform equicontinuity of translations:

$$\sup_{t \in [0,T]} \|\rho(t, \cdot + z) - \rho(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leqslant \|\rho_0(\cdot + z) - \rho_0\|_{L^1(\mathbb{R}^d)} + c|z|^{\alpha - 1},$$

- uniform equicontinuity in time: $\|\rho(t) - \rho(s)\|_{L^1(\mathbb{R}^d)} \leqslant C\omega(|t-s|)$,
- uniform tightness by a generalized moment bound.

Existence of $\frac{\delta U}{\delta m}$ and the linearized system

In order to get existence and regularity of $\frac{\delta U}{\delta m}, D_y \frac{\delta U}{\delta m}, \mathcal{L}_y \frac{\delta U}{\delta m}$ we use estimates for the following forward-backward linear system:

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Theorem (Well-posedness of the linearized system)

Assume (K), (F2), (G2), *and (roughly)*

- $\Gamma \in C([t_0, T], C_b^1(\mathbb{R}^d))$ *and* $0 \leqslant \Gamma \leqslant C\mathrm{I_d}$, $\quad V \in L^\infty([t_0, T], C_b^{2+\sigma}(\mathbb{R}^d))$,
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Then, the following system has a unique solution:

$$\begin{cases} -\partial_t z - \mathcal{L}z + V(t,x)\cdot Dz = \langle \frac{\delta F}{\delta m}(x, m(t)), \rho(t)\rangle + b(t,x) & \text{in } (t_0, T)\times\mathbb{R}^d, \\ \partial_t \rho - \mathcal{L}^*\rho - div(\rho V) - div(m\Gamma Dz + c) = 0 & \text{in } (t_0, T)\times\mathbb{R}^d, \\ z(T,x) = \langle \frac{\delta G}{\delta m}(x, m(T)), \rho(T)\rangle + z_T(x), \quad \rho(t_0) = \rho_0. \end{cases}$$

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Recall: $C_b^{-\gamma}(\mathbb{R}^d) = (C_b^\gamma(\mathbb{R}^d))^*$ for $\gamma \geqslant 0$.

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General recipe for solving: Cardaliaguet–Delarue–Lasry–Lions (Δ on torus).

Linearized system – comments

- On the proof:
  - Approximate the data and use the Leray–Schauder theorem.
    Problem: since we are in the whole space, ρ_0 and c may be so bad (e.g. Banach limits) that convolving with a C_c^∞ function does not regularize them.
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$$\rho_0 = \delta_y \qquad \Longrightarrow \qquad z(t_0, x) = \frac{\delta U}{\delta m}(t_0, x, m_0, y),$$

$$\rho_0 = \partial^\alpha \delta_y \qquad \Longrightarrow \qquad z(t_0, x) = \partial_y^\alpha \frac{\delta U}{\delta m}(t_0, x, m_0, y).$$

In the worst case we use two derivatives in y, so $|\alpha| = 2 \implies \rho_0 \in C_b^{-2}(\mathbb{R}^d)$.

Linearized system – more comments

- Irregular c appear while studying continuity in m_0 and t_0 of $\frac{\delta U}{\delta m}$, $D_y \frac{\delta U}{\delta m}$, $D_y^2 \frac{\delta U}{\delta m}$.

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- Ricciardi: in the linearized system result use L^1 instead of a uniform bound in time for $c \implies$ less regularity required from the data.
- To apply and improve/fix that idea we prove the following result.

Lemma

Assume that (K) *holds,* $V_1 \in C([0,T], C_b^2(\mathbb{R}^d))$, $V_2 \in C([0,T], (\mathcal{M}(\mathbb{R}^d), d_0))$ *and bounded in total variation, and* $\rho_0 \in C_b^{-2}(\mathbb{R}^d)$. *Then the problem*

$$\begin{cases} \partial_t \rho - \mathcal{L}\rho - div(\rho V_1) - div(V_2) = 0, & \text{on } (0,T)\times\mathbb{R}^d, \\ \rho(0) = \rho_0. \end{cases}$$

has a distributional solution ρ *such that* $\rho \in C((0,T], C_b^{\gamma-2}(\mathbb{R}^d)) \cap B([0,T], C_b^{-2}(\mathbb{R}^d))$ *for every* $\gamma \in (0,\alpha)$ *and*

$$\sup_{t\in(0,T]} \| t^{\frac{\gamma}{\alpha}} \rho(t) \|_{C_b^{\gamma-2}(\mathbb{R}^d)} \leqslant C(V_1)\Big(\sup_{t\in[0,T]} \|V_2(t)\|_{TV} + \|\rho_0\|_{C_b^{-2}(\mathbb{R}^d)} \Big).$$

If ρ_0 *is measure representable, then* $\rho(t)$ *is as well for all* $t \in [0,T]$.

Thank you for your attention!

Existence for the master equation

Recall that $U(t_0,x,m_0)=u(t_0,x)$ where (u,m) solves the system (MFG). For $h>0$,

$$\frac{U(t_0+h,x,m_0)-U(t_0,x,m_0)}{h}=\frac{U(t_0+h,x,m(t_0+h))-U(t_0,x,m_0)}{h}$$
$$-\frac{U(t_0+h,x,m(t_0+h))-U(t_0+h,x,m_0)}{h}=I_1^h-I_2^h.$$

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Note that $U(t_0 + h, x, m(t_0 + h)) = u(t_0 + h, x)$, hence by H–J,

$$I_1^h \underset{h \to 0^+}{\longrightarrow} \partial_t u(t_0, x) = -\mathcal{L}u + H(x, Du) - F(x, m).$$

Existence for the master equation

Recall that $U(t_0, x, m_0) = u(t_0, x)$ where (u, m) solves the system (MFG). For $h > 0$,

$$\frac{U(t_0+h,x,m_0) - U(t_0,x,m_0)}{h} = \frac{U(t_0+h,x,m(t_0+h)) - U(t_0,x,m_0)}{h} - \frac{U(t_0+h,x,m(t_0+h)) - U(t_0+h,x,m_0)}{h} = I_1^h - I_2^h.$$

Note that $U(t_0+h, x, m(t_0+h)) = u(t_0+h, x)$, hence by H–J,

$$I_1^h \xrightarrow[h \to 0^+]{} \partial_t u(t_0, x) = -\mathcal{L}u + H(x, Du) - F(x, m).$$

By the fundamental theorem of calculus for m and F–P $(\partial_t m - \mathcal{L}^* m - \operatorname{div}(m D_p H(x, Du)) = 0)$,

$$I_2^h = \frac{1}{h}\int_0^1 \int_{\mathbb{R}^d} \overbrace{\frac{\delta U}{\delta m}(t_0+h, x, \lambda m(t_0+h) + (1-\lambda)m_0, y)}^{t \text{ independent, use as test function in F–P}} (m(t_0+h) - m_0)(dy)\, d\lambda$$

$$\xrightarrow[h \to 0^+]{} \int_{\mathbb{R}^d} \left(H_p(y, D_y U(t,y,m)) D_y \frac{\delta U}{\delta m}(t,x,m,y) - \mathcal{L}_y \frac{\delta U}{\delta m}(t,x,m,y) \right) m(dy).$$

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$$\begin{cases} \partial_t \widetilde{m}(t) - \mathcal{L}^* \widetilde{m}(t) - \operatorname{div}\left(\widetilde{m}(t) D_p H(x, D_x V(t, x, \widetilde{m}(t))\right) = 0, & \text{in } [t_0, T] \times \mathbb{R}^d, \\ \widetilde{m}(t_0) = m_0. \end{cases}$$

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3. Then $(v, \widetilde{m})$ solves the same MFG system as (u, m), so by uniqueness for (MFG) $(u, m) = (v, \widetilde{m})$ and therefore $V = U$.