Numerical methods for fractional HJB equations: Improved error bounds and weakly degenerate equations

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Hamilton-Jacobi-Bellman Equation – introduction

Bellman equations:

$$u_t + \sup_{\alpha \in \mathcal{A}} \Big\{ f^\alpha - b^\alpha . \nabla u - tr[\sigma^\alpha \sigma^{\alpha T} D^2 u] \Big\} = 0$$
$$u(0, x) = u_0(x)$$

Controlled SDE:

$$dX_s = \sigma^{\alpha_s}(X_s) dB_s + b^{\alpha_s}(X_s) ds \quad \text{and} \quad X_t = x.$$

Define

$$u(t, x) := \sup_\alpha E\Big[\int_t^T f(\alpha_s, X_s) ds + u_0(X_T)\Big]$$

- The value function $u(t, x)$ would be the solution of dynamic programming equation which is of Bellman type.
- Terminal condition: $u(T, x) = u_0(x)$. (Change of variable $t \rightsquigarrow T - t$, to initial condition)

Hamilton-Jacobi-Bellman Equation – introduction

We consider the SDE with jump

$$dX_s = \sigma^{\alpha_s}(X_s)dB_s + b^{\alpha_s}(X_s)ds + \int_{|z|>0} \eta^{\alpha_s}(X_{s-}, z)\tilde{N}(ds, dz)$$

↑
Jump length

where $E[\tilde{N}(A, [0,1])] = \nu(A)$(Lévy measure).

The value function $u(t, x)$ would be the solution of Non-local Bellman equations

$$u_t + \sup_{\alpha \in \mathcal{A}} \left\{ -f^\alpha - b^\alpha . \nabla u - tr[\sigma^\alpha \sigma^{\alpha T} D^2 u] - \mathcal{I}^\alpha[u] \right\} = 0,$$

$$\mathcal{I}^\alpha[u] = \int_{|z|>0} \big(u(t,x) - u(t, x + \eta^\alpha(x,z)) + \eta^\alpha(x,z).\nabla u(t,x) \big) \ \nu(dz)$$

Hamilton-Jacobi-Bellman Equation – introduction

- value function of infinite horizon control $\rightarrow$ Elliptic HJB equations.

$$u(x) = \sup_{\alpha} \int_0^{\infty} e^{-\lambda} s f(\alpha_s, X_s) ds$$

solves,

$$\lambda u(x) + \sup_{\alpha \in \mathcal{A}} \Big\{ -f^{\alpha} - b^{\alpha}.\nabla u - tr[\sigma^{\alpha} \sigma^{\alpha T} D^2 u] - \mathcal{I}^{\alpha}[u] \Big\} = 0$$

- Here we will concentrate on purely non-local equation ($\sigma^{\alpha} = 0$).

Hamilton-Jacobi-Bellman Equation – introduction

Typical Assumption:

- $u_0, \sigma^\alpha, b^\alpha, \eta^\alpha, f^\alpha$ are continuous is t, x, α. And, Lipschitz in t, x uniformly in α.
- $\int_{|z|>0} \min\{1, |z|^2\}\ \nu(dz) < \infty$. eg. $\nu(dz) = \frac{dz}{|z|^{N+\sigma}}$, for $\sigma \in (0, 2)$.

  ↑ Fractional Laplacian $(-\triangle)^{\sigma/2}$

- Order σ of fractional operator if $\frac{d\nu}{dz} \sim \frac{K}{|z|^{N+\sigma}}$, $|z| < 1$.
- No uniform ellipticity condition in general unless specified.

The solutions are interpreted in viscosity sense(Not smooth!). Solutions are Hölder continuous of exponent less or equal to one in general.

Monotone Numerical Scheme

Define the approximate equation by $S_h(t, x, u_h, [u_h]) = 0$.

- **Monotonicity**: $\frac{\partial S_h}{\partial u_h} \geq 0$ and $\frac{\partial S_h}{\partial [u_h]} \leq 0$
- **Consistency**: For any smooth functions

$$|S_h(\phi, [\phi]) - \text{Eqn}(\phi)| \leq K(||D^n \phi||) o(h).$$

- L^∞**- stability**: $||u_h||_{L^\infty} \leq C$.

Convergence Result: by Barles and Souganidis(1991). General Result for fully non-linear equations.

Similar convergence results hold for non-local Bellman and Isaacs equations as well.

1st order equations: Rate of convergence

$$u_t + F(t, x, u, Du) = 0$$

Result: rate $O(\Delta t^{\frac{1}{2}} + h^{\frac{1}{2}})$

Crandal-Lions,1981 $\rightarrow$ F independent of t, x, u.

Souganidis,1984 $\rightarrow$ General case.

- Method: Doubling of Variable technique - Modification of comparison principle result.
- Works if F is not convex. Isaac Eq.
- Does not work for 2nd order equations.

Rate of convergence: 2nd order Bellman equations

$$u_t + \sup_{\alpha \in \mathcal{A}} \left\{ f^\alpha + c^\alpha u - b^\alpha . \nabla u - tr[\sigma^\alpha \sigma^{\alpha T} D^2 u] \right\} = 0$$

Method: Shaking the coefficient(Regularization)+Comparison principle + Lipschitz type bounds on numerical soln!

Available Results:

- Sub-optimal rates without Lipschitz bound on schemes:
  - First result by Krylov(1997,2000).
  - Improvement on rate by Barles and Jakobsen(2005).
  - Best known rate for general case: $O(h^{1/5})$ by Barles and Jakobsen(2007).
- Optimal error bound for specific schemes :
  - Finite Difference Schemes by Krylov(2005).
  - Semi Lagrangian schemes by Barles and Jakobsen(2002).

Rate of convergence: 2nd order Bellman equations

$$u_t + \sup_{\alpha \in \mathcal{A}} \left\{ f^\alpha + c^\alpha u - b^\alpha . \nabla u - tr[\sigma^\alpha \sigma^{\alpha T} D^2 u] \right\} = 0$$

Method: Shaking the coefficient(Regularization)+Comparison principle + Lipschitz type bounds on numerical soln!

Shaking the coefficient $\rightarrow$ Argument similar to Jensen's inequality

Limitation: Not applicable for general non-convex case.

Rate of convergence: Non-convex Case

1D: Degenerate Isaac equations: Jakobsen(2004).

N D: Bellman + Obstacle :

- Jakobsen(2006).
- Bonnans, Maroso, Zidani(2006).

The method close to Bellman case!!

Uniformly elliptic cases: Algebraic rate.

- Caffarelli and Souganidis (2008), Turanova(2015), Krylov(2015).

The method adopts eliptic regularity.(Need: Uniform elipticity assump.)
Hence not applicable for Degenerate case.

Rate of convergence: Non-local Bellman Equation

$$u_t + \sup_{\alpha}\{-f^\alpha + c^\alpha u - b^\alpha.\nabla u - \mathcal{I}^\alpha[u]\} = 0.$$

Results:

Chioma, Karlsen and Jakobsen (2008).

Biswas, Karlsen and Jakobsen (2007, 2010).

Method: Non-local adaptation of "Krylov's" Shaking the coefficient technique.

$$u_t + \underset{\substack{\uparrow \\ \text{Non-convex}}}{F}(t, x, u, \nabla u, \mathcal{I}[u]) = 0 \quad \text{(e.g. Isaacs equations)}$$

$$u_t + \sup_{\alpha,\beta}\{-f^{\alpha,\beta} + c^{\alpha,\beta} u - b^{\alpha,\beta}.\nabla u - \mathcal{I}^{\alpha,\beta}[u]\} = 0.$$

Results:

Biswas, Chowdhury and jakobsen (2019). $\mathcal{O}(h^{\frac{2-\sigma}{2\sigma}})$

Method: Adaptation of doubling of variable technique for non-local equation.

Approximation of Nonlocal operator

$$\mathcal{I}^\alpha[\phi](x) = \int_{|z|<\delta} \Big(\phi(x+\eta^\alpha(z)) - \phi(t,x) - \eta^\alpha(z)\cdot\nabla\phi(x)\Big)\,\nu_\alpha(dz) + \int_{|z|>\delta} \Big(\phi(x+\eta^\alpha(z)) - \phi(t,x)\Big)\,\nu_\alpha(dz)$$

$$:= \mathcal{I}^\alpha_\delta[\phi](x) + \mathcal{I}^{\alpha,\delta}[\phi](x).$$

$\uparrow$

singular term

- STEP 1: Approximate $\mathcal{I}^\alpha_\delta[\phi](x)$ by small diffusion

$$\mathcal{I}^\alpha_\delta[\phi](x) \rightsquigarrow \operatorname{tr}(a^\alpha_\delta D^2\phi), \quad a^\alpha_\delta = \frac{1}{2}\int_{|z|<\delta} \eta^\alpha(z)\eta^\alpha(z)^T\,\nu_\alpha(dz)$$

- STEP 2: Approximate small diffusion by monotone difference scheme $(\operatorname{tr}(a^\alpha_\delta D^2\phi) \rightsquigarrow \mathcal{L}^{\alpha,\delta}_{k,h}[\phi])$

$$\mathcal{L}^{\alpha,\delta}_{k,h}[\phi](x) = \sum_{i=1}^{N} \frac{i_h\left[\phi(x+k(\sqrt{a^\alpha_\delta})_i)\right] + i_h\left[\phi(x-k(\sqrt{a^\alpha_\delta})_i)\right] - 2\phi(x)}{2k^2}$$

Approximation of Nonlocal operator

$$i_h[\phi](x) := \sum_{j\in\mathbb{Z}^N} \omega_j(x)\phi(x_j), \quad \omega_j \geq 0, \sum_{j\in\mathbb{Z}^N} \omega_j(x) = 1$$

$\uparrow$

linear/multilinear interpolation

$$\mathcal{I}^{\alpha,\delta}[\phi](x) \rightsquigarrow \mathcal{I}_h^{\alpha,\delta}[\phi] = \sum_{\mathbf{j}\in\mathbb{Z}^N} \big(\phi(x+x_{\mathbf{j}}) - \phi(x)\big)\kappa_{h,\mathbf{j}}^{\alpha,\delta};$$

$\uparrow$

$$\kappa_{h,\mathbf{j}}^{\alpha,\delta} = \int_{|z|>\delta} \omega_{\mathbf{j}}(\eta^\alpha(z);h)\nu_\alpha(dz)$$

- Consistency Error bound:

$$\left\|\mathcal{I}^\alpha[\phi] - (\mathcal{L}_{k,h}^{\alpha,\delta}[\phi] + \mathcal{I}_h^{\alpha,\delta}[\phi])\right\|$$

$$\leq C\Big(\underbrace{\delta^{4-\sigma}\|D^4\phi\|_0}_{\text{small diffusion}} + \underbrace{\delta^{2(2-\sigma)}k^2\|D^4\phi\|_0 + \frac{h^2}{k^2}\|D^2\phi\|_0}_{\text{Monotone difference}} + \frac{h^2}{\delta^\sigma}\|D^2\phi\|_0\Big)$$

$\uparrow$

Assuming ν symmetric and η^α odd

Our Framework: Nonlocal HJB equation

$$\sup_{\alpha\in\mathcal{A}} \{f^\alpha(x) + c^\alpha(x)u(x) - \mathcal{I}^\alpha[u](x)\} = 0 \quad \text{in} \quad \mathbb{R}^N,$$

- Non-Degenerate: For every α

$$\Big(\frac{d\nu_\alpha}{dz}, \eta^\alpha(z)\Big) \to \Big(\frac{c_1}{|z|^{N+\sigma}}, c_2 z\Big) \qquad \text{for} \quad z \text{ near } 0.$$

- Strongly-Degenerate: $\mathcal{I}^\alpha$ are degenerate for every α.

$$0 \le \frac{d\nu_\alpha}{dz} \le \frac{C}{|z|^{M+\sigma}} \qquad \text{for} \quad z \text{ near } 0$$

  * It is possible to have different degeneracy for different α.

- Weakly-Degenerate: There exists atleast one α such that $\mathcal{I}^\alpha$ is nondegenerate.
  * More regular solutions than strongly-degenerate, but the equation is not uniformly elliptic (not classical in general)

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- Weakly-Degenerate: There exists atleast one α such that $\mathcal{I}^\alpha$ is nondegenerate.
  There exists $\beta > \sigma - 1$ such that $f^\alpha \in C^{1,\beta}$
  and $\|f^\alpha\|_{1,\beta} \leq K$

Our Framework: Nonlocal HJB equation

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Regularity of solutions

- Non-Degenerate: For every α

$$\left(\frac{d\nu_\alpha}{dz}, \eta^\alpha(z)\right) \to \left(\frac{c_1}{|z|^{N+\sigma}}, c_2 z\right) \qquad \text{for} \quad z \text{ near } 0. \qquad\qquad u \in C^{2\sigma+\alpha}$$

- Strongly-Degenerate: $\mathcal{I}^\alpha$ are degenerate for every α.

$$0 \le \frac{d\nu_\alpha}{dz} \le \frac{C}{|z|^{M+\sigma}} \qquad \text{for} \quad z \text{ near } 0 \qquad\qquad u \in Lip$$

- Weakly-Degenerate: There exists atleast one α such that $\mathcal{I}^\alpha$ is nondegenerate.
  There exists $\beta > \sigma - 1$ such that $f^\alpha \in C^{1,\beta}$ and $\|f^\alpha\|_{1,\beta} \le K$ $\qquad (-\triangle)^{\sigma/2}u \in L^\infty$

Main Results: Error Estimates

- Monotone Numerical Scheme:

$$\sup_{\alpha\in\mathcal{A}} \Big\{ f^\alpha(x) + c^\alpha(x)U(x) - \mathcal{L}^{\alpha,\delta}_{k,h}[U](x) - \mathcal{I}^{\alpha,\delta}_{h}[U](x) \Big\} = 0$$

Theorem (C*-Jakobsen)

(Strongly degenerate Case) *Let u be the viscosity solution and u_h the solution of the approximate equation, then there exists a constant $C > 0$ such that for any $0 < \sigma < 2$ we have*

$$|u - u_h| \le C\, h^{\frac{4-\sigma}{4+\sigma}}.$$

Theorem (C*-Jakobsen)

(Weakly-degenerate Case) *Let u be the viscosity solution and u_h the solution of the approximate equation, then there exists a constant $C > 0$ such that for any $0 < \sigma < 2$ we have*

$$|u - u_h| \le \begin{cases} C\, h^{\frac{4-\sigma}{4+\sigma}} & \textit{for} \quad 0 < \sigma \le 1 \\ C\, h^{\frac{\sigma(4-\sigma)}{4+\sigma}} & \textit{for} \quad 1 < \sigma < 2. \end{cases}$$

Main Results: Error Estimates

- **(Strongly degenerate Case)**
  - (i) For $\sigma > 1$, the rate decreases as the order σ increases.
  - (ii) The rate approaches $\frac{1}{3}$ as $\sigma \to 2^-$ and 1 as $\sigma \to 0^+$.

- **(Weakly degenerate Case)**
  - (i) For $\sigma > 1$, the rate of convergence is always more than $\mathcal{O}(h^{\frac{1}{2}})$,
  - (ii) the rate approaches $\mathcal{O}(h^{\frac{2}{3}})$ when $\sigma \to 2^-$.

- Wellposedness of approximate solutions:
  - Existence of unique solution via Banach fixed point argument
  - Monotonicity of scheme => Comparison Principle , if u_h subsolution, v_h super solution, then $u_h \leq v_h$.

Idea of the Proofs: Strongly degenerate case

- u_h is the solution of approximate problem. Let, $\big(\rho_\varepsilon\big)_{\varepsilon>0}$ be the standard mollifier on $\mathbb{R}^N$ and define $u_{\varepsilon,h} = u_h * \rho_\varepsilon$.
- Estimate for convolution

$$\|u_h - u_{\varepsilon,h}\|_0 \le \varepsilon\|Du_h\|_0 \quad \text{and,} \quad \|D^k u_{\varepsilon,h}\|_0 \le \frac{C\|u_h\|_{0,1}}{\varepsilon^{k-1}}$$

- for every α

$$f^\alpha(x) + c^\alpha(x)u_h(x) - \mathcal{L}^{\alpha,\delta}_{k,h}\, u_h(x) - \sum_{\mathbf{j}\in\mathbb{Z}^N} \big(u_h(x+x_{\mathbf{j}}) - u_h(x)\big)\kappa^{\alpha,\delta}_{h,\mathbf{j}} \le 0.$$

- Convolve by ρ_ε,

$$f^\alpha(x) + c^\alpha(x)u_{\varepsilon,h}(x) - \mathcal{I}^\alpha[u_{\varepsilon,h}](x)$$
$$\le \Big\|\mathcal{I}^\alpha[u_{\varepsilon,h}] - \big(\mathcal{L}^{\alpha,\delta}_{k,h}\, u_{\varepsilon,h} + \mathcal{I}^{\alpha,\delta}_h[u_{\varepsilon,h}]\big)\Big\|_0 + (CK^2+K)\varepsilon$$
$$= C\Big(\delta^{4-\sigma}\,\frac{1}{\varepsilon^3} + h\,\frac{1}{\varepsilon} + k^2\,\delta^{2(2-\sigma)}\,\frac{1}{\varepsilon^3} + \frac{h^2}{k^2}\frac{1}{\varepsilon} + \frac{h^2}{\delta^\sigma}\,\frac{1}{\varepsilon}\Big) + (CK^2+K)\varepsilon := M_{\varepsilon,\delta}$$

Idea of the Proofs: Strongly degenerate case

- $u_{h,\varepsilon} - \frac{C}{\lambda} M_{\varepsilon,\delta}$ is a subsolution of

$$\sup_{\alpha} \left\{ f^{\alpha}(x) + c^{\alpha}(x) v(x) - \mathcal{I}^{\alpha}[v](x) \right\} = 0$$

- By comparison result $u_{h,\varepsilon} - \frac{C}{\lambda} M_{\varepsilon,\delta} \leq u$, and hence $u_h - u \leq K(\varepsilon + M_{\varepsilon,\delta})$

- Similarly, $u - u_h \leq K(\varepsilon + M_{\varepsilon,\delta})$,
  starting with u as solution of the equation + convolution + Consistency Error bound to show $u_\varepsilon = u * p_\varepsilon$ satisfying

$$f^{\alpha}(x) + c^{\alpha}(x)\, u_\varepsilon(x) - \mathcal{L}^{\alpha,\delta}_{k,h}\, u_\varepsilon(x) - \mathcal{I}^{\alpha,\delta}_{h}[u_\varepsilon](x) \leq K(\varepsilon + M_{\varepsilon,\delta})$$

- we optimize the choice of k, δ and ε in order to get the results. First by choosing $k^2 = \frac{h\varepsilon}{\delta^{2-\sigma}}$ and then $\varepsilon = \frac{h}{\delta^{\frac{\sigma}{2}}}$

$$|u - u_h| \leq C\Big(\delta^{4+\frac{\sigma}{2}} h^{-3} + \delta^{\frac{\sigma}{2}} + \delta^2 h^{-1} + \frac{h}{\delta^{\frac{\sigma}{2}}}\Big).$$

- Result follows by choosing $\delta = h^{\frac{4}{4+\sigma}}$ for any $\sigma \in (0, 2)$. □

Idea of the Proofs: Weakly degenerate case

Main Goals:

- For viscosity and approximate solution u and u_h ,

$$(-\triangle)^{\sigma/2}u,\, \mathcal{I}_h[u_h] \in L^\infty.$$

- Improved estimates related to regularized u_h and u_h ,

$$\|u_h^{(\varepsilon)} - u_h\|_0 \approx K\varepsilon^\sigma$$

Idea of the Proofs: Weakly degenerate case

Sketch of the proof of $(-\triangle)^{\sigma/2}u \in L^\infty$:

- For simplification take $\mathcal{I}^\alpha[\phi] = a^\alpha(-\triangle)^{\sigma/2}u$ for $a^\alpha \geq 0$.
  define $\Delta^{\sigma,r}[\phi](x) = \int_{|z|>r} \big(\phi(x+z) - \phi(x)\big) \frac{dz}{|z|^{N+\sigma}}$
- If u_r is a solution of

$$\lambda u(x) + \sup_{\alpha \in \mathcal{A}} \{f^\alpha(x) - a^\alpha \Delta^{\sigma,r}[u](x)\} = 0 \quad \text{(defined pointwise)}$$

  then $\|u - u_r\|_0 \leq C r^{1-\frac{\sigma}{2}}$
- for any $\varepsilon > 0$ there exists $\bar{\alpha} \in \mathcal{A}$ such that

$$\lambda\, u_r(x) + f^{\bar{\alpha}}(x) - a^{\bar{\alpha}} \Delta^{\sigma,r} u_r(x) \geq -\varepsilon,$$

$$\text{also,} \quad \lambda\, u_r(x+y) + f^{\bar{\alpha}}(x+y) - a^{\bar{\alpha}} \Delta^{\sigma,r} u_r(x+y) \leq 0 \quad \text{for any } y$$

- Subtracting and integrating over $|y| > r$ we get

$$-\lambda\, a^{\alpha_0} \Delta^{\sigma,r}[u_r](x) - a^{\alpha_0} \Delta^{\sigma,r}[f^{\bar{\alpha}}](x) + a^{\alpha_0} \Delta^{\sigma,r}\big[-a^\alpha \Delta^{\sigma,r}[u_r]\big](x) \geq -\tilde{\varepsilon},$$

Idea of the Proofs: Weakly degenerate case

- $-a^{\alpha_0}\Delta^{\sigma,r}[u_r]$ is a supersolution of

$$\lambda v + \sup_{\alpha\in\mathcal{A}}\{-a^{\alpha_0}\Delta^{\sigma,r}[f^\alpha](x) + a^\alpha\Delta^{\sigma,r}[v]\} = 0. \quad (-\frac{C}{\lambda} \text{ is a subsolution})$$

- By comparison principle $a^{\alpha_0}\Delta^{\sigma,r}[u_r] \leq \frac{C}{\lambda}$
- On the other hand,

$$-a^{\alpha_0}\Delta^{\sigma,r}[u_r] \leq \lambda u_r(x) + \sup_{\alpha\in\mathcal{A}}\{f^\alpha(x) - a^\alpha\Delta^{\sigma,r}[u_r](x)\} + \sup_{\alpha\in\mathcal{A}}\|f^\alpha\|_0 + \lambda\|u_r\|_0$$

- Weak degeneracy assumption, $a^{\alpha_0} > 0$, combining two inequalities $\|\Delta^{\sigma,r}[u_r]\| \leq \frac{K}{a^{\alpha_0}}$
- Using limiting argument (careful calculation as both operator and function converging) we get $(-\triangle)^{\sigma/2}u \in L^\infty$. $\square$

- for general nonlocal operator involving η^α, the proof is more involved but follows.
- Similar estimate holds for approximate solution, $\mathcal{I}_h^{\alpha_0}[u_h] \in L^\infty$.

Idea of the Proofs: Weakly degenerate case

Another crucial part:

Lemma

If $v \in C^{1,\beta}(\mathbb{R}^N)$ for $\beta \in (0,1]$ and ρ is a radial function and $v^{(\varepsilon)} = v * \rho_\varepsilon$, then for any $m \geq 2$, there are C and K, independent of ε,

$$\|v^{(\varepsilon)} - v\|_0 \leq C\varepsilon^{1+\beta}\|v\|_{1,\beta} \qquad \text{and} \qquad \|D^m v^{(\varepsilon)}\|_0 \leq \frac{K}{\varepsilon^{m-1-\beta}}\|v\|_{1,\beta}$$

- Follows from

$$|v^{(\varepsilon)}(x) - v(x)| = \Big| \int_{\mathbb{R}^N} (v(x-y) - v(x) - y \cdot \nabla v(x))\rho_\varepsilon(y)\, dy \Big|$$
$$\leq C\|v\|_{1,\beta} \int_{\mathbb{R}^N} |y|^{1+\beta}\rho_\varepsilon(y)\, dy \leq C\|v\|_{1,\beta}\varepsilon^{1+\beta}.$$

Question: Similar result involving u_h? (u_h Lipschitz and $\mathcal{I}_h^{\alpha_0} u_h \in L^\infty$)

Answer: $u_h \notin C^{1,1-\sigma}$!! To take specific function as molifier to extract from regularity structure.

Idea of the Proofs: Weakly degenerate case

- $\tilde{K}^\sigma(t,x) := \mathcal{F}^{-1}\big(e^{-t|\cdot|^\sigma}\big)(x)$ be the fractional heat kernel .
- Define the convolution $v^{[\varepsilon]} := v(\cdot) * \tilde{K}^\sigma(\varepsilon^\sigma, \cdot)(x)$
- $\tilde{K}^\sigma(t,x)$ smooth and having right properties for our estimate (Ref. Quirós talk)

Lemma

Assume $\varepsilon > 0$, $\sigma > 1$, $\beta \in (\sigma - 1, 1)$, *and* $v \in C^{1,\beta}(\mathbb{R}^N)$. *Then*

$$\|v^{[\varepsilon]} - v\|_0 \leq C\varepsilon^\sigma.$$

If $v \in C^{0,1}(\mathbb{R}^N)$, *and define* $\varepsilon_1 = \frac{\varepsilon}{2^{\frac{1}{\sigma}}}$. *Then*

$$\|D^m v^{[\varepsilon]}\|_0 \leq \frac{C}{\varepsilon^{m-1}}\|v\|_{0,1} \quad \textit{and} \quad \|D^m v^{[\varepsilon]}\|_0 \leq \frac{C}{\varepsilon^{m-\sigma}}\|v^{[\varepsilon_1]}\|_{1,\sigma-1}.$$

- Main challenge is to get precise estimate for $\|u_h^{[\varepsilon]} - u_h\|_0$
- For simplicity we again choose $\mathcal{I}^\alpha[\phi] = -a^\alpha(-\triangle)^{\sigma/2}[\phi]$ and choose the approximate operator by 'power of discrete Laplacian'

$$\left|(-\Delta_h)^{\frac{\sigma}{2}}\phi(x) - (-\Delta)^{\frac{\sigma}{2}}\phi(x)\right| \leq Ch^2\Big(\|D^4\phi\|_0 + \|\phi\|_0\Big),$$

Idea of the Proofs: Weakly degenerate case

Lemma

$$\|u_h^{[\varepsilon]} - u_h\|_0 \leq K\Big(\varepsilon^\sigma \|(-\Delta_h)^{\frac{\sigma}{2}}[u_h]\|_0 + h^{\frac{2}{4-\sigma}} \|u_h\|_{0,1}\Big).$$

Proof.

S_t is the semigroup corresponding to fractional heat Kernel and $u_h^{[\varepsilon]} = S_{\varepsilon^\sigma}(u_h)$.

$$|S_t(u_h) - S_s(u_h)| = |S_t(v) - S_s(v)| = \Big|\int_s^t \partial_r[S_r(v)]dr\Big| = \Big|\int_s^t (-\Delta)^{\frac{\sigma}{2}}[S_r(v)]dr\Big|$$

$$\leq \int_s^t K\Big(\|(-\Delta_h)^{\frac{\sigma}{2}} u_h\|_0 + \frac{h^2}{r^{\frac{3}{\sigma}}}\|u_h\|_{0,1}\Big)\, dr$$

$$\leq K\Big(t\,\|(-\Delta_h)^{\frac{\sigma}{2}} u_h\|_0 + \frac{h^2}{s^{\frac{3-\sigma}{\sigma}}}\|u_h\|_{0,1}\Big).$$

Further, we see

$$|S_s(u_h) - u_h| = \Big|\int_{\mathbb{R}^N} \Big(u_h(x - s^{\frac{1}{\sigma}} y) - u_h(x)\Big) K^\sigma(y)\, dy\Big| \leq K s^{\frac{1}{\sigma}} \|u_h\|_{0,1}.$$

The result follows by taking $t = \varepsilon^\sigma$ and $s = h^{\frac{2\sigma}{4-\sigma}}$. □

Error Bound: Weakly Degenerate case

- take a mollifier and denote $u^{(\varepsilon)} = u * \rho_\varepsilon$. Use regularity of u and estimates to get

$$\lambda\, u^{(\varepsilon)} + \sup_{\alpha\in\mathcal{A}} \left\{ f^\alpha(x) + a^\alpha(-\Delta_h)^{\frac{\sigma}{2}} u^{(\varepsilon)} \right\}$$
$$\leq K\varepsilon^\sigma + Ch^2\Big(\|D^4 u^{(\varepsilon)}\|_0 + \|u^{(\varepsilon)}\|_0 \Big).$$

- Considering as a subsolution + comparison principle + $\|u^{(\varepsilon)} - u\|_0 \leq K\varepsilon^\sigma$

$$u(x) - u_h(x) \leq K\Big(\varepsilon^\sigma + \frac{h^2}{\varepsilon^{4-\sigma}} \Big)$$

- For the lower bound (precise!) mollify u_h by the '*fractional heat kernel*'

$$u_h^{[\varepsilon]} - u \leq \frac{C}{\lambda}\Big(\varepsilon^\sigma + h^2 \|D^4 u_h^{[\varepsilon]}\|_0 + h^2 \|u_h^{[\varepsilon]}\|_0 \Big).$$

- We use the estimates $\|D^m u_h^{[\varepsilon]}\|_0$ and $\|u_h^{[\varepsilon]} - u_h\|_0$

$$u_h - u \leq C\Big(\varepsilon^\sigma + h^{\frac{2}{4-\sigma}} + \frac{h^2}{\varepsilon^{4-\sigma}}\Big(1 + \frac{h^2}{\varepsilon^3}\Big)\Big) = C\Big(\varepsilon^\sigma + h^{\frac{2}{4-\sigma}} + \frac{h^2}{\varepsilon^{4-\sigma}} + \frac{h^4}{\varepsilon^{7-\sigma}} \Big).$$

- Choose $\varepsilon = h^{\frac{1}{2}}$ to show $\|u_h - u\|_0 \leq K h^{\frac{\sigma}{2}}$.

Summery:

THANK YOU