Hölder and maximal regularity for Hamilton-Jacobi equations

Marco Cirant

Università di Padova

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Goals

For $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ satisfying

$$-\mathrm{tr}(A\, D^2 u) + |Du|^\gamma \in L^q(\Omega)$$

what can be said about Hölder regularity of u? (when $\gamma > 2, A > 0$)

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About L^q-maximal regularity? Is it true that

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For $u : \Omega \subseteq \mathbb{R}^n \times (0,T) \to \mathbb{R}$ satisfying

$$\partial_t u - \mathrm{tr}(A\, D^2 u) + |Du|^\gamma \in L^q(\Omega \times (0,T))$$

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- Maximal regularity: conjectured by P.-L. Lions $\sim$ '12-'14 to hold iff

$$q > d\frac{\gamma-1}{\gamma} \quad =: q_0$$

Gain of regularity

Since $-\Delta u = f - |Du|^\gamma$, by Calderón-Zygmund

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Using Gagliardo-Nirenberg,

$$\|Du\|_{L^{q^*}} \lesssim \|u\|^{\theta}_{W^{2,q}} [u]^{1-\theta}_{\alpha} \lesssim \left(\|Du\|^{\gamma}_{L^{\gamma q}} + \|f\|_{L^q}\right)^{\theta} [u]^{1-\theta}_{\alpha}$$

and

$$\gamma\theta < 1 \quad \Leftrightarrow \quad \alpha > \frac{\gamma - 2}{\gamma - 1}$$

Scaling

If $\qquad -\Delta u + |Du|^{\gamma} = f,$

the α-Hölder scaling $v(x) = \varepsilon^{-\alpha} u(\varepsilon x)$ solves

$$-\Delta v + \varepsilon^{(\alpha-1)\gamma+2-\alpha} |Dv|^{\gamma} = \varepsilon^{2-\alpha} f(\varepsilon x)$$

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$$-\Delta v = f_\varepsilon + o_\varepsilon(1)|Dv|^\gamma$$

α-Hölder bounds depending on L^q-norm of f, $q > d/2$: [LU].

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- Superquadratic case: $\gamma > 2$

$$|Dv|^\gamma = f_\varepsilon + o_\varepsilon(1)\Delta v$$

universal α-Hölder bounds : [Dall'Aglio-Porretta] for

$$\alpha \leq \frac{\gamma-2}{\gamma-1} := \alpha, \quad q \leq q_0 := \frac{d}{\gamma'}$$

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$\frac{\gamma-2}{\gamma-1}$-Hölder is "sharp": $u(x) = c|x|^{\frac{\gamma-2}{\gamma-1}}$ is a weak sol. of

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- $\gamma > 1$, [Capuzzo Dolcetta-Leoni-Porretta]

$$-\mathrm{tr}(A(x)D^2u) + |Du|^\gamma = f$$

Lipschitz bounds depending on $W^{1,\infty}$-norm of f, for viscosity solutions, $A \geq 0$

Gap in α-Hölder regularity, when $f \in L^q$, $\gamma > 2$ and

$$\alpha \in \left(\frac{\gamma - 2}{\gamma - 1}, 1\right), \qquad q \in \left(\frac{d}{\gamma'}, d\right)$$

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- Need a nonperturbative argument
- Need the “strength” of nondegenerate diffusion
- Need solutions that are better than weak
- $\frac{\gamma-2}{\gamma-1}$-Hölder holds up to the boundary, better estimates may not.
- This gap is crucial in the problem of maximal regularity

joint work with A. Goffi (Padova), for the model problem

$$-\Delta u + |Du|^\gamma = f$$

Theorem

Let $f \in C^1(\mathbb{T}^d)$, $\gamma > 1$,

$$q > d\frac{\gamma - 1}{\gamma} \quad \textit{and } q > 2,$$

and $u \in C^3(\mathbb{T}^d)$ *be a classical periodic solution.*
Then, there exists $K = K(\|f\|_q, \|Du\|_1, \gamma, q, d) > 0$ *such that*

$$\|D^2 u\|_{L^q(\mathbb{T}^d)} + \|\,|Du|^\gamma\,\|_{L^q(\mathbb{T}^d)} \leq K.$$

Proof via an (integral) Bernstein method: look at the equation satisfied by

$$w = g(|Du|^2) \sim |Du|$$

on its level sets, i.e. $\{w_k = (w - k)^+ \geq 0\}$:

$$-\Delta w_k + \gamma |Du|^{\gamma-2} Du \cdot Dw_k + \frac{|D^2u|^2}{|Du|} \leq Df \cdot \frac{Du}{|Du|}.$$

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Equation can be plugged in

$$|D^2 u|^2 \geq |\Delta u|^2 = (|Du|^\gamma - f)^2$$

to yield

$$-\Delta w_k + w^{2\gamma-1} \leq Df \cdot \frac{Du}{|Du|} + \frac{f^2}{|Du|} - w^{\gamma-1} |Dw_k|.$$

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- the proof needs regular solutions
- Bernstein needs $f \in L^q$, $q > 2$
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- handling general x dependencies, e.g. $-\mathrm{tr}(A(x)D^2u) + H(x, Du)$, might be painful
- the argument may break down for different operators div form is ok, but nonlocal, parabolic, … ??

different approach?

back to Hölder

need to improve the known $\frac{\gamma-2}{\gamma-1}$-Hölder regularity.

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A remarkable Liouville theorem

Lemma ([Lions, 85])

Let A_0 be a constant, symmetric and positive definite matrix, $h_0 > 0$, and $w \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$, $q > d/\gamma'$, solve

$$-\mathrm{tr}\left(A_0 D^2 w\right) + h_0 |Dw|^\gamma = 0 \qquad \text{in } \mathbb{R}^d.$$

Then w is constant.

Note: no need of growth/sign conditions on w.

joint work with G. Verzini, for the problem

$$-\mathrm{tr}(A(x)D^2u) + H(x, Du) = f(x)$$

where

$$A \in C \cap W^{1,d}, \qquad H(x, Du) = h(x)|Du|^\gamma + \ldots$$

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Theorem

Let $q > \frac{d}{\gamma'}$. For every $M \geq 0$ there exists C such that if $u \in W^{2,q}(\Omega)$ is a strong solution, with $\|f\|_q \leq M$, then

$$\sup_{\bar{x} \neq x} \left(\mathrm{dist}(\bar{x}, \partial\Omega) \wedge \mathrm{dist}(x, \partial\Omega)\right)^{\alpha - \alpha_0} \frac{|u(\bar{x}) - u(x)|}{|\bar{x} - x|^\alpha} \leq C,$$

where

$$\alpha = 2 - \frac{N}{q} \ \wedge \ 1 \qquad > \alpha_0 = \frac{\gamma - 2}{\gamma - 1}$$

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As a straightforward consequence, we obtain a local maximal regularity result

Proof. By contradiction, pick a sequence s.t.

- $-\mathrm{tr}\left(A(x)D^2u_n\right) + H(x, Du_n) = f_n(x)$;
- $\|f_n\|_q \le M$;
- $r_n = |\bar{x}_n - x_n|$, $\left(\mathrm{d}(\bar{x}_n, \partial\Omega)\right)^{\alpha-\alpha_0} \frac{|u_n(x_n)|}{r_n^{\alpha}} \to +\infty$ as $n \to +\infty$.

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and define

$$w_n(y) := \frac{1}{|u_n(x_n)|} u_n(\bar{x}_n + r_n y), \qquad y \in \Omega_n := \frac{\Omega - \bar{x}_n}{r_n}.$$

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Step 1: $\frac{\mathrm{d}(\bar{x}_n, \partial\Omega)}{r_n} \to +\infty$, hence $\Omega_n \to \mathbb{R}^d$.
This is a consequence of $\frac{\gamma-2}{\gamma-1}$-Hölder estimates by [Dall'Aglio-Porretta]

Step 2: w_n solves

$$-\mathrm{tr}\left(A_n(y)D^2 w_n\right) + H_n\left(y, Dw_n\right) = g_n(y) \qquad \text{in } \Omega_n,$$

and

$$H_n\left(y, Dw_n\right) \sim \left(\frac{|u_n(x_n)|}{r_n^{\frac{\gamma-2}{\gamma-1}}}\right)^{\gamma-1} |Dw_n|^\gamma, \qquad g_n \xrightarrow{L^q} 0$$

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Step 3: w_n is locally bounded in $W^{2,q}$ by an interpolation argument $\rightsquigarrow$ compactness.

Step 4: in the limit, w is a nonconstant solution of

$$-\mathrm{tr}\left(A_0 D^2 w\right) + h_0|Dw|^\gamma = 0 \qquad \text{in } \mathbb{R}^d,$$

which is impossible by Liouville.

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Suitable regularizations / truncations u_n of $c|x|^{\frac{\gamma-2}{\gamma-1}}$ satisfy

$$-\Delta u_n + |Du_n|^\gamma = f_n, \qquad \|f_n\|_{L^{q_0}} \le C, \qquad \| |Du_n|^\gamma \|_{L^{q_0}} \to +\infty,$$

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Conjecture (work in progress):

$|Du|^\gamma$ remains bounded in L^q whenever
f varies in a set of uniformly L^q integrable functions.

True when $\gamma < 2$.

Parabolic

$$\partial_t u - \mathrm{tr}(A(x)D^2u) + h(x)|Du|^\gamma = f(x,t).$$

- Hölder estimates by [Cardaliaguet-Silvestre, Stokols-Vasseur], for "rough" h, A, but "incompatible" with maximal regularity .
- Hölder and maximal regularity for "nice" h, A by [C.-Goffi], under non sharp conditions on the rhs integrability:

$$f \in L^q, \qquad q \geq \bar{q} > \frac{d+2}{\gamma'}.$$

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Stationary result is based on

- $\frac{\gamma-2}{\gamma-1}$-Hölder estimates,
- Liouville theorem

both missing now.

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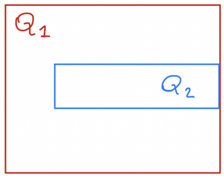
[Cardaliaguet-Silvestre] hinges on oscillation estimates:
Q_1 be the unit cylinder,



then $\quad \mathrm{osc}_{Q_2} u \leq (1-\theta)\mathrm{osc}_{Q_1} u \quad$ for suitable $Q_2 \subset Q_1$.

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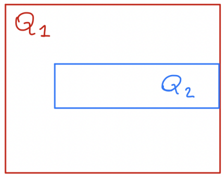

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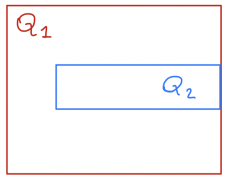
By scaling, Hölder estimates follow. Diffusion is perturbative.

Our strategy: prove diminish of suitable seminorms, that is, for $Q_2 \subset Q_1$,



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$$[\![u]\!]_{\alpha} \approx \max\left\{ \frac{|u(x,t)-u(\bar{x},t)|}{|x-\bar{x}|^{\alpha}}, \left(\frac{|u(x,t)-u(x,\bar{t})|}{|t-\bar{t}|^{\frac{\alpha}{2}}}\right)^{\frac{2}{\gamma}} \right\}$$

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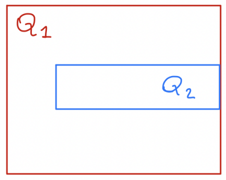

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using from the representation formula

$$u(x_0,0) = \inf_{b_s} \mathbb{E}\int_0^\tau \ell|b_s|^{\gamma'} + f(X_s,s)ds + \mathbb{E}w(X_\tau,\tau).$$

which reads

$$u(x_0,\tau) = \iint |b|^{\gamma'}\rho + \iint f\rho + \int u(0)\rho(0)$$

where

$$-\partial_t\rho - \Delta\rho + \mathrm{div}(b\rho) = 0, \qquad b = -\gamma|Du|^{\gamma-2}Du, \quad \rho(\tau) = \delta_{x_0},$$

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Crucial Lemma:

$$\|\rho\|_{L^{(d+2/\gamma')'}} \lesssim \iint |b|^{\gamma'}\rho + 1$$

+ control of ρ at the boundary of the unit cylinder.

Then, by estimating $u(x_0+h,\tau) - u(x_0,\tau)$, ...

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… we can complete the program: Hölder estimates, full maximal regularity, and Liouville theorem as a byproduct.

Ongoing work / perspectives

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- fully nonlinear problems
- nonlocal problems

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Thank you for your attention !