Spreading of populations. Travelling-wave behaviour in non-linear equations

Alejandro Gárriz (Insitut de Mathematiques de Toulouse-CNRS)

Joint work with professors Fernando Quiros (Universidad autónoma de Madrid) and Yihong Du (University of New England-Armidale), and with professor Alessandro Audrito (Politecnico di Torino)

11 de mayo de 2023

Linear models

Some history. Linear diffusion

Some history. The spreading of muskrat

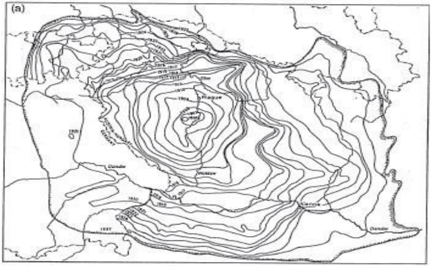
Skellam, J. G.

Random dispersal in theoretical populations.
Biometrika 38 (1951), 196–218.

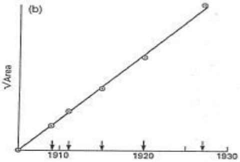
- Native to North America, brought to Europe for fur-breeding
- 1905: Five muskrats escaped from a farm near Prague
- Spreading and reproduction → entire Europe in 50 years
- Today: Millions

Some history. Skellam's Observation

$A(t)$: Area of muskat's range at time t



Range expansion of muskrat from 1905-1927 (after Elton)



Square root of area occupied by muskrat versus time (after Skellam)

Skellam's observation

$$t \to \sqrt{A(t)} \sim radius\big(A(t)\big) \text{ is linear (constant spreading speed)}$$

Some history. The PDE models

This constant spreading speed suggested that the appropiate tools to model the spreading of the population were

random dispersal + natural selection + travelling wave solutions

$$\begin{cases} u_t = \underbrace{\Delta u}_{\text{random dispersal}} + \underbrace{u(1-u)}_{\text{natural selection}}, & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases}$$

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Travelling waves with constant speed c are solutions of the form $V(\xi)$ with $\xi = x \cdot \mu - ct$, $\mu \in \mathbb{R}^N$, that satisfy

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In particular, we are interested in wavefront solutions , which are travelling waves that also satisfy

$$V(-\infty) = 1, \quad V(\infty) = 0$$

Some history. Known results

- Fisher, R. A. *The wave of advance of advantageous genes.* Ann. Eugenics 7 (1937), 355–369.
- Kolmogorov, A. N.; Petrovskii, I. G.; Piscunov, N. S. *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application á un problème biologique.* Moscow Univ. Bull. Math., Série Internat., Sec. A, Math. et Méc. 1(6) (1937), 1–25.

$$\begin{cases} u_t = \Delta u + f(u), \\ f(0) = f(1) = 0, \quad f(u) > 0, 0 < u < 1, \\ f'(0) > 0, \quad f'(u) < f'(0), 0 < u \leq 1 \end{cases}$$

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Theorem

$\exists\,!$ (up to translations) wavefront $V_c \Leftrightarrow c \geq c^* = \sqrt{2f'(0)}$

Some history. Known results

KPP went even further. Let $V_{c^*}(0) = 1/2$ (normalization) and $u_0(x)$ a Heaviside initial datum.

Theorem

Let $S(t)$, the centering term, be s.t. $u(S(t), t) = 1/2$. Then

$$u(x + S(t), t) \to V_{c^*}(x), \quad S'(t) \to c^* \text{ as } t \to \infty$$

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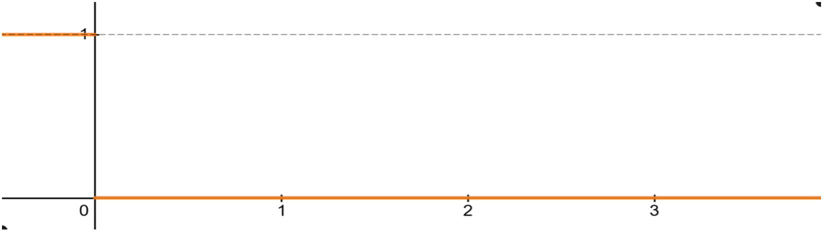

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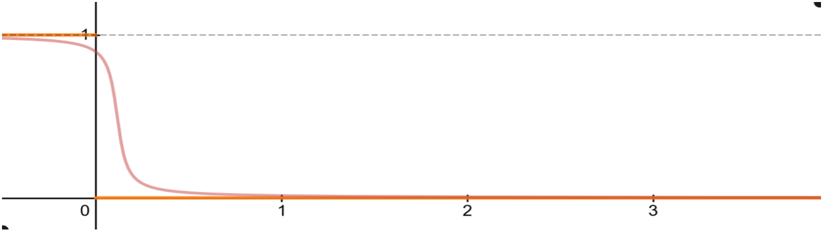

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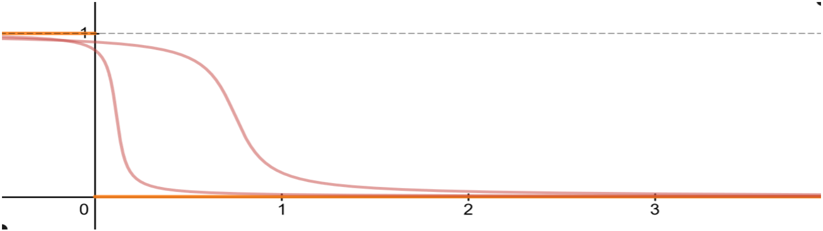

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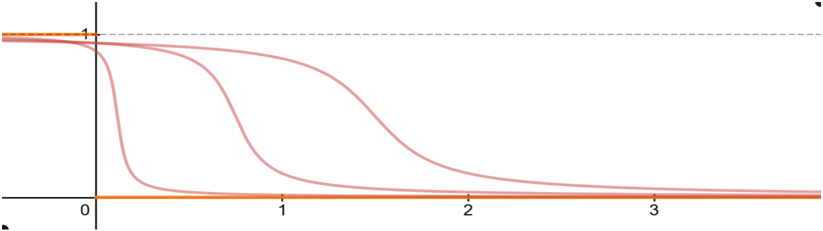

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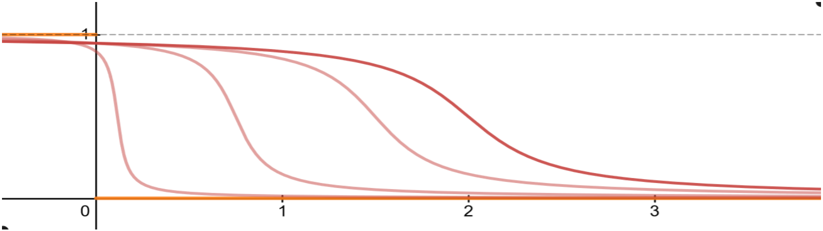

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- Dimension $N \geq 1$
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Theorem. Critical speed of propagation c^*

When there is spreading:

$$0 \leq c < c^* : \inf_{|x| \leq ct} u \to 1$$

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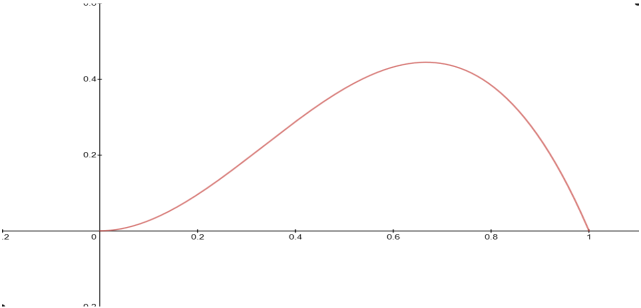
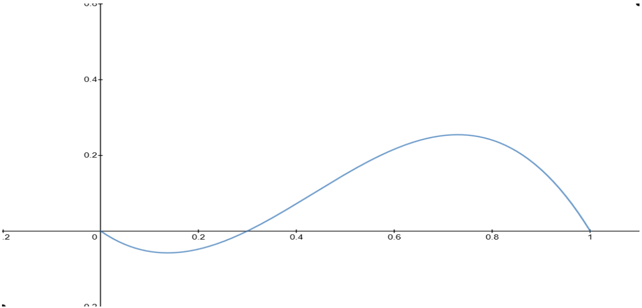
The reaction terms

Monostable reaction



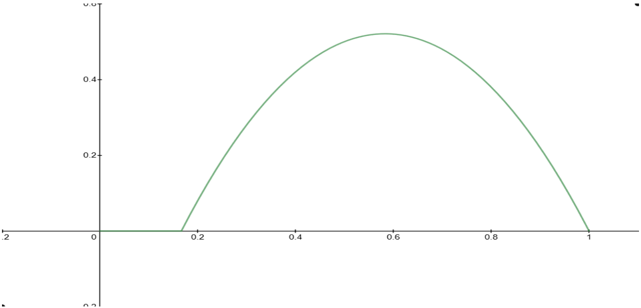
The reaction term

Bistable reaction



The reaction term

Combustion reaction



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The behaviour of the centering term $S(t)$ was also studied for other initial data and other reaction terms (which allowed $c^* > \sqrt{2f'(0)}$). We differentiate between pulled ($c^* = \sqrt{2f'(0)}$) and pushed ($c^* > \sqrt{2f'(0)}$) wavefronts.

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[Uchiyama, JMKU'1978]

Heaviside-like ini. data, pulled case $\rightarrow S(t) = c^* t - \frac{3}{2c^*} \log t + O(\log \log t)$

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Let us focus now in the convergence in dimension $N > 1$. If we consider radially symmetric initial data then we study our equation in radial coordinates $r = |x|$:

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[Roquejoffre, Rossi & Roussier-Michon, DCDS'2019]

$$u(x,t) \sim V_{c^*}\left(|x| - c^* t + \frac{N+2}{2c^*}\log t + s(x/|x|)\right)$$

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[Rossi PAMS'2017]

But s is, in general, not constant. A counter-example is showcased

Index

Nonlinear diffusion

Nonlinear Diffusion in $\mathbb{R}^N$

Nonlinear diffusion

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- Degenerate diffusion - Vanishing diffusivity. Finite speed of propagation, often called **slow** diffusion, since it maintains the finitness of the support of the solution.
- Singular diffusion - Blow-up in the diffusivity. Very fast speed of propagation, thus called **fast** diffusion. Not the focus of this talk.

Wavefronts in Nonlinear diffusion

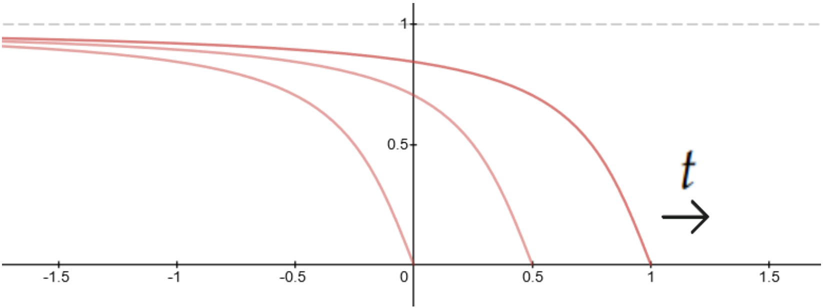
The change on the diffusivity affects, of course, the shape of the wavefront solutions of the problem. Tipically,

Typical result

$\exists$! (up to translations) wavefront $\Leftrightarrow c \geq c^* > 0$.
Moreover, in the degenerate regime, if

- $c > c^*$: Positive wavefronts ($V_{c^*} > 0$ for all $\xi \in \mathbb{R}$)
- $c = c^*$: Finite wavefronts ($V_{c^*}(\xi) \equiv 0$ for all $\xi \geq \xi^*$)

Finite Wavefront



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- **Biology** - Growth of population depending on its density and a Pearl-Verhaulst type reaction.

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- **Astronomy** - Propagation of intergalatic civilizations.

  - *Newman, W. I.; Sagan, C. Galactic civilizations: population dynamics and interstellar diffusion. Icarus 46 (1981), 293–327.*

The model

Without further delay, let us show the model that will be the focus of this talk

The model

For $m > 0, p > 1$ s.t. $m(p-1) > 1$ (**slow diff.**) , let us study the equation

$$\begin{cases} u_t = \Delta_p u^m + h(u) \quad \text{in } Q := \mathbb{R}^N \times \mathbb{R}_+, \\ u(\cdot, 0) = u_0 \geq 0 \quad \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

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$$\begin{cases} h(0) = 0, \quad h'(1) < 0 \\ h(u) \leq 0 \text{ if } u \in [0, a], \\ h(u) > 0 \text{ if } u \in (a, 1), \\ h(u) < 0 \text{ if } u > 1, \end{cases}$$

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Arising questions

- **Spreading VS vanishing**: When does the solution u spread the value 1 along the medium?

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- **Uniform convergence**: Which shape does the solution take when propagating?

The first step is to characterize the wavefront solutions of (1) with speed σ, a function $V(\xi)$ with $\xi = x \cdot \mu - \sigma t$, $\mu \in \mathbb{R}^N$, that satisfies

$$\Delta_p(V^m) + \sigma V' + h(V) = 0, \quad V(-\infty) = 1, \quad V(\infty) = 0 \tag{3}$$

Wavefronts for the equation

The existence and characterization of the TW, in particular those which are wavefronts, is a topic that deserves a presentation by itself. For the sake of brevity, let us present just the result we need for now.

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Theorem (G.): Wavefronts for equation (1)

There exists a minimal speed $\sigma^* = \sigma^*(m, p, h) > 0$ such that equation (1) has an unique (up to translations) distinct monotonic "change of fase type" wavefront satisfying

$$\lim_{\xi \to -\infty} V_{\sigma^*}(\xi) = 1, \quad V_{\sigma^*}(\xi) \equiv 0 \text{ for } \xi \geq \xi_0 \quad \text{and} \quad 0 \leq V_{\sigma^*} < 1$$

for a certain $\xi_0 \in \mathbb{R}$.

- Gárriz, A. *Singular integral equations with applications to travelling waves for doubly nonlinear diffusion.* Preprint. Available at `arXiv:2001.11109`. Accepted for publication.

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Our next result states that for every reaction h there exist certain initial data of compact support for which our solution propagates. It depends on how much mass u_0 has and how concentrated it is.

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Theorem (G., Du, Quirós): Condition on the initial data

There exists a three-parameter (σ, η, ρ) family of functions v such that if

$$u(x,0) \geq v(x - x_0; \sigma, \eta, \rho)$$

for some $x_0 \in \mathbb{R}^N$ and admissible $\sigma, \eta, \rho > 0$, then u converges to 1 uniformly on compact sets.

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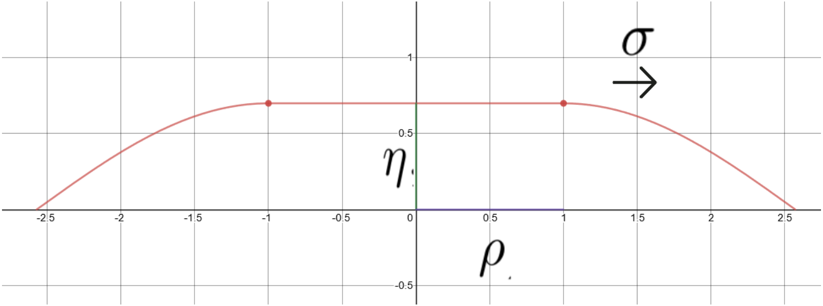
An example of a function of this three-parameter family.



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Next, we see that certain reactions always lead to propagation, regardless on the mass of the initial datum. It depends on the behaviour that h presents near $u = 0$ compared to the **Fujita exponent**. This is called the **hair-trigger effect**.

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Theorem (G., Du, Quirós): Condition on the reaction

Suppose that

$$\liminf_{u\to 0} \frac{h(u)}{u^{m(p-1)+p/N}} > 0.$$

and that $u \neq 0$.
Then u converges to 1 uniformly on compact sets.

Speed of propagation

Theorem (G., Du, Quirós): Speed of propagation

Whenever spreading happens, for any $\sigma \in (0, \sigma^*)$

$$\lim_{t\to\infty} \min_{|x-x_0|\leq \sigma t} u(x,t) = 1.$$

and for any $\sigma > \sigma^*$

$$\lim_{t\to\infty} u(x,t) = 0 \quad \text{for} \quad |x - x_0| \geq \sigma t.$$

Moving too slow will translate to $\sigma < \sigma^*$ (saturated enviroment), and too fast to $\sigma > \sigma^*$ (empty enviroment).

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Moreover, there can appear logarithmic corrections in the speed of the free boundary, as we have seen. How do we address this problem?

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Moreover, there can appear logarithmic corrections in the speed of the free boundary, as we have seen. How do we address this problem?
Remark: We maintain the slow-diffusion regime hypothesis $m(p-1) > 1$ but from here on we also assume that $p \geq 2$.

Convergence of solutions of compact support.

The main result regarding this question is the following.

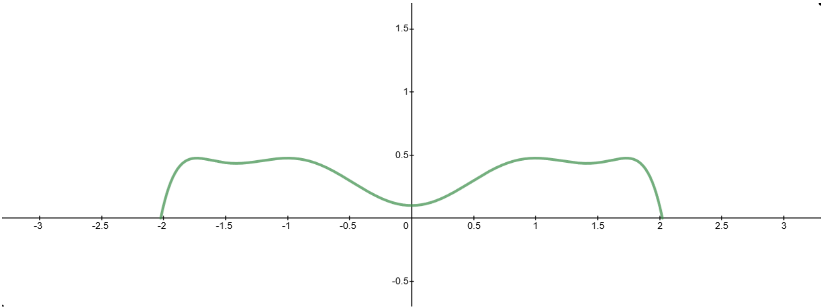
Theorem (G., Du, Quirós): Main result

Let $m(p-1) > 1$ and $p \geq 2$. Let u be a spreading solution of (1) corresponding to a bounded, radially symmetric and compactly supported initial data u_0, and let $\eta(t)$ be the function describing its interface. Then there is a constant r_0 such that

$$\lim_{t\to\infty} \sup_{r\geq 0} |u(r,t) - V_{\sigma^*}(r - \sigma^* t + (N-1)\sigma_\# \log t - r_0)| = 0 \quad \text{and}$$
$$\lim_{t\to\infty} \eta(t) - \sigma^* t + (N-1)\sigma_\# \log t = r_0,$$

where $\sigma_\#$ is a certain positive constant.

Convergence of solutions of compact support.



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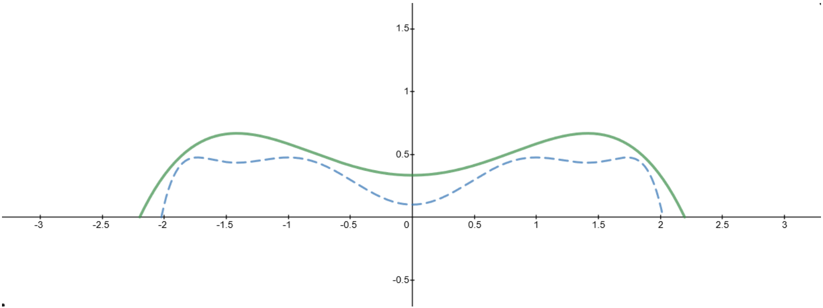

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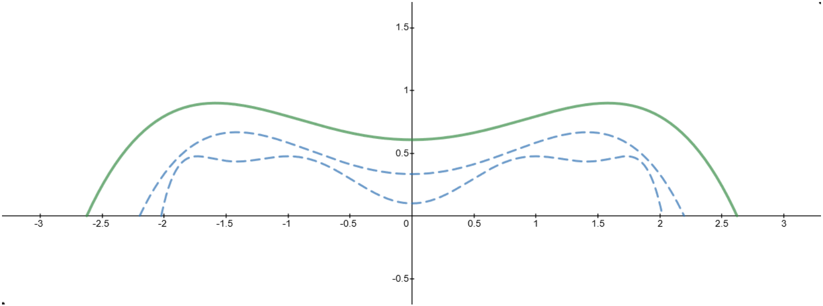

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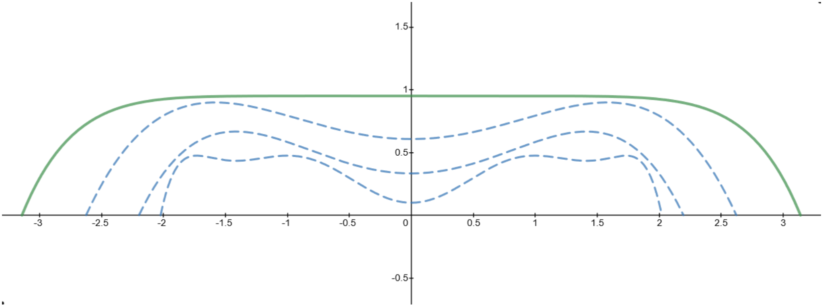

Convergence of solutions of compact support.

The key idea for the proof is the following. Let us consider the equation in radial coordinates. Taking $r := |x|$ we get

$$u_t = \left(|(u^m)_r|^{p-2}(u^m)_r\right)_r + \frac{N-1}{r}|(u^m)_r|^{p-2}(u^m)_r + h(u),$$

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but near the free boundary, due to the previous result, $r \sim \sigma^* t$, and hence

$$\frac{N-1}{r} \approx \gamma(t) := \frac{N-1}{\sigma^* t}.$$

Convergence of solutions of compact support.

Therefore it is natural to study the equation

$$u_t = \left(|(u^m)_r|^{p-2}(u^m)_r\right)_r + \gamma|(u^m)_r|^{p-2}(u^m)_r + h(u), \tag{4}$$

for a fixed small $\gamma > 0$ (since t will be big), and we can show that it has finite wavefront with speed $\sigma(\gamma)$.

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$$\eta(t) = \text{ínf}\{r > 0 : u(x,t) = 0 \text{ if } |x| > r\}$$

we conjecture that

$$\eta'(t) \approx \sigma(\gamma(t)) \approx \sigma(0) + \sigma'(0)\gamma(t) = \sigma^* + \sigma'(0)\frac{N-1}{\sigma^* t}$$

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for large times, and hence

$$\eta(t) \approx \sigma^* t - (N-1)\sigma_{\#} \log t \quad \text{as } t \to \infty,$$

with $\sigma_{\#} = -\sigma'(0)/\sigma^* > 0$.

Convergence of solutions of compact support.

It is here where we see a logarithmic correction (similar to Stokes') appearing, for dimensions bigger than 1, in the free boundary term. This correction is provoked, from an analytical point of view, by the extra convection term in radial coordinates.

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Notice also that, most notably, there are no pulled solutions . The compactness of the support forbids the *pioneers to pull from the front*. Every solution is *pushed forward by what's behind the front*.

Bibliography

Bibliography:

The results presented here can be found in

- Du, Y.; Gárriz, A.; Quirós, F. *Travelling-wave behaviour in doubly nonlinear reaction-diffusion equations.* Preprint. Available at `arXiv:2009.12959`

Nonlinear Diffusion

Nonlinear Diffusion in a tubular domain

The problem posed in the tube

This time we consider a bounded domain $D \subset \mathbb{R}^N$ and study the equation

$$\begin{cases} u_t = \Delta u^m, \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ u = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases}$$

where $\Omega = D \times \mathbb{R} \subset \mathbb{R}^{N+1}$.

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where $\Omega = D \times \mathbb{R} \subset \mathbb{R}^{N+1}$.
We consider $v(x,\tau) = t(\tau)^{\frac{1}{m-1}} u(x,\tau), \tau = \ln t$ and thus the equation becomes

$$\begin{cases} v_\tau = \Delta v^m + \frac{1}{m-1} v, \quad (x,t) \in \Omega \times \mathbb{R}, \\ v = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}, \end{cases}$$

with initial data $v_0(x)$ compactly supported in the tube.

The problem posed in the tube

The key idea here is to consider separated variables $x = (z, y) \in D \times \mathbb{R}$ and show that there exists a solution $\varphi(z, y, \tau) = \varphi(z, y - c^*\tau)$ that we call *travelling wave in the tube* and satisfies, for $\xi = y - c^*t$,

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$$\Delta_x \varphi + c^* \partial_\xi \varphi + \frac{1}{m-1} \varphi = 0,$$

having also the properties

$$\partial_\xi \varphi \leq 0, \quad \lim_{\xi \to -\infty} \frac{\varphi(z,\xi)}{\Phi(z)} = 1 \quad \text{and} \quad \sup_{z \in D} \varphi(z, \xi) = 0 \text{ for all } \xi \geq \xi_0$$

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for a certain $\xi_0 \in \mathbb{R}$, where $\Phi(z)$ is the solution to the problem

$$\begin{cases} \Delta \Phi^m + \frac{1}{m-1}\Phi = 0, \quad z \in D \\ \Phi = 0, \quad z \in \partial D \end{cases}$$

The problem posed in the tube

The existence of tavelling-wave solutions was studied in [Vázquez, CCM'2007] and the characterization of the critical speed c^* and the long time behaviour of solutions away from the boundaries in [Gilding & Goncerzewicz, IFB'2015], but two key questions remained open:

- Uniform convergence in the whole domain Ω for non-convex free boundaries
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- Uniform convergence in the whole domain Ω for non-convex free boundaries
- Convergence in relative error to the profile $\Phi(z)$ in bounded domains (ACHIEVED)

The last question was answered in a recent work with A. Audrito and F. Quirós:

- Audrito, A.; Gárriz, A.; Quirós, F. *Convergence in relative error for the Porous Medium equation in a tube.* Preprint. Available at `arXiv:2204.08224`. Accepted for publication.

The problem posed in the tube

Theorem (Audrito, G., Quirós):

The solution v satisfies, for every $c \in (0, c_*)$,

$$\lim_{\tau \to +\infty} \sup_{z \in D,\ |y| \le c\tau} \left| \frac{v(z, y, \tau)}{\Phi(z)} - 1 \right| = 0. \quad (5)$$

For every $c > c_*$, there exists $\tau_c > 0$ such that

$$v(z, y, \tau) = 0 \quad \text{in} \ \ D \times \{|y| \ge c\tau\}, \quad \forall\, \tau \ge \tau_c. \quad (6)$$

Finally, there exists $T > 0$ (depending only on m, D, u_0 and $t_0 > 0$) such that for every $\tau > T$, the free boundary of v is made by two disjoint locally Hölder hypersurfaces.

Nonlinear diffusion

Open Questions

Open questions. Nonlinear diffusion

But, to discuss open problems, let us go back to equation

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The first two questions have been recently addressed by B. Lou and M. Zhou for some particular reaction terms in the case $p = 2$, see arXiv:2211.00001 .

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At this moment I am looking forward to studying these questions in detail for the doubly non-linear diffusion equation.

Thanks

Thanks for your attention!