Optimal existence theory for a weighted PME with large data

MATTEO MURATORI

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Joint work with T. Petitt (Politecnico di Milano)

An introduction to the weighted problem

We study the following Cauchy problem for a weighted (or inhomogeneous) porous medium equation in the Euclidean space:

$$\begin{cases} \rho(x)\,\partial_t u = \Delta(u^m) & \text{in } \mathbb{R}^n \times (0,T)\,, \\ u = u_0 & \text{on } \mathbb{R}^n \times \{0\}\,, \end{cases} \tag{WPME}$$

where $m > 1$, $n \geq 3$ (for simplicity), $T \in (0, +\infty]$ is the lifetime of the solution and $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \rho)$ satisfies suitable growth assumptions.

From the physical point of view, $u \geq 0$ may represent the relative density of a substance that diffuses in a medium whose density is given by ρ, according to a nonlinear diffusion law.

This equation was originally proposed by Kamin and Rosenau [CPAM 1981] to model heat transfer in an inhomogeneous medium.

As it costs little, we will also deal with sign-changing solutions, in which case it is implicitly understood that $u^m \equiv \mathrm{sign}(u)|u|^m$.

The measurable and positive function ρ is also commonly referred to as a weight, and it is assumed to satisfy the following two-sided bound:

$$k\,(1+|x|)^{-\gamma} \le \rho(x) \le K\,|x|^{-\gamma} \qquad \text{for a.e. } x \in \mathbb{R}^n\,, \tag{W}$$

for some exponent $\gamma \in [0,2)$ and positive constants k, K.

Following e.g. [Reyes and Vázquez, CPAA 2008 and 2009] and [Kamin, Reyes and Vázquez, DCDS 2010], where the same problem was thoroughly investigated in the case of $L^1(\mathbb{R}^n,\rho)$ initial data, the requirement $\gamma < 2$ can be seen as a slow-decay assumption on ρ.

This suggests that, up to suitable "dimensional" modifications, solutions should behave similarly to their unweighted counterparts.

For instance, in the case of the exact power

$$\rho(x) \equiv |x|^{-\gamma}$$

Barenblatt solutions are still explicit and have the form

$$\mathcal{U}(x,t) = t^{-\frac{n-\gamma}{(n-\gamma)(m-1)+2-\gamma}} \left(A - b\,|x|^{2-\gamma}\, t^{-\frac{2-\gamma}{(n-\gamma)(m-1)+2-\gamma}} \right)_+^{\frac{1}{m-1}}, \qquad \text{(B)}$$

where $A > 0$ depends on the mass and $b > 0$ depends only on n, m, γ.

Such self-similar solutions were computed by Reyes and Vázquez [CPAA 2009], and in the same paper it was shown that they asymptotically attract $L^1(\mathbb{R}^n, |x|^{-\gamma})$ positive solutions.

Note that, formally, they can be obtained from the unweighted Barenblatts via the substitutions

$$n \longrightarrow n - \gamma \qquad \text{and} \qquad 2 \longrightarrow 2 - \gamma\,.$$

The large-data problem in the unweighted setting

In a remarkable paper, Bénilan, Crandall and Pierre [Indiana UMJ 1984] deeply studied the case $\rho \equiv 1$. They established that if the initial datum satisfies the average-growth condition

$$\|u_0\|_{1,1} := \sup_{R \geq 1} R^{-\frac{2}{m-1}-n} \int_{B_R} |u_0(x)| \, dx < +\infty \tag{AG}$$

then a (very weak) solution to the corresponding Cauchy problem for the PME exists, up to a lifetime

$$T \equiv T(u_0) \sim \frac{1}{\|u_0\|_{1,1}^{m-1}} \, .$$

At a pointwise level, this means that the "maximum" admissible growth rate for the nonlinear diffusion process should be

$$|u_0(x)| \sim |x|^{\frac{2}{m-1}} \qquad \text{as } |x| \to +\infty \, .$$

Such a rate is optimal from at least two viewpoints. First, the following explicit family of blow-up solutions exists:

$$u_T(x,t) = \left[\alpha \, \frac{T^{\frac{n(m-1)}{n(m-1)+2}}}{(T-t)^{\frac{n(m-1)}{n(m-1)+2}}} + b \, \frac{|x|^2}{T-t} \right]^{\frac{1}{m-1}} ,$$

where $\alpha \geq 0$ and $T > 0$ are free parameters, while $b > 0$ depends only on n, m. Note that T is actually the precise lifetime of u_T.

In particular, for $\alpha = 0$ we have a separable solution that exhibits the so-called complete blow-up:

$$\lim_{t \to T^-} u_T(x,t) = +\infty \qquad \forall x \in \mathbb{R}^n \setminus \{0\} .$$

As we will see, this kind of solutions is well suited to be adapted to the weighted setting.

Second, in a celebrated paper Aronson and Caffarelli [TAMS 1983] showed that condition (AG) is actually necessary for existence. More precisely, they proved that every nonnegative solution satisfies

$$\int_{B_R} u_0(x)\,dx \le C\left[t_0^{-\frac{1}{m-1}} R^{\frac{2}{m-1}+n} + t_0^{\frac{n}{2}} u(x_0,t_0)^{1+\frac{n(m-1)}{2}}\right] \qquad \forall R>0\,, \tag{AC}$$

where $C = C(n,m) > 0$, x_0 is arbitrary and t_0 is any time smaller than T. Note that the initial trace u_0 is in general a Radon measure.

However, it seems hard to prove the analogue of (AC) for (WPME).

As for uniqueness of “large” solutions, Bénilan-Crandall-Pierre showed it subject to an L^1_{loc} continuity assumption plus the pointwise bound

$$|u(x,t)| \le C_\varepsilon \left(1+|x|\right)^{\frac{2}{m-1}} \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times (\varepsilon, T-\varepsilon)\,,$$

for all $\varepsilon > 0$ small enough and some constant $C_\varepsilon > 0$.

Towards our problem: the notion of solution

We will mostly work with very weak solutions, in the following sense.

Definition

Let $n \geq 3$, $m > 1$, $T \in (0, +\infty]$ and ρ satisfy (W) *for some $\gamma \in [0, 2)$ and $k, K > 0$. Let $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \rho)$. We say that a function u is a (very weak) solution of problem* (WPME) *if*

$$u \in C\big([0, T); L^1_{\mathrm{loc}}(\mathbb{R}^n, \rho)\big)\,, \qquad u^m \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times (0, T))\,, \qquad u(0) = u_0$$

and

$$-\int_0^T \int_{\mathbb{R}^n} u\, \partial_t \phi\, \rho(x) dxdt = \int_0^T \int_{\mathbb{R}^n} u^m\, \Delta\phi\, dxdt$$

for all $\phi \in C^\infty_c(\mathbb{R}^n \times (0, T))$.

As we will see shortly, in fact the constructed solution enjoys better integrability and continuity properties.

Existence

Before stating our main existence result, let us set some key notations:

$$\|u_0\|_{1,r} := \sup_{R\ge r} R^{-\frac{2-\gamma}{m-1}-(n-\gamma)} \int_{B_R} |u_0(x)|\, \rho(x) dx \,,$$

$$\ell(u_0) := \lim_{r\to+\infty} \|u_0\|_{1,r} \,,$$

$$\|u_0\|_{\infty,r} := \sup_{R\ge r} R^{-\frac{2-\gamma}{m-1}} \|u_0\|_{L^\infty(B_R)} \,, \qquad \text{for each } r \ge 1 \,.$$

We also let X_1 and X_∞ denote the (Banach) spaces of all measurable functions u_0 such that $\|u_0\|_{1,r}$ and $\|u_0\|_{\infty,r}$ is finite, respectively.

Theorem E

Let $n \ge 3$, $m > 1$ and ρ satisfy (W) *for some $\gamma \in [0,2)$ and $k, K > 0$. Let $u_0 \in X_1$. Then there exists a solution u of problem* (WPME) *with*

$$T(u_0) = \frac{C_1}{[\ell(u_0)]^{m-1}} \quad \text{if } \ell(u_0) \neq 0 \,, \qquad T(u_0) = +\infty \quad \text{if } \ell(u_0) = 0 \,.$$

Theorem E (…continued)

Moreover, upon setting

$$T_r(u_0) := \frac{C_1}{\|u_0\|_{1,r}^{m-1}} \qquad \text{for each } r \ge 1 \, ,$$

for all $t \in (0, T_r(u_0))$ *we have the bounds*

$$\|u(t)\|_{1,r} \le C_2 \, \|u_0\|_{1,r} \, , \qquad \|u(t)\|_{\infty,r} \le C_3 \, t^{-\frac{n-\gamma}{(n-\gamma)(m-1)+2-\gamma}} \, \|u_0\|_{1,r}^{\frac{2-\gamma}{(n-\gamma)(m-1)+2-\gamma}} \, .$$

Furthermore, if v *is the constructed solution associated with another initial datum* $v_0 \in X_1$*, the stability estimate*

$$\|u(t) - v(t)\|_{1,r} \le C_4 \, \exp\!\left(C_5 \, t^{\frac{2-\gamma}{(n-\gamma)(m-1)+2-\gamma}} \right) \|u_0 - v_0\|_{1,r}$$

holds for all $t \in (0, T_r(u_0) \wedge T_r(v_0))$.

Here the (positive) constants C_1, C_2, C_3 *depend on* n, m, γ, k, K*, the constant* C_4 *on* n, m, γ *and the constant* C_5 *on* $n, m, \gamma, k, K, \|u_0\|_{1,r}, \|v_0\|_{1,r}$.

The overall proof strategy mimics that of Bénilan-Crandall-Pierre, the main idea being to approximate u_0 via the truncations

$$u_{0j} := (u_0 \wedge j) \vee (-j) \cdot \chi_{B_j} \in L^1(\mathbb{R}^n, \rho) \cap L^\infty(\mathbb{R}^n) \qquad \forall j \in \mathbb{N},$$

and carefully taking limits as $j \to \infty$ on the corresponding sequence of (approximate) solutions $\{u_j\}$ to (WPME).

To this aim, a key role is played by the above smoothing effect:

$$\|u(t)\|_{\infty,r} \leq C_3\, t^{-\frac{n-\gamma}{(n-\gamma)(m-1)+2-\gamma}} \|u_0\|_{1,r}^{\frac{2-\gamma}{(n-\gamma)(m-1)+2-\gamma}} . \tag{S}$$

The latter is shown to hold for $\{u_j\}$ by using a Moser-type argument that takes advantage of the weighted Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |f(x)|^{\frac{2(n-\gamma)}{n-2}} \rho(x) dx \right)^{\frac{n-2}{n-\gamma}} \leq C_{n,\gamma,K} \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx \qquad \forall f \in C_c^1(\mathbb{R}^n),$$

which easily follows from (W) by interpolation.

The starting point to prove (S) is the Bénilan-Crandall inequality

$$\rho(x)\,\partial_t u_j = \Delta\left(u_j^m\right) \geq -\frac{\rho(x)\,u_j}{(m-1)\,t}\,,$$

which is satisfied provided $u_j \geq 0$ (this restriction can however be overcome). We emphasize that such inequality holds in quite general frameworks, as it is due to a pure time scaling argument.

Instead, Bénilan-Crandall-Pierre took advantage of the (stronger) Aronson-Bénilan inequality

$$\Delta\left(u^{m-1}\right) \geq -\frac{m-1}{m}\,\frac{n}{n(m-1)+2}\,\frac{1}{t}\,,$$

which is not available in weighted settings.

The smoothing effect is fundamental to ensure the local boundedness of $\{u_j\}$, so one can pass to the limit as $j \to \infty$ (using monotonicity as well) to obtain a solution u of

$$\rho(x)\,\partial_t u = \Delta(u^m)\,.$$

How can we retrieve continuity in $L^1_{\mathrm{loc}}(\mathbb{R}^n, \rho)$? It turns out that X_1 is continuously embedded in the weighted $L^1(\mathbb{R}^n, \Phi_\alpha\, \rho)$ space provided

$$\Phi_\alpha(x) = \left(1 + |x|^2\right)^{-\alpha}, \qquad \alpha > \tfrac{2-\gamma}{2(m-1)} + \tfrac{n-\gamma}{2}\,.$$

Moreover, the stability estimate

$$\left\|u_i(t) - u_j(t)\right\|_{L^1(\mathbb{R}^n, \Phi_\alpha\, \rho)} \leq \exp\left(C_6\, t^{\frac{2-\gamma}{(n-\gamma)(m-1)+2-\gamma}} \right) \left\|u_{0i} - u_{0j}\right\|_{L^1(\mathbb{R}^n, \Phi_\alpha\, \rho)}$$

holds for all $i, j \in \mathbb{N}$ and all $t \in (0, T_r(u_0))$, for a suitable constant $C_6 > 0$ depending on $n, m, \gamma, k, K, r, \alpha, \|u_0\|_{1,r}$.

This is crucial since $\left\{u_{0j}\right\}$ is not Cauchy in X_1, but it is in $L^1(\mathbb{R}^n, \Phi_\alpha\, \rho)$.

However, if $\ell(u_0) = 0$, the sequence is also Cauchy in X_1, so that $u \in C([0, +\infty); X_1)$ and

$$\operatorname*{ess\,lim}_{|x| \to +\infty} |x|^{-\frac{2-\gamma}{m-1}}\, u(x,t) = 0 \qquad \forall t > 0\,.$$

Uniqueness

Under suitable additional assumptions, which are fulfilled by the constructed solutions, we have the following uniqueness result.

Theorem U

Let $n \geq 3$, $m > 1$, $T \in (0, +\infty]$ and ρ satisfy (W) *for some $\gamma \in [0, 2)$ and $k, K > 0$. Let u and v be any two solutions of problem* (WPME)*, corresponding to the same initial datum $u_0 \in X_1$, such that*

$$u, v \in L^\infty_{\mathrm{loc}}((0, T); X_\infty)\,, \qquad u, v \in L^\infty_{\mathrm{loc}}([0, T); X_1)\,.$$

Then $u = v$.

The strategy of proof, still inspired by Bénilan-Crandall-Pierre, takes advantage of a duality method that can be traced back to Hilbert and whose adaptation to nonlinear diffusions was first implemented by Aronson, Crandall and Peletier [NA 1982] and Pierre [NA 1982].

Such method is employed to show an important intermediate result.

Proposition

Let $n \geq 3$, $m > 1$, $T \in (0, +\infty]$ and ρ satisfy (W) *for some $\gamma \in [0,2)$ and $k, K > 0$. Let u and v be any two solutions of problem* (WPME)*, corresponding to the same initial datum $u_0 \in X_\infty$, such that*

$$|u(x,t)| \vee |v(x,t)| \leq C\,(1 + |x|)^{\frac{2-\gamma}{m-1}} \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times (0,T)\,, \qquad \text{(A)}$$

for some $C > 0$. Then $u = v$.

In order to prove it, the basic idea is to test the very weak formulation satisfied by $(u - v)$ with the solution ξ of the backward dual problem

$$\begin{cases} \rho(x)\,\partial_t \xi + a(x,t)\,\Delta\xi = 0 & \text{in } \mathbb{R}^n \times (0,T)\,, \\ \xi = \omega & \text{on } \mathbb{R}^n \times \{T\}\,, \end{cases} \qquad \text{where } a = \frac{u^m - v^m}{u - v}\,\chi_{u \neq v}$$

and $\omega \in C_c^\infty(\mathbb{R}^n)$ is an arbitrary nonnegative function. An additional issue that we have to tackle is the fact that ρ can be singular at $x = 0$.

If u is the constructed solution and v is any other solution complying with the assumptions of Theorem U, then Theorem E and the above proposition ensure that

$$\|u(t) - v(t+\epsilon)\|_{L^1(\mathbb{R}^n, \Phi_\alpha\, \rho)} \le \exp\left(C_6\, t^{\frac{2-\gamma}{(n-\gamma)(m-1)+2-\gamma}} \right) \|u_0 - v(\epsilon)\|_{L^1(\mathbb{R}^n, \Phi_\alpha\, \rho)} \quad \text{(E)}$$

for all $t, \epsilon > 0$ small enough. Moreover, the smoothing effect yields

$$|v(x, t+\epsilon)| \le C\, t^{-\frac{n-\gamma}{(n-\gamma)(m-1)+2-\gamma}} \, (1 + |x|)^{\frac{2-\gamma}{m-1}} \,,$$

for a suitable positive constant C independent of ϵ.

This crucial pointwise bound can be exploited, by means of dominated convergence and the $L^1_{\mathrm{loc}}(\mathbb{R}^n, \rho)$ continuity of v, to show that

$$\lim_{\epsilon \to 0^+} \|u_0 - v(\epsilon)\|_{L^1(\mathbb{R}^n, \Phi_\alpha\, \rho)} = 0 \,,$$

which implies $u(t) = v(t)$ due to (E). The smallness assumption on t is then removed via a standard iteration argument.

Blow-up and optimality

Inspired by the unweighted case, we can show finite-time blow-up.

Theorem B

Let $n \geq 3$, $m > 1$, $T \in (0, +\infty)$ and ρ be a radial function satisfying (W) for some $\gamma \in [0, 2)$ and $k, K > 0$. Let $u_0 \in X_\infty$ fulfill

$$W_{\beta_1}(x) \leq u_0(x) \leq W_{\beta_2}(x) \qquad \text{for a.e. } x \in \mathbb{R}^n$$

for some $\beta_2 > \beta_1 > 0$, where the positive radial function W_β solves

$$\Delta\left(W_\beta^m\right) = \tfrac{1}{T(m-1)}\, \rho\, W_\beta \quad \text{in } \mathbb{R}^n\,, \qquad W_\beta(0) = \beta > 0\,.$$

Then the solution u of problem (WPME) blows up pointwise at $t = T$, that is

$$\operatorname*{ess\,lim}_{t \to T^-} u(x,t) = +\infty \qquad \text{for a.e. } x \in \mathbb{R}^n\,.$$

Moreover, for positive constants $\underline{C}, \overline{C}$ depending only on n, m, γ, k, K, we have

$$\underline{C}\, \left[\ell(u_0)\right]^{1-m} \leq T \leq \overline{C}\, \left[\ell(u_0)\right]^{1-m}.$$

In the special case $\rho(x) \equiv |x|^{-\gamma}$, there is again an explicit family of blow-up solutions of (WPME):

$$u_T(x,t) = \left[\alpha \, \frac{T^{\frac{(n-\gamma)(m-1)}{(n-\gamma)(m-1)+2-\gamma}}}{(T-t)^{\frac{(n-\gamma)(m-1)}{(n-\gamma)(m-1)+2-\gamma}}} + b \, \frac{|x|^{2-\gamma}}{T-t} \right]^{\frac{1}{m-1}},$$

where $\alpha \geq 0$ and $T > 0$ are free parameters, while $b > 0$ depends only on n, m, γ. Note that for $\alpha = 0$ we recover a separable solution corresponding to the extremal choice $\beta = 0$ in the above theorem.

By means of related barrier techniques, we are also able to show that if $u_0 \geq 0$ has a supercritical growth, namely

$$\operatorname*{ess\,lim}_{|x|\to+\infty} \frac{u_0(x)}{|x|^{\frac{2-\gamma}{m-1}}} = +\infty \, ,$$

then there exists no nonnegative solution of (WPME).

However, it is still unknown whether a sharp result in the spirit of Aronson-Caffarelli can be retrieved.

Some work in progress on asymptotics

In a project that we are about to finish, in collaboration with T. Petitt and F. Quirós (Universidad Autónoma de Madrid), we investigate the asymptotic behavior of solutions to (WPME) for nonintegrable data.

More specifically, we consider $L^1_{\mathrm{loc}}(\mathbb{R}^n, \rho)$ initial data satisfying

$$\operatorname*{ess\,lim}_{|x|\to+\infty} |x|^\alpha \, u_0(x) = c \, , \qquad c > 0 \, ,$$

for some $\alpha \in (0, N - \gamma)$, and a density ρ that complies with (W) and behaves asymptotically like $|x|^{-\gamma}$, i.e.

$$\operatorname*{ess\,lim}_{|x|\to+\infty} |x|^\gamma \, \rho(x) = \kappa \, , \qquad \kappa > 0 \, .$$

In the unweighted case, first Alikakos and Rostamian [Israel JM 1984] and then Kamin and Ughi [JMAA 1987] proved that (if in addition $u_0 \geq 0$) the corresponding solutions converge to self-similar profiles.

More precisely, they showed (in particular) that

$$\lim_{t\to+\infty} t^{\frac{\alpha}{\alpha(m-1)+2}} \left|u(x,t) - \mathcal{U}_\alpha(x,t)\right| = 0$$

uniformly in x on balls $B_{R(t)}$ of growing radius

$$R(t) = \mathcal{R}\, t^{\frac{1}{\alpha(m-1)+2}}$$

for every $\mathcal{R} > 0$, where $\mathcal{U}_\alpha$ is the solution of the singular problem

$$\begin{cases} \partial_t \mathcal{U}_\alpha = \Delta(\mathcal{U}_\alpha^m) & \text{in } \mathbb{R}^n \times (0,+\infty)\,, \\ \mathcal{U}_\alpha = c\,|x|^{-\alpha} & \text{on } \mathbb{R}^n \times \{0\}\,. \end{cases}$$

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THANKS FOR YOUR ATTENTION!