Continuity results for the obstacle problem to porous medium type equations

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The porous medium equation (PME)

For an open set $\Omega \subset \mathbb{R}^n$, $0 < T < \infty$, and a parameter $m > 0$ consider

$$\text{(PME)} \qquad \partial_t u - \Delta\big(|u|^{m-1}u\big) = 0 \quad \text{in } \Omega_T := \Omega \times (0,T)$$

- Special case $m = 1$: (PME) is the heat equation
- Formally, (PME) $\Leftrightarrow \partial_t u - m \operatorname{div}\left(|u|^{m-1}\nabla u\right) = 0$
- Thus, (PME) is degenerate if $m > 1$ and singular if $0 < m < 1$
- In the singular case, (PME) is also known as the fast diffusion equation

Properties of local weak solutions

	singular case	degenerate case
Propagation of perturbations	infinite speed	finite speed
Compact support	impossible	possible
Locally bounded	if $m > \frac{(n-2)_+}{n+2}$	yes

Weak sub-/super-solutions

Definition

We say that u is a local weak sub-/super-solution to (PME) if

$$|u|^{m-1}u \in L^2_{\mathrm{loc}}(0,T;H^1_{\mathrm{loc}}(\Omega))$$

and

$$\iint_{\Omega_T} -u\partial_t\varphi + \nabla\left(|u|^{m-1}u\right)\cdot\nabla\varphi \,\mathrm{d}x\mathrm{d}t \leq / \geq 0$$

holds for all non-negative test functions $\varphi \in C_0^\infty(\Omega_T)$. It is a local weak solution if it is both a local weak sub- and super-solution.

Example worse than local weak super-solution

- For $m > \frac{(n-2)_+}{n}$ the Barenblatt solution is defined by

$$\mathfrak{B}(x,t) = \begin{cases} t^{-\lambda}\left(C - \frac{\lambda(m-1)}{2mn}\frac{|x|^2}{t^{\frac{2\lambda}{n}}}\right)_+^{\frac{1}{m-1}}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

  in which $\lambda = \frac{n}{n(m-1)+2}$ and $C > 0$.
- $\mathfrak{B}$ is a local weak solution to (PME) in the upper half-space $\mathbb{R}^n \times (0,\infty)$.
- However, $\mathfrak{B}$ is not a local weak super-solution in any domain containing the origin, since then $|\nabla \mathfrak{B}^m| \notin L^2$.
- Even worse examples exist.

A more general notion of super-solutions I

The Barenblatt solution is a super-solution to (PME) in the following sense:

Definition (m-supercaloric functions)

A function $u\colon \Omega_T \to (-\infty, \infty]$ is called m-supercaloric if

(i) u is lower semicontinuous in Ω_T

(ii) u is finite in a dense subset of Ω_T

(iii) u satisfies the comparison principle in interior cylinders, i.e. if $\Omega'_{t_1,t_2} \Subset \Omega_T$ and $h \in C(\overline{\Omega'}_{t_1,t_2})$ is a continuous solution with $u \geq h$ on $\partial_p \Omega'_{t_1,t_2}$, then $u \geq h$ in Ω'_{t_1,t_2}.

A more general notion of super-solutions II

We want to study

- Integrability properties
- Sobolev space properties
- Classification of these functions
- etc...

Connection between notions of super-solution

In the degenerate case $m > 1$:

Theorem (Kinnunen & Lindqvist 2008)

If u is a locally bounded m-supercaloric function in Ω_T, then it is a weak super-solution in Ω_T.

Proof sketch

1) Since u is lower semicontinuous, there exists sequence of smooth functions $(\psi_k)_{k\in\mathbb{N}}$ such that $\psi_k < \psi_{k+1} < u$ for every k and $\psi_k \to u$ pointwise.
2) Use ψ_k as an obstacle in order to find a weak super-solution u_k above ψ_k, with $u_k = \psi_k$ on the parabolic boundary.
3) Prove that $u_1 \leq u_2 \leq \ldots \leq u$. To this end, we want u_k to be continuous and that u_k is a weak solution in $\{u_k > \psi_k\}$ so that we can use a comparison principle.
4) Show that $\nabla |u_k|^{m-1}u_k \to \nabla |u|^{m-1}u$ by a compactness argument which is available due to the local boundedness of u.

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Question

Are solutions to the obstacle problem to (PME) continuous up to the boundary if the obstacle function is continuous?

Known Hölder continuity results - obstacle free case

Local regularity:

- DiBenedetto & Friedman 1985: non-negative solutions, degenerate case
- DiBenedetto, Gianazza & Vespri 2012: non-negative solutions, singular case
- Liao 2020: signed solutions, degenerate & singular case

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Regularity up to the parabolic boundary:

- DiBenedetto 1986: degenerate case
- Kinnunen, Lindqvist & Lukkari 2016: degenerate case, non-negative solutions, Perron's method
- Björn, Björn, Gianazza & Siljander 2018: non-cylindrical domains, degenerate case, non-negative solutions, barrier characterization of regular boundary points

Known Hölder continuity results - obstacle problems

Obstacle problems to quasilinear equations:

- Struwe & Vivaldi 1985: variational inequalities involving quasilinear operators with quadratic growth, Hölder regularity (interior and up to the parabolic boundary)
- Choe 1993: quasilinear equations involving operators with quadratic growth, interior $C^{1,\alpha}$-regularity

Obstacle problems to porous medium type equations: Interior Hölder continuity, non-negative solutions

- Bögelein, Lukkari & Scheven 2017: degenerate case
- Cho & Scheven 2020: singular case

Equations of porous medium type

Setting $v = |u|^{m-1}u$, (PME) is equivalent to

$$\partial_t\big(|v|^{q-1}v\big) - \Delta v = 0 \quad \text{in } \Omega_T,$$

where $q = \frac{1}{m} \rightsquigarrow$ degenerate case $0 < q < 1$, singular case $q > 1$

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More generally, consider

$$\partial_t\big(|u|^{q-1}u\big) - \mathbf{A}(x,t,u,\nabla u) = 0 \quad \text{in } \Omega_T$$

with a Carathéodory function $\mathbf{A}\colon \Omega_T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$,

$$\begin{cases} \mathbf{A}(x,t,u,\zeta)\cdot\zeta \geq C_o|\zeta|^2, \\ |\mathbf{A}(x,t,u,\zeta)| \leq C_1|\zeta| \end{cases}$$

Local solutions I

For an obstacle function ψ define

$$K_\psi(\Omega_T) := \big\{v \in C^0((0,T); L^{q+1}_{\mathrm{loc}}(\Omega)) \cap L^2_{\mathrm{loc}}(0,T; H^1_{\mathrm{loc}}(\Omega)) : v \geq \psi \text{ a.e. in } \Omega_T\big\}$$

Definition

$u \in K_\psi(\Omega_T)$ is a local weak solution to the obstacle problem if

$$\langle\!\langle \partial_t\big(|u|^{q-1}u\big), \varphi(v-u)\rangle\!\rangle + \iint_{\Omega_T} \mathbf{A}(x,t,u,\nabla u) \cdot \nabla\left(\varphi(v-u)\right) \,\mathrm{d}x\mathrm{d}t \geq 0$$

holds true for all comparison maps $v \in K_\psi(\Omega_T)$ with time derivative $\partial_t v \in L^{q+1}_{\mathrm{loc}}(\Omega_T)$ and cutoff functions $\varphi \in C_0^\infty(\Omega_T; \mathbb{R}_{\geq 0})$.

Local solutions II

The time term is defined by

$$\langle\!\langle \partial_t\big(|u|^{q-1}u\big), \varphi(v-u)\rangle\!\rangle$$
$$:= \iint_{\Omega_T} \left[\partial_t\varphi\left(\tfrac{q}{q+1}|u|^{q+1} - |u|^{q-1}uv\right) - \varphi|u|^{q-1}u\partial_t v\right] \mathrm{d}x\mathrm{d}t.$$

Boundary values

Assumptions:

- Boundary values $g \in C^0((0,T); L^{q+1}(\Omega)) \cap L^2(0,T; H^1(\Omega))$ with $\partial_t g \in L^{q+1}(\Omega)$
- Initial values $g_o \in L^{q+1}(\Omega)$
- Compatibility conditions $g \geq \psi$, $g_o \geq \psi(\cdot, 0)$ a.e.

Solutions attaining the initial/boundary values: Local solution $u \in C^0((0,T); L^{q+1}(\Omega)) \cap L^2(0,T; H^1(\Omega))$ such that

- $u - g \in L^2(0,T; H_0^1(\Omega))$
- $$\frac{1}{h} \int_0^h \int_\Omega |u - g_o|^{q+1} \,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } h \downarrow 0$$

Interior Hölder continuity

Theorem (Moring & S. 2022)

Let u be a (signed) bounded local weak solution to the obstacle problem to the porous medium type equation with $q \in (0, \infty)$ and a Hölder continuous obstacle function $\psi \in C^{0,\beta,\frac{\beta}{2}}(\Omega_T)$ for some $\beta \in (0, 1)$. Then u is locally Hölder continuous.

Continuity up to the parabolic boundary

Additional assumptions:

- Positive geometric density condition: there exist $\alpha_* \in (0,1)$ and $\rho_o > 0$, such that for all $x_o \in \partial\Omega$ and $\rho \in (0, \rho_o]$ there holds $|\Omega \cap B_\rho(x_o)| \le (1-\alpha_*)|B_\rho(x_o)|$.
- $\psi \in C^0(\overline{\Omega_T})$ with modulus of continuity ω_ψ
- $g \in C^0(\overline{\Omega_T})$, $g_o \in C^0(\overline{\Omega})$ with moduli of continuity ω_g, ω_{g_o}

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Theorem (Moring & S. 2023)

Let u be a (signed) weak solution to the obstacle problem to the porous medium type equation with $q \in (0, \infty)$ and the obstacle function and initial/boundary values as above. Then u is continuous up to the parabolic boundary with a modulus of continuity depending on n, q, C_o, C_1, $\|u\|_\infty$, α_*, ω_ψ, ω_g/ω_{g_o}.

The linear elliptic case

Consider weak solutions to

(LE) $$\operatorname{div}(\mathbf{A} \cdot Du) = 0 \quad \text{in } E,$$

where $x \mapsto \mathbf{A}(x) = (a_{i,j}(x))_{1 \le i,j \le N}$ is measurable and

$$0 < \lambda|\zeta|^2 \le \sum_{i,j=1}^{N} a_{i,j}(x)\zeta_i\zeta_j \le \Lambda|\zeta|^2 \qquad \forall \zeta \in \mathbb{R}^N \setminus \{0\}.$$

Hölder continuity: Independent results by De Giorgi (1957) and Nash (1958; + parabolic version), new proof by Moser (1960).

Structure of De Giorgi's proof

1. From L^2 to L^∞:

$$\sup_{B_\varrho(x_o)} |u| \le C \left(\fint_{B_{2\varrho}(x_o)} |u|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}$$

2. From L^∞ to $C^{0,\alpha}$:

$$\operatorname*{ess\,osc}_{B_\varrho(x_o)} u \le C \Big(\operatorname*{ess\,osc}_{B_R(x_o)} u \Big) \left(\frac{\varrho}{R} \right)^\alpha$$

for any $0 < \varrho < R$, $B_R(x_o) \Subset E$

From L^∞ to $C^{0,\alpha}$

$u \in C^{0,\alpha}$ means that the oscillation is reduced in a "dyadic" way if the radius is reduced in a "dyadic" way:

$$\exists\, c, \eta \in (0,1) : \operatorname*{ess\,osc}_{B_{cR}(x_o)} u \leq \eta \operatorname*{ess\,osc}_{B_R(x_o)} u,$$

because by iteration

$$\begin{aligned}\operatorname*{ess\,osc}_{B_{c^n R}(x_o)} u \leq \eta^n \operatorname*{ess\,osc}_{B_R(x_o)} u &= c^{n \log_c \eta} \operatorname*{ess\,osc}_{B_R(x_o)} u \\ &= \left(\frac{c^n R}{R}\right)^{\log_c \eta} \operatorname*{ess\,osc}_{B_R(x_o)} u.\end{aligned}$$

Set $\alpha := \log_c \eta$ and consider $\varrho \approx c^n R$.

Reduction of oscillation

Assume that $|u| \leq 1$ in $B_1 = B_1(0)$. De Giorgi showed that

$$|\{u_+ = 0\} \cap B_1| \geq \tfrac{1}{2}|B_1| \quad \Rightarrow \quad \exists\, k : \sup_{B_{\frac{1}{2}}} u_+ \leq 1 - 2^{-(k+1)}$$

and analogously

$$|\{u_- = 0\} \cap B_1| \geq \tfrac{1}{2}|B_1| \quad \Rightarrow \quad \exists\, k : \sup_{B_{\frac{1}{2}}} u_- \leq 1 - 2^{-(k+1)}$$

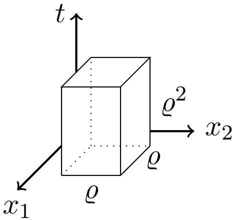
Measure theoretical alternatives on the left-hand side $\Rightarrow$ Reduction of oscillation in any case

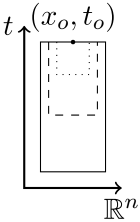
Parabolic PDEs

- De Giorgi's scheme could be extended to parabolic PDEs

$$\partial_t u - \operatorname{div}(\mathbf{A} \cdot Du) = 0 \quad \text{in } E_T$$

  by Ladyzhenskaja and Ural'ceva (1964), mainly because the equation is homogeneous (i.e. u is solution $\Rightarrow \lambda u$ with $\lambda \in \mathbb{R}$ is solution)
- Oscillation decay estimate with balls replaced by standard parabolic cylinders $K_\varrho \times (-\varrho^2, 0]$





Intrinsic scaling

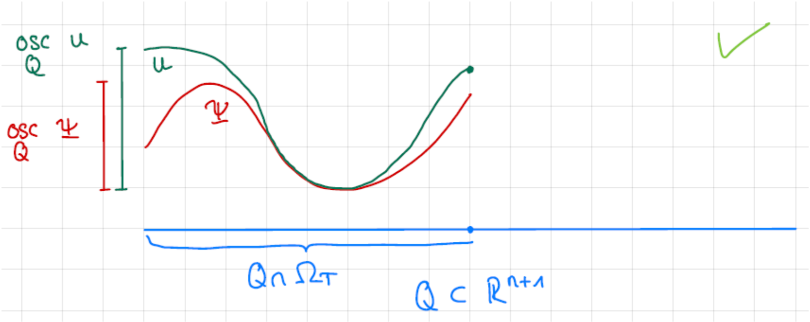
- The porous medium type equation is not homogeneous (unless $q = 1$)
- De Giorgi's scheme cannot be extended with standard parabolic cylinders $K_\varrho \times (-\varrho^2, 0]$
- Use intrinsic scaling (introduced by DiBenedetto):

$$Q_\varrho(\boldsymbol{\omega}^{q-1}) = K_\varrho \times (-\boldsymbol{\omega}^{q-1}\varrho^2, 0],$$

where

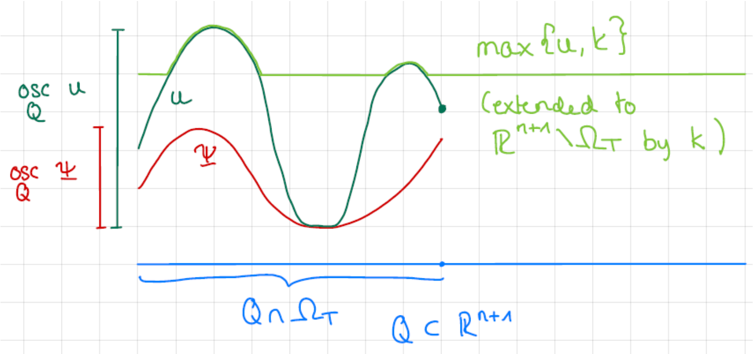
$$\operatorname*{osc}_{Q_\varrho(\boldsymbol{\omega}^{q-1})} u \leq \boldsymbol{\omega}$$

Comparison with the obstacle and boundary values I



Alternative 1: The oscillation of u is controlled by the oscillation of the obstacle function or the boundary values

Comparison with the obstacle and boundary values II



Alternative 2: For a suitable level k, $\max\{u,k\}$ ($\min\{u,k\}$) is a local sub-solution (super-solution) to the obstacle free PME in Q

Thank you for your attention!