High order adaptive methods for computing exit time of Itô–diffusions

Håkon Hoel and Sankarasubramanian Ragunathan

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UNIVERSITY OF OSLO
Mathematics for Uncertainty Quantification
RWTH AACHEN UNIVERSITY

Overview

Problem Description

- Consider the bounded, open and connected domain $D \subset \mathbb{R}^d$ and a d-dimensional process:

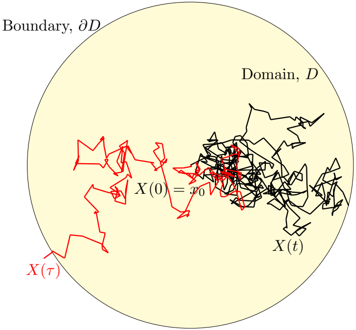
$$\begin{aligned} \mathrm{d}X(t) &= a(X(t))\,\mathrm{d}t + b(X(t))\,\mathrm{d}W(t) \\ X(0) &= x_0 \in D \end{aligned}$$

- **Goal:** Efficient numerical for approximations of exit time

$$\tau := \inf\left\{t \geq 0 \;\middle|\; X(t) \notin D\right\} \wedge T$$

where $T > 0$ is given.

- **Main ideas:**
  1. Small timestep size close to the boundary ∂D, larger elsewhere.
  2. High-order strong Itô–Taylor schemes.



Assumptions

A

The domain boundary ∂D is $\mathcal{C}^4$ continuous.

B

1. Coefficients satisfy $a \in \mathcal{C}^3(\mathbb{R}^d, \mathbb{R}^d)$ and $b \in \mathcal{C}^3(\mathbb{R}^d, \mathbb{R}^{d\times d})$
2. There exists constants $\widehat{C}_b \geq \hat{c}_b > 0$ such that

$$\hat{c}_b \, |\xi|^2 \leq \xi^\top \big(b(x) b^\top(x)\big) \xi \leq \widehat{C}_b \, |\xi|^2 , \qquad \forall (x, \xi) \in \overline{D} \times \mathbb{R}^d$$

Remark

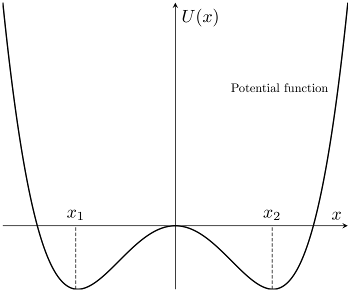
Sufficient: τ is well-defined when a, b uniformly Lipschitz, and $D \in \mathcal{B}(\mathbb{R}^d)$.

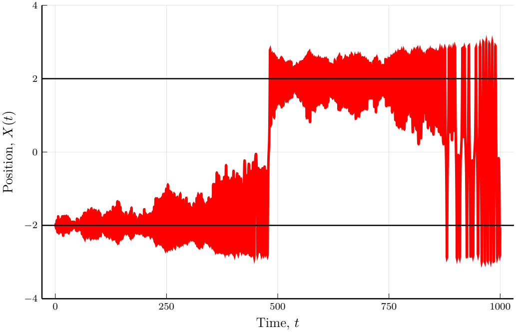
Additional regularity needed for numerical methods.

Applications: Langevin dynamics

Mean first-passage time between pseudo-stable states x_1 and x_2 for

$$dX(t) = -\nabla U(X(t))dt + dW(t), \qquad \tau := \inf\{t \geq 0 \mid X(t) = 0\}$$





Application: American put option

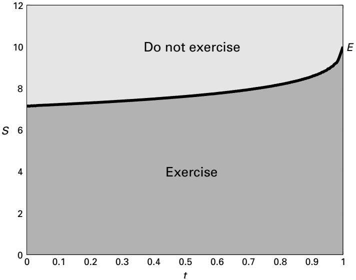
- Holder of option has the right (but not the obligation) to sell a stock $X(t)$ at price $E > 0$ any time between $[0, T]$.

- Price of this option[1]:

$$\sup_{\tau:\Omega\to[0,T]} \mathbb{E}[\max(E - X(\tau), 0)]$$

Under some assumptions, there exists $(D_t)_{t\in[0,T]} \subset \mathbb{R}$ s.t.

$$\tau(\omega) = \inf\{t \in [0, T] \mid X(t) \notin D_t\}\,.$$



[1] D. J. Higham, *An introduction to financial option valuation: mathematics, stochastics and computation*, (Cambridge University Press, 2004), vol. 13.

Methods for exit times

- Methoods
  - Feynman–Kac PDE
    - Computationally expensive as dimension d increases
  - Monte Carlo methods
    - Scales well with dimension d **(Method of choice)**
  - Walk on spheres
    - Needs good approximation of sphere hitting distributions of SDE.

The Monte Carlo method with Euler–Maruyama

Approximate true exit time by

$$\nu := \inf\left\{t \in [0, T] \;\middle|\; \overline{X}(t) \notin D\right\} \wedge T$$

where $\overline{X}(t)$ is a numerical solution of SDE

$$dX(t) = a(X(t))dt + b(X(t))dW$$

Standard method is **Euler–Maruyama**:

$$\overline{X}(t_{n+1}) = a(\overline{X}(t_n))\Delta t + b(\overline{X}(t_n))\underbrace{(W(t_{n+1}) - W(t_n))}_{\Delta W_n},$$

where $\Delta W_n \overset{iid}{\sim} N(0, \Delta t\, I_d)$.

Extension to continuous time: $\overline{X}(t) = \overline{X}(t_n) \quad \forall t \in [t_n, t_{n+1})$.

Monte Carlo methods for exit times I:

Results for $\overline{X}(t)$ and $\nu(\overline{X}(\cdot))$ computed with the **Euler–Maruyama** using constant step size $\Delta t = h$:

- Weak error[2]: $\left|\mathbb{E}\left[\nu - \tau\right]\right| = \mathcal{O}(h^{1/2})$.
  (NB! compare to $\left|\mathbb{E}\left[X(T) - \overline{X}(T)\right]\right| = \mathcal{O}(h)$.)

- Strong error[3]: $\mathbb{E}\left[|\nu - \tau|\right] = \mathcal{O}(h^{1/2})$.

- Multilevel Monte Carlo method for approximating mean-exit time[4] achieves

$$\mathbb{E}\left|E_{MLMC}[\nu] - \mathbb{E}\,\tau\right|^2 = \mathcal{O}(\mathrm{Work}^{-1}\log(\mathrm{Work})^{3/2}),$$

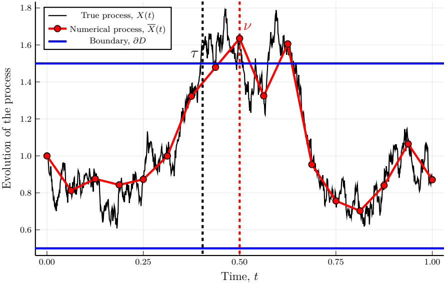
vs single level $\quad \mathbb{E}\left|E_{MC}[\nu] - \mathbb{E}\,\tau\right|^2 = \mathcal{O}(\mathrm{Work}^{-1/2})$

[2] E. Gobet, *Stochastic processes and their applications* **87**, 167–197 (2000).
[3] B. Bouchard *et al.*, *Bernoulli* **23**, 1631–1662 (2017).
[4] M. B. Giles, F. Bernal, *SIAM JUQ* **6**, 1454–1474 (2018).

Monte Carlo methods for exit times II:

- **Main source of error:** The discrete process $\overline{X}(t)$ misses first exit of the true process $X(t)$.
- **Correction through boundary shifting:** shrink the domain D for discrete problem[5][6] by magnitude $\mathcal{O}(h^{1/2})$.

Improves weak convergence rate: $$|\mathbb{E}[\nu - \tau]| = \mathcal{O}(h)$$



[5] M. Broadie *et al.*, *Mathematical finance* **7**, 325–349 (1997).

[6] E. Gobet, S. Menozzi, *Stochastic processes and their applications* **112**, 201–223 (2004).

Overview

Strong Itô–Taylor schemes

Itô SDE

$$\mathrm{d}X(t) = a(X(t))\,\mathrm{d}t + b(X(t))\,\mathrm{d}W(t)$$

Approximation

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} \underbrace{a(X(t))}_{\approx a(X(t_n))}\,\mathrm{d}t + \int_{t_n}^{t_{n+1}} \underbrace{b(X(t))}_{\approx b(X(t_n))}\,\mathrm{d}W(t)$$

motivates Euler–Maruyama scheme (Itô–Taylor strong $1/2$)

$$\overline{X}(t_{n+1}) = a(\overline{X}(t_n))\Delta t + b(\overline{X}(t_n))\Delta W_n \quad n \geq 0$$

Strong Itô–Taylor schemes

Itô SDE

$$\mathrm{d}X(t) = a(X(t))\,\mathrm{d}t + b(X(t))\,\mathrm{d}W(t)$$

Approximation

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} \underbrace{a(X(t))}_{\approx a(X(t_n))}\,\mathrm{d}t + \int_{t_n}^{t_{n+1}} \underbrace{b(X(t))}_{\approx b(X(t_n))}\,\mathrm{d}W(t)$$

Additional correction term:

$$\int_{t_n}^{t_{n+1}} b(X(t)) - b(X(t_n))\,\mathrm{d}W(t) \approx \int_{t_n}^{t_{n+1}} b'(X(t_n))b(X(t_n))(W(t) - W(t_n))\,\mathrm{d}W(t)$$

Leads to Milstein scheme (Itô–Taylor strong 1)

$$\overline{X}(t_{n+1}) = a(\overline{X}(t_n))\Delta t + b(\overline{X}(t_n))\Delta W_n + b'(\overline{X}(t_n))b(\overline{X}(t_n)) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW(s)dW(t)$$

... etc.

Overview numerical method

- We simulate the SDE using the order γ strong Itô–Taylor scheme. Notation:

$$\overline{X}(t_{n+1}) = \Psi_\gamma\big(\overline{X}(t_n), \Delta t_n\big), \qquad n = 0, 1, \dots \tag{1}$$
$$\overline{X}(0) = x_0$$

- The size of timestep $\Delta t_n = \Delta t\big(\overline{X}(t_n)\big)$ is **chosen as function of the current state $\overline{X}(t_n)$.**

- We consider $\gamma \in \{1, 1.5\}$, and "strong order γ" implies that

$$\mathbb{E}[\max_{t_n} |\overline{X}(t_n) - X(t_n)|] = \mathcal{O}(h^\gamma)$$

- **Continuous time extension of numerical solution:**

$$\overline{X}(t) = \overline{X}(t_n), \quad \forall t \in [t_n, t_{n+1})$$

Strong Itô–Taylor schemes – explicit form

If the diffusion mapping b is a diagonal matrix, then [7]

$$\begin{aligned}\big(\Psi_\gamma(\overline{X}(t_n),\Delta t_n)\big)_i = {} & \overline{X}_i(t_n) + a(\overline{X}(t_n))\Delta t_n + b_{ii}(\overline{X}(t_n))\Delta W_n^i + \\ & \frac{1}{2}\mathcal{L}^i b_{ii}(\overline{X}(t_n))\big((\Delta W_n^i)^2 - \Delta t_n\big) + \frac{1}{2}\mathcal{L}^0 a_i(\overline{X}(t_n))\Delta t_n^2 + \\ & \mathcal{L}^0 b_{ii}(\overline{X}(t_n))\big(\Delta W_n^i \Delta t_n - \Delta Z_n^i\big) + \mathcal{L}^i a_i(\overline{X}(t_n))\Delta Z_n^i \\ & + \frac{1}{2}\mathcal{L}^i\mathcal{L}^i b_{ii}(\overline{X}(t_n))\Big(\frac{1}{3}(\Delta W_n^i)^2 - \Delta t_n\Big)\Delta W_n^i\end{aligned}$$

$$\text{with} \qquad \mathcal{L}^i = b_{ii}(x)\partial_{x_i} \quad \text{and} \quad \mathcal{L}^0 = \sum_{i=1}^d a_i\partial_{x_i} + \frac{1}{2}b_{ii}^2\partial_{x_i x_i}$$

(Red part): the *Milstein* scheme, (red part) + (blue part): $\gamma = 1.5$.

NB! Cost involved in sampling the random variables $(\Delta W_n, \Delta Z_n)$ is $\mathcal{O}(1)$ where

$$\Delta Z_n^i := \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_2} \mathrm{d}W^i(s_1)\,\mathrm{d}s_2$$

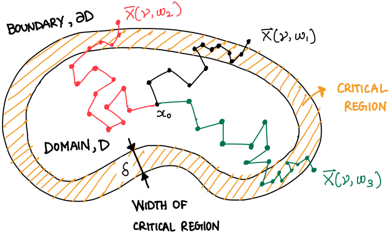
[7] Kloeden and Platen, *Numerical Solution of Stochastic Differential Equations*, 2011

Adaptive time-stepping idea

- **Step size parameter:** $h > 0$
- **Critical region parameter:** $\delta > 0$

- and the Milstein scheme:

$$\Delta t_n := \begin{cases} h, & \text{if } d\big(\overline{X}(t_n), \partial D\big) > \delta \\ h^2, & \text{if } d\big(\overline{X}(t_n), \partial D\big) \leq \delta \end{cases}$$

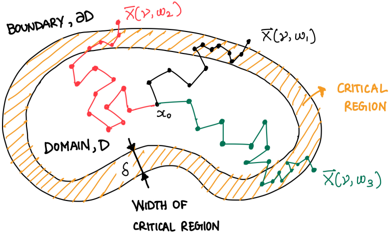


Adaptive time-stepping idea

- **Step size parameter:** $h > 0$
- **Critical region parameter:** $\delta > 0$

- and the Milstein scheme:

$$\Delta t_n := \begin{cases} h, & \text{if } d\left(\overline{X}(t_n), \partial D\right) > \delta \\ h^2, & \text{if } d\left(\overline{X}(t_n), \partial D\right) \leq \delta \end{cases}$$



- Order 1.5 scheme with $\delta_1 > \delta_2 > 0$:

$$\Delta t_n := \begin{cases} h, & \text{if } d\left(\overline{X}(t_n), \partial D\right) > \delta_1 \\ h^2, & \text{if } d\left(\overline{X}(t_n), \partial D\right) \in [\delta_2, \delta_1) \\ h^3, & \text{if } d\left(\overline{X}(t_n), \partial D\right) \leq \delta_2 \end{cases}$$

Adaptive time-stepping idea

Our goal: Achieve strong convergence rate γ:

$$\mathbb{E}\left[|\nu - \tau|\right] = \mathcal{O}(h^{\gamma}), \qquad \gamma = 1, 1.5$$

essentially at same asymptotic cost as when using uniform timesteps $\Delta t = h$, namely $\mathcal{O}(h^{-1})$.

Conflict between cost and accuracy: Critical regions(s) must be

- **large enough** to achieve accuracy goal
- **small enough** to have low computational cost

Lemma 1 (Exiting domain D using smallest timestep ($\gamma = 1$))

Given $h > 0$*, define*

$$\delta(h) := \sqrt{8 \widehat{C}_b d h \log(h^{-1})}.$$

Then probability that stepsize is largest at exit time is

$$\mathbb{P}\left(\Delta t(\overline{X}(\nu-)) = h\right) = \mathcal{O}(h^1),$$

Proof idea for $\gamma = 1$: An exit of D at time t_n with large timestep $\Delta t_n = h$ is contained in event

$$|\overline{X}(t_{n+1}) - \overline{X}(t_n)| > \delta$$

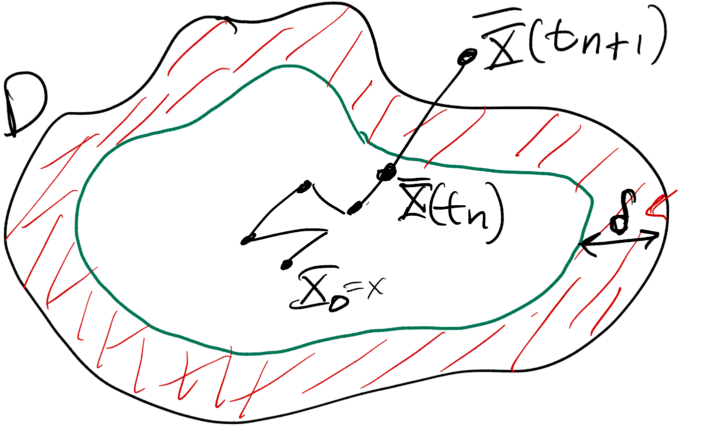
Leading order approximation

$$|\overline{X}(t_{n+1}) - \overline{X}(t_n)| = |b(\overline{X}(t_n))\Delta W_n| + \mathcal{O}(h) \lessapprox \sqrt{\widehat{C}_b}|\Delta W_n|.$$

And $\delta(h)$ chosen sufficently large to ensure that

$$\mathbb{P}(\sqrt{\widehat{C}_b}|\Delta W_n| > \delta) = \mathcal{O}(h^2).$$

$$\Delta t(\bar{X}(\nu-)) = \Delta t(\bar{X}(t_n)) = h$$



Lemma 1 (Similar exit result for $\gamma = 1.5$ method)

Given $h > 0$, define size of the two critical regions by

$$\delta_1(h) := \sqrt{12\widehat{C}_b dh \log(h^{-1})} \text{ and } \delta_2(h) := \sqrt{16\widehat{C}_b dh^2 \log(h^{-1})}.$$

Then probability that stepsize at exit time is not smallest

$$\mathbb{P}\left(\Delta t(\overline{X}(\nu-)) > h^3\right) = \mathcal{O}(h^\gamma)$$

- This result is helpful for bounding overshoot error

$$\mathbb{E}[d(\overline{X}(\nu), \partial D)] \lesssim \mathbb{E}[\sqrt{\Delta t(\overline{X}(\nu-))}] \lesssim h^\gamma$$

- Overshoot and exit-state error of $|X(\tau) - \overline{X}(\nu)|$ relates to exit-time error $|\tau - \nu|$ through Feynman–Kac PDEs.

Overview

Feynman–Kac PDE

Recall that $\tau := \inf\left\{t \geq 0 \,\middle|\, X(t) \notin D\right\} \wedge T$

Definition 2 (Time-adjusted exit time)

is the exit time of the diffusion process $X(s)$ that goes through $(t, x) \in [0, T] \times \mathbb{R}^d$ and with time starting from t:

$$\tau^{t,x} := \left(\inf\left\{s \geq t \,\middle|\, X(s) \notin D \text{ and } X(t) = x\right\} - t\right) \wedge (T - t)$$

The **mean exit time** of the time-adjusted exit is defined as

$$u(t, x) := \mathbb{E}\left[\tau^{t,x}\right], \qquad \forall (t, x) \in [0, T] \times \overline{D}.$$

Feynman–Kac PDE

Proposition 1 (Gilbarg and Trudinger 2015, Gobet and Menozzi 2010)

Given $a, b \in C^3$ and domain D with C^4 boundary, the function $u(t,x) := \mathbb{E}\left[\tau^{t,x}\right]$ is the unique solution in $\mathcal{C}^{1,2}\big([0,T]\times D\big) \cap \mathcal{C}\big([0,T]\times \overline{D}\big)$ of backward PDE

$$\begin{aligned} \partial_t u &= -a\cdot\nabla u - \frac{1}{2}\operatorname{tr}\left(bb^\top \nabla^2 u\right) - 1 && \text{in} \quad (0,T)\times D, \\ u &= 0 && \text{on} \quad \big([0,T]\times\partial D\big)\cup\big(\{T\}\times D\big). \end{aligned}$$

And there exists a **uniform boundary gradient Lipschitz** *constant $L > 0$ such that*

$$|u(t,x) - u(t,y)| \le L\,|x-y|, \qquad \forall (t,x,y)\in[0,T]\times D\times\partial D$$

Motivation

$$\text{1D SDE:} \qquad dX(s) = a(X)ds + b(X)dW(s) \qquad s > t, \quad X(t) = x$$

Let $u(t,x)$ solve backward PDE

$$u_t + a(x)u_x + \frac{b^2(x)}{2}u_{xx} = -1, \qquad u|_{(0,T]\times\partial D} = 0.$$

Itô's rule

$$du(s, X(s)) = (u_t + au_x + \frac{b^2}{2}u_{xx})ds + au_x dW(s) = -ds + au_x dW(s)$$

Motivation

1D SDE: $$dX(s) = a(X)ds + b(X)dW(s) \qquad s > t, \quad X(t) = x$$

Let $u(t,x)$ solve backward PDE

$$u_t + a(x)u_x + \frac{b^2(x)}{2}u_{xx} = -1, \qquad u|_{(0,T]\times\partial D} = 0.$$

Itô's rule

$$\int_t^\tau du(s,X(s)) = \int_t^\tau (u_t + au_x + \frac{b^2}{2}u_{xx})ds + au_x dW(s) = \int_t^\tau -ds + au_x dW(s)$$

$$\implies \underbrace{u(\tau, X(\tau))}_{=0} - \underbrace{u(t, X(t))}_{u(t,x)} = -(\tau - t) + \int_t^\tau au_x dW(s)$$

$$\implies u(t,x) = \mathbb{E}[\tau - t] = \mathbb{E}[\tau^{x,t}].$$

Overview

Tiny-timestep exits of D

Theorem 3 (Strong convergence rate)

Given critical regions of the size(s) given earlier, then for any $\xi > 0$ *there exists* $\exists C_\xi > 0$ *s.t.*

$$\mathbb{E}\left[|\nu - \tau|\right] \leq C_\xi h^{\gamma-\xi}, \qquad \gamma \in \{1, 1.5\}$$

Notation:

$$\mathbb{E}\left[|\nu - \tau|\right] = \mathcal{O}(h^{\gamma-})$$

Proof idea: Split error into two parts:

$$\mathbb{E}\left[|\tau - \nu|\right] = \mathbb{E}\left[|\tau - \nu| \, \mathbb{1}_{\nu<\tau}\right] + \mathbb{E}\left[|\tau - \nu| \, \mathbb{1}_{\nu>\tau}\right] =: I + II$$

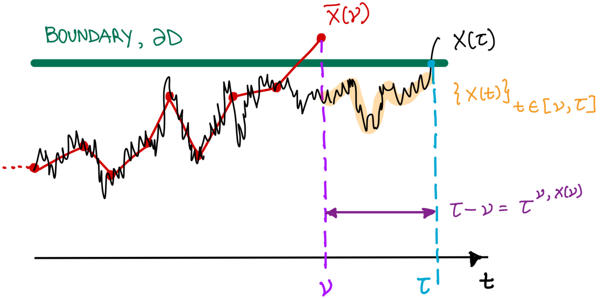
Proof I

For term I,

$$\nu < \tau \leq T \implies X(\nu) \in D \text{ and } \overline{X}(\nu) \in D^C$$

Since numerical sol exits at mesh point $\nu \in \{t_n\}_n$, we can write

$$(\tau - \nu)\mathbb{1}_{\nu<\tau} = \tau^{\nu, X(\nu)}\mathbb{1}_{\nu<\tau}$$



Proof II

Moreover, $\exists y(\omega) \in \partial D$ s.t.

$$|X(\nu) - y| \leq \left|X(\nu) - \overline{X}(\nu)\right|, \quad \text{and} \quad \tau^{\nu,y} = 0, \qquad \forall \omega \in \{\nu < \tau\}$$

This yields,

$$\begin{aligned}
I &= \mathbb{E}\left[(\tau - \nu)\mathbb{1}_{\nu<\tau}\right] \\
&= \mathbb{E}\left[\tau^{\nu,X(\nu)}\mathbb{1}_{\nu<\tau}\right] \\
&\leq \mathbb{E}\left[\,\mathbb{E}\left[\tau^{\nu,X(\nu)} - \tau^{\nu,y} \;\middle|\; \mathcal{F}_\nu\right]\mathbb{1}_{\nu<\tau}\right] \\
&= \mathbb{E}\left[(u(\nu, X(\nu)) - u(\nu, y))\mathbb{1}_{\nu<\tau}\right] \\
&\leq L\,\mathbb{E}\left[\left|X(\nu) - y\right|\mathbb{1}_{\nu<\tau}\right] \qquad \text{(Boundary Lip Feynm-K)} \\
&\leq L\,\mathbb{E}\left[\left|X(\nu) - \overline{X}(\nu)\right|\right] \overset{\text{order of scheme}}{=} \mathcal{O}(h^\gamma)
\end{aligned}$$

Proof II

Moreover, $\exists y(\omega) \in \partial D$ s.t.

$$|X(\nu) - y| \leq \left|X(\nu) - \overline{X}(\nu)\right|, \quad \text{and} \quad \tau^{\nu,y} = 0, \qquad \forall \omega \in \{\nu < \tau\}$$

This yields,

$$\begin{aligned}
I &= \mathbb{E}\left[(\tau - \nu)\mathbb{1}_{\nu<\tau}\right] \\
&= \mathbb{E}\left[\tau^{\nu,X(\nu)}\mathbb{1}_{\nu<\tau}\right] \\
&\leq \mathbb{E}\left[\,\mathbb{E}\left[\tau^{\nu,X(\nu)} - \tau^{\nu,y} \mid \mathcal{F}_\nu\right]\mathbb{1}_{\nu<\tau}\right] \\
&= \mathbb{E}\left[(u(\nu, X(\nu)) - u(\nu, y))\mathbb{1}_{\nu<\tau}\right] \\
&\leq L\,\mathbb{E}\left[|X(\nu) - y|\mathbb{1}_{\nu<\tau}\right] \qquad \text{(Boundary Lip Feynm-K)} \\
&\leq L\,\mathbb{E}\left[\left|X(\nu) - \overline{X}(\nu)\right|\right] \overset{\text{order of scheme}}{=} \mathcal{O}(h^\gamma)
\end{aligned}$$

Term II relates to overshoot:

$$II = \mathbb{E}\left[|\tau - \nu|\,\mathbb{1}_{\{\nu>\tau\}}\right] \leq C\,\mathbb{E}[d(\overline{X}(\nu), \partial D)] + \mathcal{O}(h^{\gamma-\xi}) = \mathcal{O}(h^{\gamma-\xi}).$$

Adaptive time-stepping: Computational cost

Definition 4 (Cost of numerical realization)

Cost of computing one path $\{\overline{X}(t)\}_{t\in[0,\nu]}$ defined by

$$\mathrm{Cost}(\overline{X}) := \int_0^{\nu} \frac{1}{\Delta t(\overline{X}(t))}\,\mathrm{d}t \quad = \#\text{timesteps used on } [0,\nu],$$

where we assume $\mathrm{Cost}(\Psi_\gamma(x,\Delta t)) = \mathcal{O}(1)$ for one-step solver.

Theorem 5 (Computational cost)

For both numerical methods $\gamma \in \{1, 1.5\}$ *it holds that*

$$\mathbb{E}[\mathrm{Cost}(\overline{X})] = \mathcal{O}(h^{-1}\log(h^{-1}))$$

Proof ideas for $\gamma = 1$: I

Events

- $\{\Delta t(\overline{X}(t)) = h\} \cap \{t < \nu\}$ set of paths not in critical region at time t
- $\{\Delta t(\overline{X}(t)) = h^2\} \cap \{t < \nu\}$ in critical region at time t.

$$\mathbb{E}\left[\,\mathrm{Cost}(\overline{X})\right] = \mathbb{E}\left[\int_0^{\nu} \frac{1}{\Delta t(\overline{X}(t))}\,\mathrm{d}t\right]$$

$$= \int_0^T \frac{\mathbb{E}\left[\mathbb{1}_{\{t<\nu\}\cap\{\Delta t(\overline{X}(t))=h\}}\right]}{h} + \frac{\mathbb{E}\left[\mathbb{1}_{\{t<\nu\}\cap\{\Delta t(\overline{X}(t))=h^2\}}\right]}{h^2}\,\mathrm{d}t$$

$$\leq \frac{T}{h} + \frac{1}{h^2}\underbrace{\int_0^T \mathbb{E}\left[\mathbb{1}_{\{t<\nu\}\cap\{d(\overline{X}(t),\partial D)\leq\delta\}}\right]\mathrm{d}t}_{\text{occupation time in critical region}}$$

Proof ideas for $\gamma = 1$: II

Occupation time for $\overline{X}$ bounded by larger-set occuption time for X:

$$\int_0^T \mathbb{E}\left[\mathbb{1}_{\{t<\nu\}\cap\{d(\overline{X}(t),\partial D)\le\delta\}}\right]\mathrm{d}t \le \int_0^T \mathbb{E}\left[\mathbb{1}_{\{t<\tau_{\widetilde{D}}\}\cap\{d(X(t),\partial\widetilde{D})\le 2\delta\}}\right]\mathrm{d}t$$

$$= \int_0^T \int_{\widetilde{D}} p(t,x)\mathbb{1}_{d(x,\partial\widetilde{D})\le 2\delta}(x)\,\mathrm{d}x\,\mathrm{d}t$$

where $\tilde{D} = D \oplus B(0,\delta/2)$ and "density"
$p(t,x) = \mathbb{P}(\{X(t)\in dx\}\cap\{t<\tau_{\widetilde{D}}\})/dx$ solves FP equation

$$\begin{aligned}
p_t &= -\nabla\cdot(ap) + \frac{1}{2}\nabla\cdot\left(bb^\top\nabla p\right) && \text{in} \quad (0,T]\times\widetilde{D}, \\
p &= 0 && \text{on} \quad [0,T]\times\partial\widetilde{D} \\
p(0,x) &= \delta(x-x_0) && x\in\widetilde{D} \quad (\text{fixed } x_0\in D)
\end{aligned}$$

Proof ideas for $\gamma = 1$: III

Key property: $p(t,\cdot) = 0$ on $\partial\widetilde{D}$ and p Lipschitz near boundary implies that $|p(t,x)| \le L\delta$ inside critical region. Therefore

$$\begin{aligned}\int_{\widetilde{D}} p(t,x)\mathbb{1}_{d(x,\partial\widetilde{D})\le 2\delta}(x)\,\mathrm{d}x &\le L\delta \int_{\widetilde{D}} \mathbb{1}_{d(x,\partial\widetilde{D})\le 2\delta}\,\mathrm{d}x \\ &= \mathcal{O}(\delta^2) \\ &= \mathcal{O}(h\,|\log(h)|)\end{aligned}$$

Conclusion:

$$\begin{aligned}\mathbb{E}\left[\,\mathrm{Cost}(\overline{X})\right] &\le \frac{T}{h} + \frac{1}{h^2} \underbrace{\int_0^T \mathbb{E}\left[\mathbb{1}_{\{t<\nu\}\cap\{d(\overline{X}(t),\partial D)\le\delta\}}\right]\mathrm{d}t}_{=\mathcal{O}(h|\log(h)|)} \\ &= \mathcal{O}(h^{-1}|\log(h)|)\end{aligned}$$

Overview

Numerical example

Consider the SDE with *linear drift* and *non-linear diffusion* given by

$$\mathrm{d}X_1 = 0.1X_2\,\mathrm{d}t + 0.25(\cos(X_1)+3)\,\mathrm{d}W_1$$
$$\mathrm{d}X_2 = 0.1X_1\,\mathrm{d}t + 0.25(\cos(X_2)+3)\,\mathrm{d}W_2$$

where

$$X(0) = (3,3), \quad \text{domain} \;\; D = B(X(0),3) \subset \mathbb{R}^2, \text{ and } T = 10.$$

Seek to verify

$$\mathbb{E}[|\tau-\nu|] = \mathcal{O}(h^{\gamma-}) \quad \text{and} \quad \mathbb{E}[\mathrm{Cost}\,\overline{X}] = \mathcal{O}(h^{-1}|\log(h)|)$$

Monte Carlo approximation of strong rate:

$$\mathbb{E}\left[|\tau-\nu|\right] \approx \mathbb{E}_{\mathrm{MC}}\left[|\nu_h - \nu_{2h}|\right]$$

with 10^7 iid MC samples $\nu_h(\omega)$ and $\nu_{2h}(\omega)$ **sharing the same driving noise**.

Weak rate reference solution

"Exact" weak rate estimate: $\left|\mathbb{E}\left[\tau - \nu_h\right]\right| \approx \left|\mathbb{E}_{\mathrm{MC}}\left[\tau - \nu_h\right]\right|$

where pseudo-reference solution $\mathbb{E}[\tau] = \mathbb{E}[\tau^{0,x_0}] = u(0, x_0)$,
is obtained by solving Feynman–Kac PDE using **Gridap.jl FEM**[8].

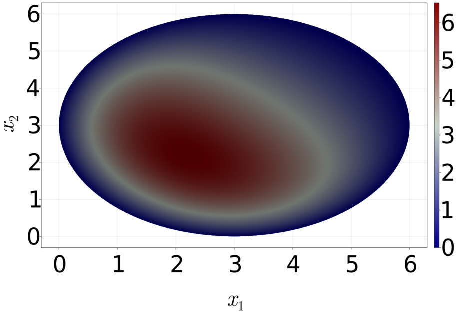
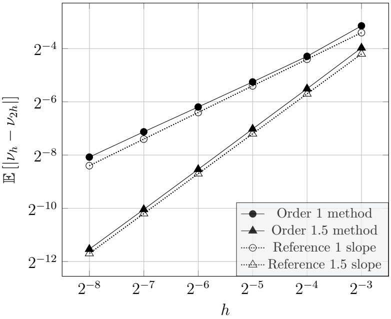


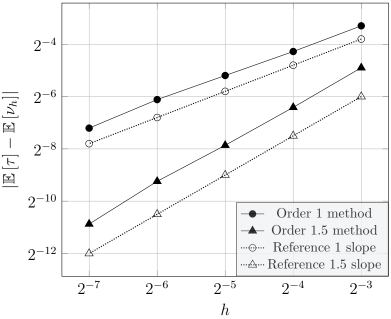
Figure: FEM solution of $u(0,(x_1,x_2)) = \mathbb{E}[\tau^{0,(x_1,x_2)}])$

[8] F. Verdugo, S. Badia, *Computer physics communications* **276**, 108341 (July 2022).

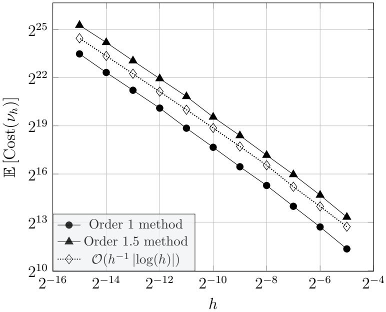
Strong error



Weak error



Computational complexity



Conclusion

- Our method improves error rate in $\mathbb{E}[|\tau - \nu|] = \mathcal{O}(h^\gamma)$ from literature $1/2$ to $\gamma = 1.5$ at nearly same cost
- Our theoretical efficiency gains hinge on restrictive assumptions:
  - diffisuion b being near-diagonal
  - ∂D being C^4
  - Cost per evaluation of Ψ_γ is $\mathcal{O}(1)$

Extension by Sankar:

- Multilevel Monte Carlo method

$$\mathbb{E}_{MLMC}[\nu] = \sum_{\ell=1}^{L} \mathbb{E}_{M_\ell}[\nu_{h_\ell} - \nu_{h_{\ell-1}}] + \mathbb{E}_{M_0}[\nu_{h_0}]$$

- Approximating payoff functions dependent on exit state and exit time, i.e. $g(\tau, X_\tau)$

Extensions I:

- Arbitrary order approximations at near-linear cost: $\mathcal{O}(h^{-1}\log(h^{-1}))$ using strong Itô–Taylor scheme of any order $\gamma > 1.5$ (Sankar's PhD thesis for GBM and Wiener processes.)

- Exit times for time-dependent or lower-regularity domains D

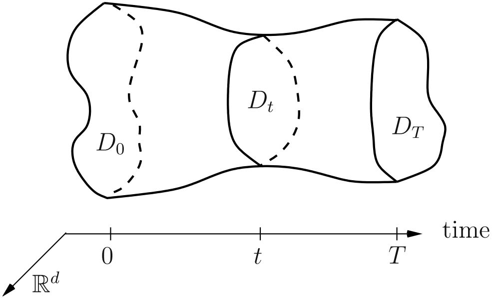


Figure: (Gobet and Menozzi, 2010)

Reflected diffusion I

$$dX(t) = a(X)dt + b(X)dW(t) + \nu(X)dL(t) \qquad t \in [0, T],$$

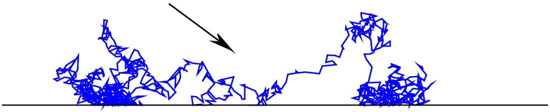
in domain D where $L(t)$ is the local time near ∂D and $\nu : \partial D \to \mathbb{R}^d$ inward pointing normal.

For half-space $(0, \infty)$, $\nu = 1$ and above is limit of splitting method:

$$\hat{X}_{n+1} = a(X_n)\Delta t_n + b(X_n)\Delta W_n$$

and projection onto D:

$$X_{n+1} = \hat{X}_{n+1} + 1 \times \underbrace{\max(0, -\hat{X}_{n+1})}_{\approx \Delta L_n}.$$



Reflected diffusion II

Challenge: For uniform timesteps above method yields

$$\max_n \mathbb{E}[|X(t_n) - \overline{X}(t_n)|^2] \le C|\Delta t \log(\Delta t)|^{1/2} \qquad \text{(Slominski 2001)}$$

Ongoing project (M. Giles and J. Meo): Use similar adaptive timestepping ideas to achieve

$$\max_n \mathbb{E}[|X(t_n) - \overline{X}(t_n)|^2] \le C|\Delta t \log(\Delta t)|$$

for application in multilevel Monte Carlo.

Connection to PDE: $u(t,x) = \mathbb{E}[g(X^{t,x}(T))]$ solves

$$\begin{aligned} \partial_t u &= -a \cdot \nabla u - \frac{1}{2} \operatorname{tr}\left(bb^\top \nabla^2 u\right) && \text{on} \quad (0,T) \times D \\ \nu \cdot \nabla u &= 0 && \text{on} \quad [0,T] \times \partial D \\ u(T,x) &= g(x) && \text{on} \quad D \end{aligned}$$

References

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- E. Gobet, S. Menozzi, *Stochastic processes and their applications* **120**, 130–162 (2010)
- E. Weinan *et al.*, *Applied stochastic analysis*, (American Mathematical Soc., 2021), vol. 199

Thank you for listening!