Yudovich theory for rough perturbations of Euler's equation

Based on joint works with

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Euler's equation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases}$$

models an idealized/inviscid fluid.

$$u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \qquad \text{velocity}$$
$$p : [0,T] \times \mathbb{R}^d \to \mathbb{R} \qquad \text{pressure}$$

Given u, consider Lagrangian trajectories

$$\dot{\phi}_t = u_t(\phi_t)$$

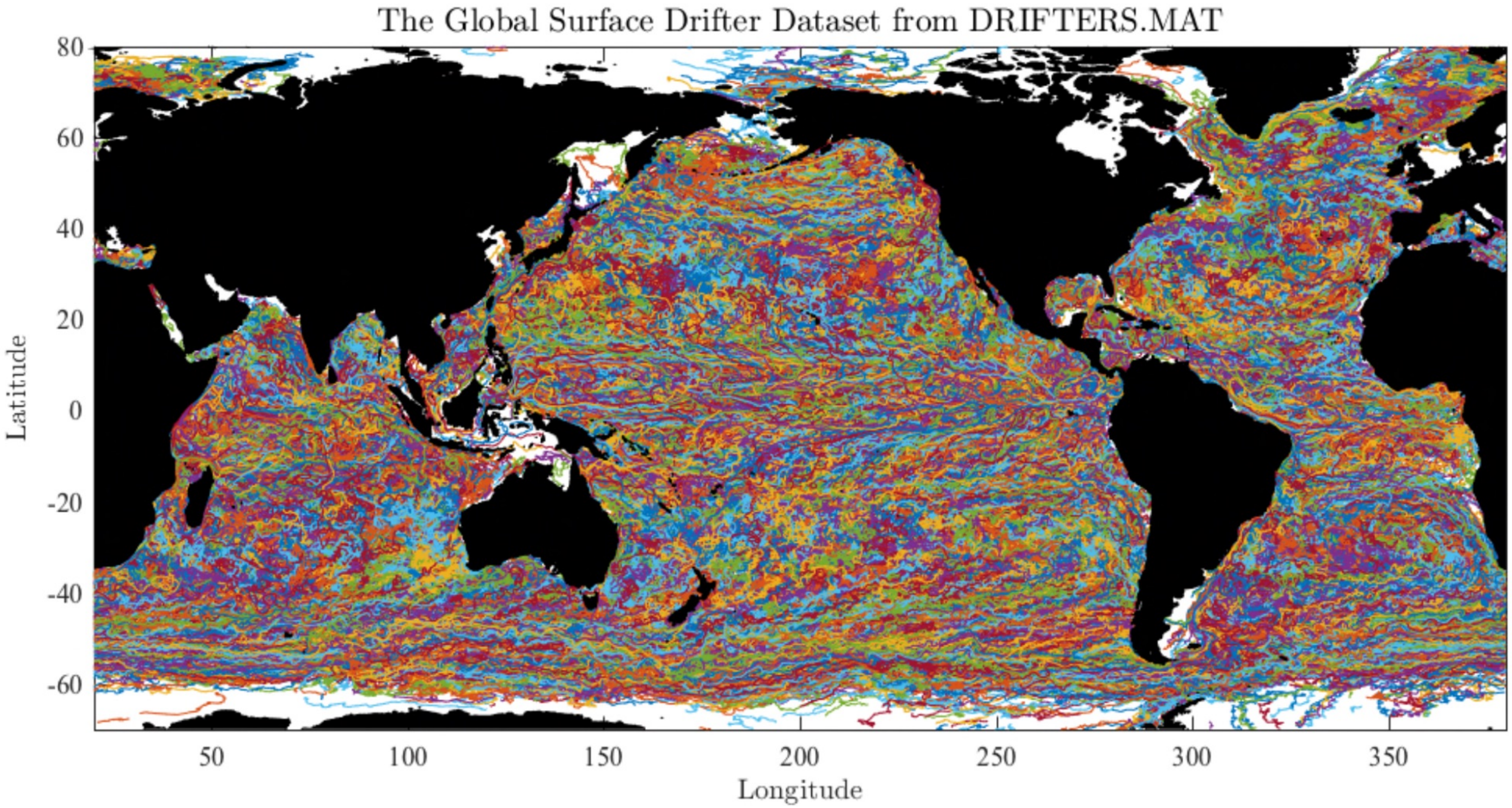

Figure 1.1. This figure shows latitude and longitude of Lagrangian trajectories of drifters on the ocean surface driven by the wind and ocean currents, as compiled from satellite observations by the National Oceanic and Atmospheric Administration Global Drifter Program. Each colour corresponds to a different drifter, see [33]. Upon looking carefully at the individual Lagrangian paths in this figure, one sees that each of them evolves as a mean drift flow, composed with an erratic flow comprising rapid fluctuations around the mean.

(from [Crisan, Holm, Flandoli, '19])

Model

$$\dot{\phi}_t = \text{"mean drift"} + \text{"erratic flow"}$$

$$= u_t(\phi_t) + \text{random term}$$

$$= u_t(\phi_t) + \xi_k(\phi_t)\dot{W}_t^k$$

- $(W_t^k)_{k=1}^K$ irregular/noisy term
- $(\xi^k)_{k=1}^K$ fixed, used to fit model to data.

$\longrightarrow$ What is the corresponding PDE?

Using techniques from geometric hydrodynamics, in

[Crisan, Holm, Leahy, N., '22]

we argue that

$$\partial_t u + u\cdot\nabla u + (\xi_k\cdot\nabla u + D\xi_k u)\dot{W}^k + \nabla p = 0$$

$$\operatorname{div} u = 0$$

$+ Du\, u$

$= \frac{1}{2}\nabla |u|^2$

preserve the geometric structure of Euler's equation and is a physically relevant model.

(Kelvin's circulation thm, enstrophy balance, ...)

Can we solve the equation?

Consider $d=2$ and look at the vorticity

$$\omega = \operatorname{curl} u = \nabla \times u = \partial_1 u^2 - \partial_2 u^1$$

Satisfies

$$\partial_t \omega + u \cdot \nabla \omega + \xi_k \cdot \nabla \omega \dot{W}^k = 0$$

The velocity can be recovered via Biot-Savart

$$u_t(x) = K * \omega_t(x) = \int K(x-y)\, \omega_t(y)\, dy, \quad K(z) = \frac{z^\perp}{2\pi |z|^2}$$

Method of characteristics:

$$\frac{d}{dt} \omega_t(\phi_t) = \partial_t \omega_t(\phi_t) + D\omega_t(\phi_t)\dot{\phi}_t$$

$$= \left(\partial_t \omega_t + u_t \cdot \nabla \omega_t + \xi_k \cdot \nabla \omega_t \dot{W}_t^k\right)(\phi_t) = 0$$

since $$\dot{\phi}_t = u_t(\phi_t) + \xi_k(\phi_t)\dot{W}_t^k$$

$\Rightarrow$ $$\omega_t(\phi_t) = \omega_0$$

$\Rightarrow$ $$\omega_t = \omega_0(\phi_t^{-1})$$

Leads to the system

$$\begin{cases} \dot{\phi}_t = u_t(\phi_t) + \xi_k(\phi_t)\dot{W}_t^k \\ u_t = K * \omega_t \\ \omega_t = \omega_0(\phi_t^{-1}) \end{cases}$$

Yudovich theory : well-posedness in the class $\omega \in L^\infty([0,T]; L^1 \cap L^\infty(\mathbb{R}^2))$

How do we understand $\xi_k(\phi_t)\dot{W}_t^k$ when W is not differentiable, e.g. a sample path of Brownian motion?

Itô theory
- best suited for memoryless noise (ocean's have memory)
- its generalizations breaks the geometric structure and are difficult (possible?) to implement

Rough paths / Rough diff. eq.

First $u \equiv 0$, eq reads

$$\dot{\phi}_t = \mathfrak{z}_k(\phi_t)\,\dot{W}_t^k$$

Integrate

$$\phi_t - \phi_s = \int_s^t \mathfrak{z}_k(\phi_r)\,\dot{W}_r^k\,dr = \int_s^t \mathfrak{z}_k(\phi_r)\,dW_r^k$$

When is integration well-defined?

Thm [Young, 1936]

If

$$|g_t - g_s| \lesssim |t-s|^\alpha, \quad |h_t - h_s| \lesssim |t-s|^\beta$$

and $\alpha + \beta > 1$

then

$$\int_0^T g_r \, dh_r = \lim_{|\pi_t| \to 0} \sum_{\pi_t} g_{t_i} (h_{t_{i+1}} - h_{t_i})$$

is well defined and

$$\left| \int_s^t g_r \, dh_r - g_s (h_t - h_s) \right| \lesssim |t-s|^{\alpha+\beta}.$$

If W is α-Hölder and $\xi_\kappa(\phi_\cdot)$ is β-Hölder;

$$\phi_t - \phi_s = \int_s^t \xi_\kappa(\phi_r)\, dW_r^\kappa \mp \xi_\kappa(\phi_s)(W_t^\kappa - W_s^\kappa)$$

$$\lesssim |t-s|^{\alpha+\beta} + \|\xi_\kappa\|_\infty |t-s|^\alpha$$

Expect ϕ to inherit regularity of W, so $\alpha = \beta$.

Need $\alpha + \alpha > 1 \Rightarrow \boxed{\alpha > \frac{1}{2}}$

For sample paths of Brownian motion have $\alpha = \frac{1}{2} - \varepsilon$ for $\varepsilon > 0$, so Young theory is out of reach.

In fact

Proposition 1.1. *There exists no separable Banach space $\mathcal{B} \subset \mathcal{C}([0,1])$ with the following properties:*

1. *Sample paths of Brownian motions lie in $\mathcal{B}$ almost surely.*
2. *The map $(f,g) \mapsto \int_0^\cdot f(t)\dot{g}(t)\,dt$ defined on smooth functions extends to a continuous map from $\mathcal{B} \times \mathcal{B}$ into the space of continuous functions on $[0,1]$.*

[T. Lyons, 91]

But ; $\int_0^t B_r^k \, dB_r^l$ can be defined

using Itô theory.

However, special care is needed:

$$\sum B_{t_i+\theta(t_{i+1}-t_i)}(B_{t_{i+1}}-B_{t_i}) \longrightarrow \begin{cases} \frac{1}{2}(B_t^2 - t) & \text{if } \theta = 0 \\ \frac{1}{2} B_t^2 & \text{if } \theta = \frac{1}{2} \end{cases}$$

So $B \longmapsto \left(\int B^k \, dB^l\right)_{k,l}$ is discontinuous.

In general, also the solution map is discontinuous

$$\begin{tikzcd}
C([0,T];\mathbb{R}^k) \arrow[r] \arrow[d, bend right] & C([0,T];\mathbb{R}^d) \\
\mathcal{C}^\alpha_g \arrow[ur, bend right, "\text{cont!}"'] &
\end{tikzcd}$$

$$B \longmapsto \phi, \quad \dot\phi_t = \sigma_k(\phi_t)\dot B^k_t$$

$$B \longmapsto \left(B, \int B \otimes dB\right)$$

→ cont!

Eg

$$\dot\phi^1_t = \dot B^k_t$$

$$\dot\phi^2_t = \phi^1_t \dot B^l_t$$

$$\Rightarrow \quad \phi^2_t = \int_0^t B^k_r \, dB^l_r$$

Rough paths

A pair $\mathbf{W} = (W, \mathbb{W})$,

$$W : [0,T] \to \mathbb{R}^K, \quad \mathbb{W} : [0,T]^2 \to \mathbb{R}^{K \times K}$$

s.t.

$$|W_t - W_s| \lesssim |t-s|^{\alpha}, \quad |\mathbb{W}_{st}| \lesssim |t-s|^{2\alpha}$$

and

Chen's relation

$$\mathbb{W}_{st} - \mathbb{W}_{su} - \mathbb{W}_{ut} = (W_t - W_u) \otimes (W_u - W_s)$$

for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ is called a **rough path.**

Think of $\mathbb{W}$ as defining

$$\int_s^t (W_r - W_s) \otimes dW_r$$

Typically need probability to give meaning to $\mathbb{W}$.

Possible for a much larger class of stochastic processes.

Once fixed, analysis will be deterministic.

Rough differential equations

$$\dot{\phi} = \xi_k(\phi)\dot{W}^k$$

$$\phi_t - \phi_s = \int_s^t \xi_k(\phi_r)\,dW_r^k$$

$$= \int_s^t \xi_k(\phi_s) + D\xi_k(\phi_s)(\phi_r - \phi_s) + O(|\phi_r - \phi_s|^2)\,dW_r$$

$$= \xi_k(\phi_s)(W_t^k - W_s^k) + D\xi_k(\phi_s)\int_s^t\int_s^r \xi_\ell(\phi_u)\,dW_u^\ell\,dW_r^k + O(|t-s|^{3\alpha})$$

$$= \xi_k(\phi_s)(W_t^k - W_s^k) + D\xi_k(\phi_s)\xi_\ell(\phi_s)\int_s^t\int_s^r dW_u^\ell\,dW_r^k + O(|t-s|^{3\alpha})$$

solution look like a Taylor expansion in W and $\mathbb{W}$.

Get

$$\mathcal{L}_e^\alpha \longrightarrow C([0,T];\mathbb{R}^d)$$

$$\mathbb{W} \longmapsto \phi : \quad \dot{\phi}_t = \mathfrak{Z}_\alpha(\phi_t)\dot{W}_t^\alpha$$

is continuous!

Drifted equation

$$\dot{\phi}_t = b_t(\phi_t) + \zeta_k(\phi_t)\,\dot{W}^k_t$$

$\zeta_k \in C^{1/\alpha + \varepsilon}$

- $\zeta_k \in C^3_b$
- $\mathbf{W} = (W, \mathbb{W})$ a rough path
- b has lower regularity (comes from a fluid)

Osgood regular vector field

Say $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is Osgood if

$$|b_t(x) - b_t(\bar{x})| \leq h(|x - \bar{x}|)$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ s.t.

$$\int_0^\varepsilon \frac{1}{h(r)}\, dr = \infty$$

E.g.
- $h(r) = cr^\alpha$ only for $\alpha = 1$
- $h(r) = r(1 - \ln r)$ (log-Lipschitz)

Thm [Galeati, Leahy, N., 24]

Suppose $\xi_\kappa \in C_b^3$, $\mathbb{W}$ α-rough path, $\alpha \in (\frac{1}{3}, \frac{1}{2}]$

and b is bounded and Osgood.

Then $$\dot{\phi}_t = b_t(\phi_t) + \xi_\kappa(\phi_t)\dot{W}_t^\kappa, \quad \phi_0 = x$$

is well-posed and $\phi \in C^\alpha([0,T];\mathbb{R}^d)$

$$\sup_{t \le T} |\phi_t(x) - \phi_t(\bar{x})| \lesssim H^{-1}(H(|x-\bar{x}|) + T)$$

where $$H(u) = \int_\varepsilon^u \frac{1}{h(r)} dr.$$

An essential feature we need for Lagrangian flows is volume preservation.

Need

i) $\operatorname{div} \xi_k = \operatorname{div} b = 0$

ii) $\mathbb{W}$ geometric, i.e. $\exists$ $W^n \in C^1([0,T];\mathbb{R}^K)$ s.t.

$$\mathbb{W}^n := \left(W^n, \int W^n \otimes \dot{W}^n \, dr\right) \longrightarrow \mathbb{W} \text{ in } \mathscr{C}^\alpha$$

E.g. • $\int B \circ dB$ (Stratonovich) is geometric

• $\int B \, dB$ (Itô) is <u>not</u>

Thm [Galeati, Leahy, N. '24]

- $\xi_k \in C^3_b$, $\operatorname{div} \xi_k = 0$
- $\mathbb{W} \in \mathscr{C}^\alpha_g$ geometric
- b Osgood, $\operatorname{div} b = 0$ (distributional)

then $\dot{\phi}_t = b_t(\phi_t) + \xi_k(\phi_t)\dot{W}^k_t$

is volume preserving

$$(\phi_t)_\# \mathcal{L}^d = \mathcal{L}^d$$

Back to fluids

Consider

$$\begin{cases} \dot{\phi}_t = u_t(\phi_t) + \xi_k(\phi_t)\dot{W}_t^k \\ u_t = K * \omega_t \\ \omega_t = \omega_0(\phi_t^{-1}) \end{cases}$$

where $K(z) = \frac{1}{2\pi} \frac{z^\perp}{|z|^2}$.

Wanted an $L^1 \cap L^\infty$-theory in ω.

When $d=2$, convolution w/ K is smoothing;

- $\sup_x |K*f(x)| \lesssim \|f\|_{L^1} + \|f\|_{L^\infty}$

- $|K*f(x) - K*f(\bar{x})| \lesssim h(|x-\bar{x}|)\left(\|f\|_{L^1} + \|f\|_{L^\infty}\right)$

where $h(r) = r(1-\ln^- r) = \begin{cases} r(1-\ln r), & 0<r<1 \\ r, & r>1 \end{cases}$

So given $\omega \in L^\infty([0,T]; L^1 \cap L^\infty(\mathbb{R}^2))$

$$u_t := K * \omega_t$$

is Osgood and thus

$$\dot{\phi}_t = u_t(\phi_t) + \xi_k(\phi_t)\dot{W}^k_t$$

is meaningful.

Moreover $\operatorname{div} K = 0$, so we get a priori estimates

- $\|\omega_t\|_{L^1} = \|\omega_0(\phi_t^{-1})\|_{L^1} = \|\omega_0\|_{L^1}$

and

- $\|\omega_t\|_{L^\infty} = \|\omega_0(\phi_t^{-1})\|_{L^\infty} \leq \|\omega_0\|_{L^\infty}$

If we now consider smooth approx. W^ε of W s.t.

$$\mathbb{W}^\varepsilon = \left(W^\varepsilon, \int W^\varepsilon \otimes \dot{W}^\varepsilon dr\right) \longrightarrow \mathbb{W}$$

and the system

$$\begin{cases} \dot{\phi}^\varepsilon_t = u^\varepsilon_t(\phi^\varepsilon_t) + \zeta_k(\phi^\varepsilon_t)\dot{W}^{\varepsilon,k} \\ u^\varepsilon_t = K * \omega^\varepsilon_t \\ \omega^\varepsilon_t = \omega_0(\phi^{\varepsilon,-1}_t) \end{cases}$$

Then a-priori estimates, smoothing properties of K + Arzela-Ascoli yields compactness.

Thm [Galeati, Leahy, N., '24]

Assume $\xi_\kappa \in C^3_b$, $\operatorname{div} \xi_\kappa = 0$ and $\mathbb{W} \in \mathscr{C}^\alpha_t$ is geometric. Then

$$\begin{cases} \dot{\phi}_t = u_t(\phi_t) + \xi_\kappa(\phi_t)\, \dot{\mathbb{W}}^\kappa_t \\ u_t = K * \omega_t \\ \omega_t = \omega_0(\phi_t^{-1}) \end{cases}$$

is well-posed in the class

$$\omega \in L^\infty([0,T]; L^1 \cap L^\infty(\mathbb{R}^2))$$

Possible to show well-posedness of

$$\partial_t \omega + u\cdot\nabla\omega + \xi_k \cdot \nabla\omega \, \dot{W}^k = 0$$

$$u = K * \omega$$

$$\omega_0 \in L^1 \cap L^\infty$$

from a purely Eulerian perspective.

Moreover, the solutions can be represented by Lagrangian trajectories;

[Galeati, Leahy, N. '24]

In fact we brought rough PDE up to speed w/ classical DiPerna-Lions theory

$$\partial_t f + b \cdot \nabla f + \xi_k \cdot \nabla f \, \dot{W}^k_t = 0 \qquad \text{(TE)}$$

and

$$\partial_t \mu + \operatorname{div}(b\mu) + \operatorname{div}(\xi_k \mu) \dot{W}^k_t = 0 \qquad \text{(CE)}$$

when b is less than Lipschitz

Thm [Galeati, Leahy, N.]

For $b \in L^1_t W^{1,1}_{loc}$, $\operatorname{div} b \in L^1_t L^\infty_x$, $\frac{b}{1+|x|} \in L^1_t (L^1_x + L^\infty_x)$

- (TE) and (CE) well-posed in $L^\infty_t L^p_x$.

- Product μf satisfies (CE)

- Moreover, if b is Osgood then

$$\rho_t = (\phi_t)_\# \rho_0 \quad \text{and} \quad f_t = f_0(\phi_t^{-1})$$

where

$$\dot{\phi}_t = b_t(\phi_t) + \xi_k(\phi_t) \dot{W}^k_t$$

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