Caloric functions for the fractional Laplacian in Lipschitz sets

Based on a joint work with Gavin Armstrong and Krzysztof Bogdan

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Fractional Laplacian and α-stable processes

Let $\alpha \in (0,2)$, $d \geq 2$, and let

$$\Delta^{\alpha/2}u(x) := -(-\Delta)^{\alpha/2}u(x) := c_{d,\alpha} \lim_{\varepsilon \to 0^+} \int_{B(0,\varepsilon)^c} \frac{u(x+y)-u(x)}{|y|^{d+\alpha}}\, dy, \quad x \in \mathbb{R}^d.$$

The formula makes sense, e.g., for $u \in C_c^2(\mathbb{R}^d)$. For those functions $\Delta^{\alpha/2}$ coincides with the generator of the semigroup P_t corresponding to the isotropic α-stable process X_t, given by the formula $P_t f(x) = \mathbb{E}^x f(X_t)$.

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We have $P_t f(x) = p_t * f(x)$, where p_t is smooth and

$$p_t(x) \approx \left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \right), \quad t > 0,\ x \in \mathbb{R}^d.$$

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Notation: $p_t(x,y) = p_t(x-y)$, $\nu(x,y) = \nu(x-y) = c_{d,\alpha}|x-y|^{-d-\alpha}$.

The Dirichlet problem for the Poisson equation

$$\begin{cases} \Delta^{\alpha/2} u(x) = f(x), & x \in D, \\ u(z) = g(z), & z \in D^c. \end{cases} \tag{DP}$$

For $f = 0$ we refer to u as an α-harmonic (or just harmonic) function.

Some equations involving the fractional Laplacian

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Solution to (FHE): caloric function.

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Goals

Assumption: $d \geq 2$, D is nonempty, open, bounded and Lipschitz.

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Plan of the talk:

- Integral representations and structure of nonnegative solutions to (FHE).
- Relation between different notions of solution to (FHE).
- (Time permitting) Boundary regularity of solutions to (DP) and (FHE) with $g \equiv 0$.

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Green function for $\Delta^{\alpha/2}$ in D:

$$G_D(x,y) = \int_0^\infty p_t^D(x,y)\,dt, \quad x,y \in \mathbb{R}^d. \qquad (G_D(x,x) = \infty,\ x \in D)$$

Note: $G_D(x,y) = G_D(y,x)$ and $G_D(x,y) = 0$ if $x \notin D$ or $y \notin D$.

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For any $x \in D$ the function $v(y,t) = p_t^D(x,y)$ solves $(\partial_t - \Delta^{\alpha/2}) v(t,y) = 0$ pointwise. Furthermore, $p_0^D(x,\cdot) = \delta_x$ in the sense that for every $f \in L^1$ we have $P_t^D f \xrightarrow[t \to 0^+]{} f$ in L^1.

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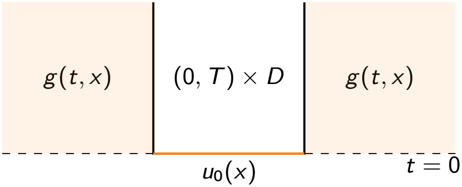
Let $G_D[f](x) = \int_D G_D(x,y) f(y)\, dy$, $x \in \mathbb{R}^d$ (Green potential of f).
If f is regular enough, then $G_D[f]$ solves (DP) pointwise with $g \equiv 0$.

Integral representations and structure of nonnegative caloric functions

Caloric functions in cylinders

$$\begin{cases} \partial_t u(t,x) = \Delta^{\alpha/2} u(t,x), & t \in (0,T),\ x \in D, \\ u(t,x) = g(t,x), & t \in (0,T),\ x \in D^c, \\ u(0,x) = u_0(x), & x \in D. \end{cases} \tag{FHE}$$



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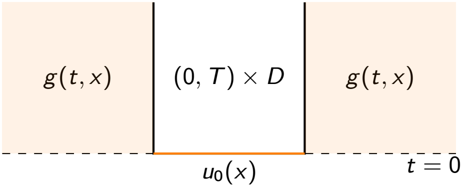

We will define solutions to (FHE) by means of a mean-value property with respect to the space-time α-stable process

$$\dot{X}_t = (-t, X_t).$$

Think of adding a drift of velocity -1 in the direction of a new coordinate.

Dirichlet problem and Markov processes

The general idea of solving (nonlocal) PDEs involving Markov operators goes back at least to Kakutani:
formally, if L is the generator of a Markov process Y_t, then a solution to

$$\begin{cases} Lu(x) = 0, & x \in D, \\ u(x) = g(x), & x \in D^c, \end{cases}$$

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Example

$L = \Delta \longrightarrow X_t$ – Brownian motion $\longrightarrow Law(X_{\tau_D})$ – harmonic measure (∂D)

$L = \Delta^{\alpha/2} \longrightarrow X_t - \alpha$-stable process $\longrightarrow Law(X_{\tau_D}) - \alpha$-harmonic measure $((\overline{D})^c)$

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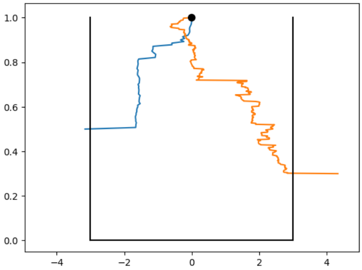
Lemma

On $C_b^{1,2}(\mathbb{R}^d)$, the generator of $\dot{X}$ coincides with $-\partial_t + \Delta^{\alpha/2}$.

First exit from a cylinder

If $G = [0,t) \times U$ then $\dot{X}$ starting at (t,x) can exit G in two ways depending on whether X leaves U before time t or not.



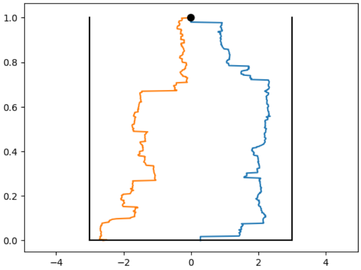


Figure: $U = (-3,3)$. On the left X leaves U before $t=1$, on the right it survives until $t=1$.

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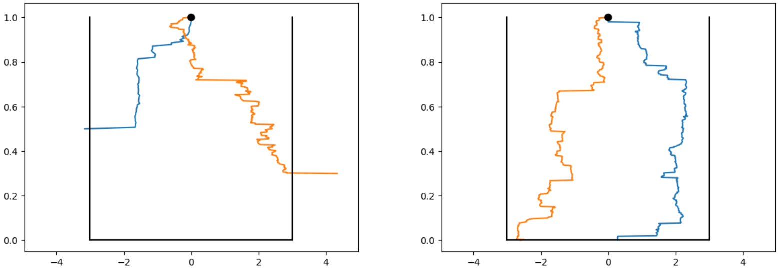

$$\mathbb{E}^{(t,x)} u(\dot{X}_{\tau_G}) = \mathbb{E}^{(t,x)}[u(X_{\tau_G});\ \tau_U \le t] + \mathbb{E}^{(t,x)}[u(X_{\tau_G});\ \tau_U > t]$$
$$= \mathbb{E}^{(t,x)}[u(\tau_U, X_{\tau_U});\ \tau_U \le t] + P_t^U u_0(x).$$

Lateral exit distribution

Ikeda–Watanabe formula: let $I \subseteq [0,\infty)$ and $A \subseteq U^c$, I, A – Borel. Then,

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Overall,

$$\mathbb{E}^{(t,x)}u(\dot{X}_{\tau_G}) = \int_U p_t^U(x,y)u_0(y)\,dy + \int_0^t \int_{U^c} u(s,z)J^U(t,x,s,z)\,dz\,ds.$$

Caloric functions in cylinders

Definition

- We say that $u \geq 0$ is **caloric** in $[0, T) \times D$ if for all $(t, x) \in (0, T) \times D$,

$$u(t,x) = \mathbb{E}^{(t,x)} u(\dot{X}_{\tau_G}) < \infty, \qquad (1)$$

holds for every open $G \subset\subset [0, T) \times D$ such that $(t, x) \in \overline{G}$.

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- If (1) holds for $(0, t) \times D$ in place of G, then we say that u is **regular caloric**.
- If $u \equiv 0$ on the *parabolic boundary*
$$D^p = D \times \{0\} \cup D^c \times (0, T),$$
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1. J. L. Doob. *Classical potential theory and its probabilistic counterpart*, 1984.
2. Z.-Q. Chen, T. Kumagai. Heat kernel estimates for stable-like processes in d-sets. *Stoch. Process. App.* 108:27–62, 2003.

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Note: by virtue of the strong Markov property we can and will verify the mean-value property only on cylinders $(0, t) \times U$, where $U \subset\subset D$ is Lipschitz.

Beyond regular caloric functions

If u is regular caloric, then it is uniquely determined by g and u_0:

$$u(t,x) = P_t^D u_0(x) + \int_0^t \int_{D^c} g(s,z) J^D(t,x,s,z)\, dz\, ds.$$

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Elliptic case ($\Delta^{\alpha/2} u = 0$):

- Hmissi (1994): explicit example of a positive singular α-harmonic function.
- Bogdan (1999): full representation of nonnegative α-harmonic functions in Lipschitz domains. Singular harmonic functions are of the form $\int_{\partial D} M_D^{x_0}(x,Q)\mu(dQ)$ with

$$M_D^{x_0}(x,Q) = \lim_{D \ni y \to Q} \frac{G_D(x,y)}{G_D(x_0,y)}, \quad x \in D,\ Q \in \partial D. \quad \text{(Martin kernel)}$$

1. K. Bogdan. Representation of α-harmonic functions in Lipschitz domains. *Hiroshima Math. J.* 29:227–243, 1999.
2. F. Hmissi. Fonctions harmoniques pour les potentiels de Riesz sur la boule unite, *Expo. Math.* 12(3):281–288, 1994.

Parabolic Martin kernel

For cones Γ with apex at 0 Bogdan–Palmowski–Wang showed that the limits below exist:

$$\lim_{\Gamma \ni y \to 0} \frac{p_t^\Gamma(x,y)}{\mathbb{P}^y(\tau_\Gamma > 1)}, \quad \lim_{\Gamma \ni y \to 0} \frac{p_t^\Gamma(x,y)}{G_\Gamma(x_0,y)}, \quad \lim_{\Gamma \ni y \to 0} \frac{p_t^\Gamma(x,y)}{p_{t_0}^\Gamma(x_0,y)}, \quad t_0, x_0 \text{ fixed.} \qquad (*)$$

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Analogue of $(*)$ for bounded Lipschitz sets: dissertation of G. Armstrong.
$C^{1,1}$ sets: Fernández-Real and Ros-Oton.

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Definition (Parabolic Martin kernel)

$$\eta_{t,Q}(x) := \lim_{D\ni y\to Q} \frac{p_t^D(x,y)}{\mathbb{P}^y(\tau_D > 1)}, \quad t>0,\ x\in D,\ Q\in\partial D.$$

Lemma

Fix $Q \in \partial D$. Then, the function $u(t,x) = \eta_{t,Q}(x)$ is singular caloric in $(0,\infty) \times D$.

Representation of caloric functions

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Fix $Q \in \partial D$. Then, the function $u(t,x) = \eta_{t,Q}(x)$ is singular caloric in $(0,\infty) \times D$.

Theorem (G. Armstrong, K. Bogdan, AR 2024)

Assume that $u \geq 0$ is caloric in $[0,T) \times D$. Then there exists unique decomposition $u = R + S$, such that $R, S \geq 0$, R is regular caloric and S is singular caloric. Furthermore,

$$R(t,x) = \mathbb{E}^{(t,x)} u(\dot{X}_{\tau_{(\mathbf{0},t)\times D}}).$$

Representation of caloric functions

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Theorem (GA–KB–AR 2024)

There exists unique Radon measure μ on $[0,T) \times \partial D$ such that

$$S(t,x) = \int_{[0,t)\times\partial D} n_{t-s,Q}(x)\, \mu(dQds). \tag{M}$$

1. G. Armstrong, K. Bogdan, A. Rutkowski. Caloric functions and boundary regularity for the fractional Laplacian in Lipschitz open sets. *Math. Ann. (online)*, 2024.

No initial condition

Definition

We say that $u \geq 0$ is **caloric** in $(0, T) \times D$ if for all $(t, x) \in (0, T) \times D$,

$$u(t,x) = \mathbb{E}^{(t,x)} u(\dot{X}_{\tau_G}) < \infty,$$

holds for every open $G \subset\subset (0, T) \times D$ such that $(t, x) \in \overline{G}$.

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Theorem (GA–KB–AR 2024)

Assume that u is caloric on $(0, T) \times D$ and let $g = u|_{D^c}$. Then there exist unique Radon measures μ on $[0, T) \times \partial D$ and μ_0 on D such that for all $0 < t < T$ and $x \in D$,

$$u(t,x) = P_t^D \mu_0(x) + \int_0^t \int_{D^c} g(s,z) J^D(t,x,s,z)\, dz\, ds + \int_{[0,t)\times\partial D} n_{t-s,Q}(x)\, \mu(dQds).$$

We also show that $\int_D \mathbb{P}^y(\tau_D > 1)\, \mu_0(dy) < \infty$.

Other related results

Elliptic setting:

1. K. Bogdan, T. Kulczycki, M. Kwaśnicki. Estimates and structure of α-harmonic functions. *Probab. Th. Rel. Fields* 140:345–381, 2008.
2. N. Abatangelo. Large s-harmonic functions and boundary blow-up solutions for the fractional Laplacian. *Discrete Contin. Dyn. Syst.* 35(12):5555-5607, 2015.

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Parabolic setting:

4. B. Barrios, I. Peral, F. Soria, and E. Valdinoci. A Widder's type theorem for the heat equation with nonlocal diffusion. *Arch. Ration. Mech. Anal.*, 213(2):629–650, 2014.
5. H. Chan, D. Gómez-Castro, J.-L. Vázquez. Singular solutions for fractional parabolic boundary value problems. *Rev. Acad. Cienc. Ser. A Math.* 116(4):159, 2022.

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Chan–Gómez-Castro–Vázquez give a quite general framework. It includes (a class of) singular solutions to $(\partial_t - \Delta^{\alpha/2})u = f$ in $C^{1,1}$ cylinders. Parabolic Martin kernel used there is:

$$\lim_{y \to Q \in \partial D} \frac{p_t^D(x,y)}{\delta_D(y)^{\alpha/2}}.$$

Relation between notions of solution to the fractional heat equation

Relation to distributional solutions

Definition

We say that $u \geq 0$ is a distributional solution to (FHE), if for every $\phi \in C_c^\infty([0,T) \times D)$ and $0 \leq s < t < T$,

$$\int_D \phi(t,x)u(t,x)\,dx = \int_D \phi(s,x)u(s,x)\,dx + \int_s^t \int_{\mathbb{R}^d} (\partial_t + \Delta^{\alpha/2})\phi(\tau,x)u(\tau,x)\,dx\,d\tau$$

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Theorem (AR, 2024)

Every caloric function is a distributional solution to (FHE) *and every distributional solution to* (FHE) *has a modification which is caloric.*

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Elliptic and related cases: Bogdan and Byczkowski, Chen.

1. K. Bogdan, T. Byczkowski. Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains. *Studia Math.*, 133(1):53—92, 1999.
2. Z.-Q. Chen. On notions of harmonicity. *Proc. Amer. Math. Soc.*, 137(10):3497—3510, 2009.
3. A. Rutkowski. Equivalence of definitions of fractional caloric functions. *ArXiv:2410.16188*, 2024.

Relation to classical solutions

A caloric function need not solve the fractional heat equation pointwise.

Example

Let

$$u(t,x) = \begin{cases} \eta_{t-1/2,Q}(x), & t > 1/2,\ x \in D, \\ 0, & \text{otherwise.} \end{cases}$$

Then u is caloric in $[0,\infty) \times D$ but it is not even Lipschitz in t at $t = 1/2$.

Note that in the above example $\mu = \delta_{1/2} \otimes \delta_Q$.

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Note that in the above example $\mu = \delta_{1/2} \otimes \delta_Q$.

(Local) smoothness in space is not an issue here.

Lemma

If u is caloric in $[0,T) \times D$, then $u(t,\cdot)$ is smooth in D for all $t \in (0,T)$.

Sufficient conditions for classical solutions

Goal: find possibly mild conditions on μ and g under which a caloric function solves (FHE) pointwise

Lemma (AR, 2024)

Let $B_r = B(0,r)$, $B = B_1$. If $x \in B_{1/2}$, then

$$|\partial_t J^B(t,x,s,z)| \lesssim \begin{cases} \frac{1}{t-s} + \frac{\delta_B(z)^{-\alpha/2}}{(t-s)^{1-(2-\alpha)/2\alpha}}, & 0 < t-s < T,\ z \in B_2 \setminus B, \\ \frac{|z|^{-d-\alpha}}{t-s}, & 0 < t-s < T,\ z \in B_2^c. \end{cases}$$

Note that the singularity in time is of order $(t-s)^{-1}$ since $(2-\alpha)/2\alpha > 0$.

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Theorem (AR, 2024)

If u is caloric with $g \in C^{\mathrm{Dini}}((0,T), L^1(1 \wedge \nu))$, and $\mu \in C^{\mathrm{Dini}}((0,T), \mathcal{M}(\partial D))$, then u is a classical solution to (FHE).

Some comments

- Recall that Dini continuity means that there exists a modulus of continuity ω such that
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- Dini-type conditions in the context of C^2 regularity for the nonlocal Poisson problem were considered by Grzywny, Kassmann and Leżaj.
- The parabolic case is much more laborious than the elliptic case. For $\Delta^{\alpha/2}$ the Poisson kernel of a ball has an explicit formula and every α-harmonic function is smooth.

1. C. C. Burch. The Dini condition and regularity of weak solutions of elliptic equations. J. Differential Equations, 30(3):308–323, 1978.
2. T. Grzywny, M. Kassmann, and Ł. Leżaj. Remarks on the nonlocal Dirichlet problem. Potential Anal., 54(1):119–151, 2021.

Boundary regularity for $\Delta^{\alpha/2}$ in Lipschitz sets

Previously known results

- Boundary Harnack principle in Lipschitz sets: Bogdan (1997).
- Green function and expected exit time estimates: Chen–Song (1998), Jakubowski (2002).
- Dirichlet heat kernel estimates:
  Bogdan–Grzywny–Ryznar (2010): for Lipschitz D,

$$p_t^D(x,y) \approx \mathbb{P}^x(\tau_D > t)p_t(x,y)\mathbb{P}^y(\tau_D > t), \quad 0 < t < T,\ x,y \in \mathbb{R}^d.$$

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  Chen–Kim–Song (2010): for D of class $C^{1,1}$,

$$p_t^D(x,y) \approx \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)p_t(x,y)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right), \quad 0 < t < T, \ x,y \in \mathbb{R}^d.$$

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In $C^{1,1}$ sets G_D and p_t^D (t fixed) decay like $\delta_D^{\alpha/2}(x)$ at ∂D. In Lipschitz sets the decay is less explicit, also for harmonic functions.

1. K. Bogdan, T. Grzywny, M. Ryznar. Heat kernel estimates for the fractional Laplacian with Dirichlet conditions. *Ann. Probab.* 38(5):1901–1923, 2010.
2. Z.-Q. Chen, P. Kim, R. Song. Heat kernel estimates for the Dirichlet fractional Laplacian. *J. Eur. Math. Soc.* 12:1307–1329, 2010.

Previously known results

- Ros-Oton–Serra (2014): if D is $C^{1,1}$, $G_D[f]/\delta_D^{\alpha/2}(x)$ is Hölder up to the boundary.
- Ros-Oton–Fernández-Real (2016): if D is $C^{1,1}$, $P_t^D u_0/\delta_D^{\alpha/2}(x)$ is Hölder up to the boundary.

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- Lian–Zhang–Li–Hong (2020): Hölder decay at the boundary for Poisson equation.
- Ding, Zhang (2024): Hölder decay at the boundary for the fractional heat equation.

1. X. Ros-Oton and J. Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl.* (9), 101(3):275–302, 2014.
2. Y. Lian, K. Zhang, D. Li, and G. Hong. Boundary Hölder regularity for elliptic equations. *J. Math. Pures Appl.* 143:311–333, 2020.
3. M. Ding and C. Zhang. A new unified method for boundary Hölder continuity of parabolic equations. *J. Geom. Anal.* 34(6):Paper No. 179, 39, 2024.

Theorem (GA–KB–AR 2024)

Let $r > 0$. There exists $p_0 > 1$ depending on d, α, r and the Lipschitz characteristics of D such that for all $p \in [1, p_0)$ there exist constants $C > 0$ and $\mu \in (0, 1]$ depending only on d, α, r, p and the Lipschitz characteristics of D such that

$$\left\| \frac{G_D(y,\cdot)}{G_D(x_0,y)} - \frac{G_D(y',\cdot)}{G_D(x_0,y')} \right\|_{L^p(D)} \leq C|y - y'|^\mu, \quad y, y' \in D \setminus B(x_0, r). \tag{2}$$

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Corollary

Let $p > p_0/(p_0 - 1)$ and let $f \in L^p(D)$. Then, the function $y \mapsto G_D f(y)/G_D(x_0, y)$ is Hölder continuous on $D \setminus B(x_0, r)$ with the Hölder constant and exponent depending only on d, α, p, r, the Lipschitz characteristics of D, and $\|f\|_{L^p(D)}$.

Remark: for the classical Poisson equation such result is known to be false if the Lipschitz constant of D is too large.

Fractional heat equation in Lipschitz sets

Theorem (GA–KB–AR 2024)

There exist $C, \gamma > 0$ depending only on d, α, T_1, T_2 and the Lipschitz characteristics of D, such that

$$\|n_{t,\cdot}(x)\|_{C^{\gamma}(D)} \leq C, \quad 0 < T_1 < t < T_2,\ x \in D.$$

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Corollary

$$\left\| \frac{P_t^D u_0(\cdot)}{\mathbb{P}^\cdot(\tau_D > 1)} \right\|_{C^\gamma(D)} \leq C\|u_0\|_{L^1(D)}, \quad 0 < T_1 < t < T_2, \; x \in D.$$

Note: similar results hold for $G_D(x_0, y)$ or $p^D_{t_0}(x_0, y)$ in place of $\mathbb{P}^y(\tau_D > 1)$.

Some comments

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- Formula (2) is the key result. Proof uses BHP, specific interior Harnack principle, and estimates of G_D and the ratios.
- Papers contain many useful estimates on p_t^D, J^D and their derivatives.

Thank you for your attention!