# Some obstacle problems with Lewy-Stampacchia's inequalities

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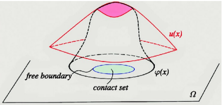
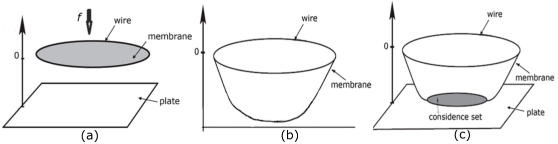



December 11-12, 2024
TMS Colloquium on PDEs - Trondheim

France

# Obstacle problems: examples

## Membrane deformation



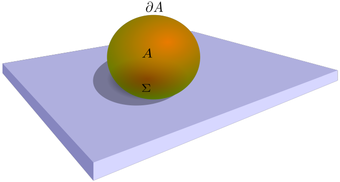

Apply a force on membrane

without plate (no obstacle)

with plate (obstacle)

## Signorini type problems

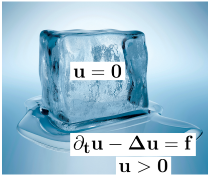
(obstacle on the boundary)

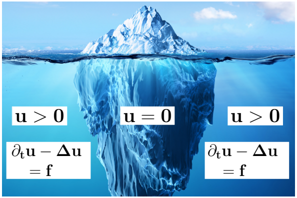



Equilibrium configuration: elastic body resting on a plane

# Obstacle problems: examples

## Stefan type problems




But also constraints in :

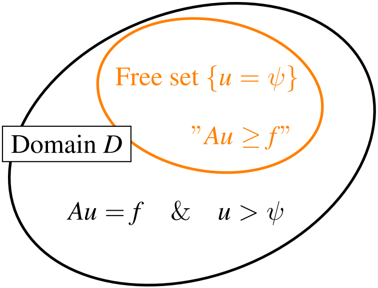
fluid flow in porous medium: contraint on the pressure to get or not a gas phase,

Model with constraints for vehicular traffic jams,

...

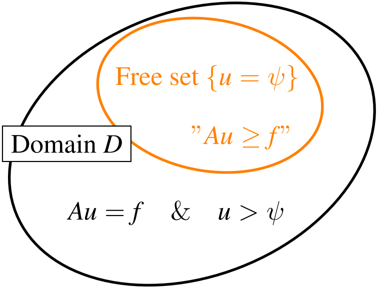
A free set: where/when the solution equals the constraint.

## Obstacle problems $\quad D \subset \mathbb{R}^d \quad$ constraint $\psi$




$A$ an operator to precise later

# Obstacle problems $\quad D \subset \mathbb{R}^d \quad$ constraint $\psi$




Penalization :

$$Au_\epsilon - \frac{1}{\epsilon}(u_\epsilon - \psi)^- = f$$

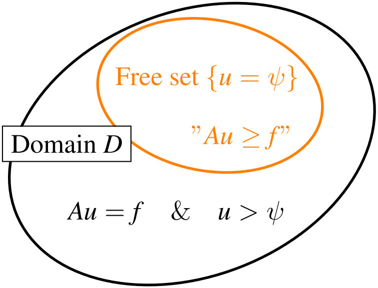
$$Au_\epsilon - f = \frac{1}{\epsilon}(u_\epsilon - \psi)^- \geq 0$$

$$\cdots \epsilon \to 0 \cdots$$

$$Au \geq f, \quad u \geq \psi$$

$$\& \quad (Au - f)(u - \psi) = 0$$

# Obstacle problems $D \subset \mathbb{R}^d$ constraint $\psi$

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$$Au \geq f, \quad u \geq \psi$$

$$\& \quad (Au - f)(u - \psi) = 0$$

$$\mu = Au - f \geq 0 \quad \& \quad \mu(u - \psi) = 0$$

Lewy-Stampacchia's inequalities: $0 \leq Au - f \leq (A\psi - f)^+$

# Elliptic case : variational inequality

$$A : V = W_0^{1,p}(D) \to V' \qquad \text{or } V = W_0^{1,p}(D) \cap L^2(D) \text{ or } V = W_0^{1,p(x)}(D)$$
$$u \mapsto Au = -\mathrm{div}\Big(a(\cdot, u, \nabla u)\Big)$$

Leray-Lions pseudomonotone operator, + coercive, + growth conditions

$$\forall \epsilon > 0, \quad \exists u_\epsilon \in V = W_0^{1,p}(D), \quad Au_\epsilon - \frac{1}{\epsilon}\big(u_\epsilon - \psi\big)^- = f \quad \text{in } V'$$

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Test with $u_\epsilon - \psi$: $\quad u_\epsilon \rightharpoonup u, \quad Au_\epsilon \rightharpoonup \chi, \quad \big(u_\epsilon - \psi\big)^- \to 0$

# Elliptic case : variational inequality

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Test with $u_\epsilon - u$: $\quad \limsup \langle Au_\epsilon, u_\epsilon - u\rangle \leq 0$

$$\Rightarrow \qquad \lim \langle Au_\epsilon, u_\epsilon\rangle = \langle Au, u\rangle \quad \& \quad \chi = Au$$

# Elliptic case : variational inequality

$A : V = W_0^{1,p}(D) \to V'$ or $V = W_0^{1,p}(D) \cap L^2(D)$ or $V = W_0^{1,p(x)}(D)$

$u \mapsto Au = -\mathrm{div}\Big(a(\cdot, u, \nabla u)\Big)$

Leray-Lions pseudomonotone operator, + coercive, + growth conditions

$$\forall \epsilon > 0, \quad \exists u_\epsilon \in V = W_0^{1,p}(D), \quad Au_\epsilon - \frac{1}{\epsilon}\big(u_\epsilon - \psi\big)^- = f \quad \text{in } V'$$

Test with $u_\epsilon - \psi$: $u_\epsilon \rightharpoonup u, \quad Au_\epsilon \rightharpoonup \chi, \quad \big(u_\epsilon - \psi\big)^- \to 0$

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$$\Rightarrow \qquad \lim \langle Au_\epsilon, u_\epsilon \rangle = \langle Au, u \rangle \quad \& \quad \chi = Au$$

$$\mu_\epsilon := \frac{1}{\epsilon}\big(u_\epsilon - \psi\big)^- = Au_\epsilon - f \quad \to \quad \mu = Au - f \geq 0 \quad \text{in } V'$$

$\ldots$ ✍ $\ldots$ $\qquad u \geq \psi \quad \& \quad \langle \mu, u - \psi \rangle = 0.$

$u_\epsilon$ cv a.e. (bd in $V$) $\qquad \nabla u_\epsilon$ cv in measure (Boccardo/Gallouët/Murat)

# Elliptic case : variational inequality

$$A : V = W_0^{1,p}(D) \to V' \qquad \text{or } V = W_0^{1,p}(D) \cap L^2(D) \text{ or } V = W_0^{1,p(x)}(D)$$

$$u \mapsto Au = -\mathrm{div}\Big(a(\cdot, u, \nabla u)\Big)$$

Leray-Lions pseudomonotone operator, + coercive, + growth conditions

$$\exists (u, \mu) \in V \times V', \quad u \geq \psi, \quad \mu \geq 0, \qquad Au - \mu = f, \quad \langle \mu, u - \psi \rangle = 0,$$

*i.e.*

$$\exists u \in V, \quad u \geq \psi, \quad \forall v \in V, \qquad v \geq \psi \implies \langle Au, v - u \rangle \geq \langle f, v - u \rangle.$$

$$0 \leq \mu = Au - f \leq (A\psi - f)^+$$

Dual-order assumption:

$$A\psi - f = h^+ - h^-, \quad h^\pm \geq 0 \quad \text{in } V'.$$

# Lewy-Stampacchia's inequalities

Dual-order assumption :
$A\psi - f = h^+ - h^-$

$Au = -\mathrm{div}[a(x, u, \nabla u)]$ pseudomonotone: (strictly monotone in $\nabla u$ only)

i. If $0 \le h^+ \in L^2(D)$.

$$\mu_\epsilon = \frac{1}{\epsilon}(u_\epsilon - \psi)^- = Au_\epsilon - f$$

Test $\quad Au_\epsilon - A\psi - \frac{1}{\epsilon}(u_\epsilon - \psi)^- = h^- - h^+ \quad$ with $\quad -\mu_\epsilon$

$$-\frac{1}{\epsilon}\langle Au_\epsilon - A\psi, (u_\epsilon - \psi)^-\rangle + \frac{1}{\epsilon^2}\int_D |(u_\epsilon - \psi)^-|^2 dx$$

$$= \underbrace{-\frac{1}{\epsilon}\langle h^-, (u_\epsilon - \psi)^-\rangle}_{\le 0} + \frac{1}{\epsilon}\int_D h^+ (u_\epsilon - \psi)^- dx \qquad \to \qquad \mu_\epsilon \text{ bounded in } L^2$$

$$0 \le \mu_\epsilon = h^+ - (h^+ - \mu_\epsilon) \le h^+ + \underbrace{(h^+ - \mu_\epsilon)^-}_{\to 0 \quad (\textit{technical}!)}$$

# Lewy-Stampacchia's inequalities

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$$= \underbrace{-\frac{1}{\epsilon}\langle h^-, (u_\epsilon - \psi)^-\rangle}_{\leq 0} + \frac{1}{\epsilon}\int_D h^+(u_\epsilon - \psi)^- dx \qquad \rightarrow \qquad \mu_\epsilon \text{ bounded in } L^2$$

$$\mu_\epsilon \rightharpoonup \mu, \qquad Au - f = \mu, \qquad 0 \leq Au - f \leq \big(A\psi - f\big)^+ = h^+.$$

# Lewy-Stampacchia's inequalities

Dual-order assumption : $A\psi - f = h^+ - h^-$

$Au = -\text{div}[a(x,u,\nabla u)]$ pseudomonotone: (strictly monotone in $\nabla u$ only)

ii. $0 \leq h_n^+ \to h^+$ in $V'$.

$$u_n \geq \psi, \quad Au_n - f_n = \mu_n, \qquad 0 \leq Au_n - f_n \leq \left(A\psi - f_n\right)^+ = h_n^+$$

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$$0 \leq \mu_n \leq h_n^+ \;\Rightarrow\; \underbrace{\mu_n \text{ bounded in } V'}_{\varphi \in V,\ \varphi = \varphi^+ - \varphi^- \ !}$$

$$\mu_n \rightharpoonup \mu, \qquad Au - f = \mu, \qquad 0 \leq Au - f \leq \left(A\psi - f\right)^+ = h^+$$

Dual-order assumption : $A\psi - f = h^+ - h^-$

$Au = -\mathrm{div}[a(x,u,\nabla u)]$ pseudomonotone: (strictly monotone in $\nabla u$ only)

$$0 \le \mu_n \le h_n^+ \;\Rightarrow\; \underbrace{\mu_n \text{ bounded in } V'}_{\varphi\in V,\ \varphi=\varphi^+-\varphi^-\ !}$$

$$\mu_n \rightharpoonup \mu, \qquad Au - f = \mu, \qquad 0 \le Au - f \le \big(A\psi - f\big)^+ = h^+$$

$V = W_0^{1,p}(D)$ [A. Mokrane & F. Murat 98-04],

$V = W_0^{1,p(x)}(D)$ [A. Mokrane & G. V. 14]

} + conditions on $a(x,u,\vec{X}) - a(x,v,\vec{X})$.

$V = W_0^{1,p(x)}(D)$ [A. Mokrane, Y. Tahraoui & G. V. 18]

$$\widetilde{a}(x,u,\nabla u) = a(x,\max(u,\psi),\nabla u)$$

& bilateral problem.

# Parabolic case. I. penalization $V = L^2(D) \cap W_0^{1,p}(D)$

$$\forall \epsilon > 0, \quad \exists u_\epsilon \in W(0,T) = \{u \in L^p(0,T,V),\ \partial_t u \in L^{p'}(0,T,V')\},$$

$$\partial_t u_\epsilon \underbrace{-\mathrm{div}\Big[a\big(t,x,\max(\psi,u_\epsilon),\nabla u_\epsilon\big)\Big]}_{\widetilde{A}(u_\epsilon)} - \frac{1}{\epsilon}\Big[\big(u_\epsilon - \psi\big)^-\Big]^{q-1} = f$$

$q = \min(2,p)$

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$$q = \min(2,p)$$

Test with $u_\epsilon - \psi$

$$\|u_\epsilon\|_{C([0,T],L^2)} + \|u_\epsilon\|_{L^p(0,T,V)} + \frac{1}{\epsilon^{1/q}}\|\big(u_\epsilon - \psi\big)^-\|_{L^q(Q)} \leq C$$

$u_\epsilon \rightharpoonup u \geq \psi$ $\qquad$ $\widetilde{A}u_\epsilon \rightharpoonup \chi$

$L^p(0,T,V)$ & $L^\infty(0,T,L^2) - *$ $\qquad$ $L^{p'}(0,T,V')$

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$$\partial_t u_\epsilon \underbrace{-\mathrm{div}\Big[a\big(t,x,\max(\psi,u_\epsilon),\nabla u_\epsilon\big)\Big]}_{\widetilde{A}(u_\epsilon)} - \frac{1}{\epsilon}\Big[\big(u_\epsilon-\psi\big)^-\Big]^{q-1} = f$$

$$q = \min(2,p)$$

Test with $u_\epsilon - \psi$

$$\|u_\epsilon\|_{C([0,T],L^2)} + \|u_\epsilon\|_{L^p(0,T,V)} + \frac{1}{\epsilon^{1/q}}\|\big(u_\epsilon-\psi\big)^-\|_{L^q(Q)} \leq C$$

$$u_\epsilon \rightharpoonup u \geq \psi \qquad \widetilde{A}u_\epsilon \rightharpoonup \chi$$

$$L^p(0,T,V)\ \&\ L^\infty(0,T,L^2) - * \qquad L^{p'}(0,T,V')$$

A three-terms equality

$$\partial_t u_\epsilon = \underbrace{f - \widetilde{A}u_\epsilon}_{bounded} + \underbrace{\frac{1}{\epsilon}\Big[\big(u_\epsilon-\psi\big)^-\Big]^{q-1}}_{\mu_\epsilon???}$$

# I. penalization

$\partial_t u_\epsilon + Au_\epsilon - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{q-1} = f$

Dual-order assumption:

$\partial_t\psi + A\psi - f = h^+ - h^-$ $\quad h^\pm \geq 0 \quad$ in $L^{p'}(0,T,V')$

Test with $-(u_\epsilon - \psi)^-$:

$$\left\langle \partial_t(u_\epsilon - \psi), -(u_\epsilon - \psi)^- \right\rangle - \left\langle Au_\epsilon - A\psi, (u_\epsilon - \psi)^- \right\rangle + \frac{1}{\epsilon}\|(u_\epsilon - \psi)^-\|^q_{L^q(D)}$$

$$= \left\langle \partial_t\psi + A\psi - f, (u_\epsilon - \psi)^- \right\rangle \cdots \leq \cdots \left\langle h^+, (u_\epsilon - \psi)^- \right\rangle \qquad (-h^- \leq 0)$$

# I. penalization

$\partial_t u_\epsilon + Au_\epsilon - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{q-1} = f$

Dual-order assumption:

$\boxed{\partial_t \psi + A\psi - f = h^+ - h^-} \quad h^\pm \geq 0 \quad \text{in } L^{p'}(0,T,V')$

Test with $-(u_\epsilon - \psi)^-$:

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$$= \left\langle \partial_t \psi + A\psi - f, (u_\epsilon - \psi)^- \right\rangle \leq \int_D h^+ (u_\epsilon - \psi)^- dx.$$

i. $0 \leq h^+ \in L^{q'}(Q)$.

$$\frac{1}{\epsilon}\|(u_\epsilon - \psi)^-\|^{q-1}_{L^q(Q)} \leq C \quad \Longrightarrow \quad \partial_t u_\epsilon \quad \text{bounded} \quad L^{p'}(0,T,V')$$

& Aubin-Lions-Simon

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$$\partial_t u_\epsilon + A u_\epsilon - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{q-1} = f$$

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& Aubin-Lions-Simon

$$\int_Q \left| [\widetilde{a}(\cdot, u_\epsilon, \nabla u_\epsilon) - \widetilde{a}(\cdot, \psi, \nabla \psi)] \nabla (u_\epsilon - \psi)^- \right| \leq C(\|h^+\|_{L^{q'}(Q)}) \epsilon^{1/q}$$

CV measure $\nabla u_\epsilon$

$$\exists (u,\mu) \in \cdots, \quad u \geq \psi, \quad \mu \geq 0, \quad \mu(u-\psi) = 0, \quad \partial_t u + Au - f = \mu \geq 0$$

# I. penalization

$$\partial_t u_\epsilon + Au_\epsilon - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{q-1} = f$$

Dual-order assumption:

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$$\ldots ✍ \ldots \left(h^+ - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{q-1}\right)^- \to 0 \qquad (\textit{technical!})$$

$$0 \leq \mu = \partial_t u + Au - f \leq \left(\partial_t \psi + A\psi - f\right)^+ = h^+$$

# II. Lewy-Stampacchia $\partial_t u_\epsilon + Au_\epsilon - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{q-1} = f$

$\partial_t \psi + A\psi - f = h^+ - h^-, \quad h^\pm \geq 0 \quad \text{in } L^{p'}(0,T,V')$

i. $0 \leq h_n^+ \in L^{q'}(Q)$. $\quad \exists (u_n, \mu_n) \in W(0,T) \times L^{p'}(0,T,V'),$

$$u_n \geq \psi, \quad 0 \leq \partial_t u_n + Au_n - f_n = \mu_n \leq h_n^+.$$

ii. $0 \leq h_n^+ \to h^+$ in $L^{p'}(0,T,V')$.

$$u_n \quad \text{bounded in} \quad C([0,T],L^2(D)) \quad \text{and} \quad L^p(0,T,V)$$

$$\Rightarrow Au_n \quad \text{bounded in} \quad L^{p'}(0,T,V')$$

$$0 \leq \mu_n \leq h_n^+ \quad \Rightarrow \quad \partial_t u_n \quad \text{bounded in} \quad L^{p'}(0,T,V').$$

$\varphi \in L^p(0,T,V), \ \varphi = \varphi^+ - \varphi^-$ !

... the result holds.

# Remarks and technical results

$A$ monotone: [F. Donati 82]

$A$ pseudomonotone: [O. Guibé, A. Mokrane, Y. Tahraoui, G. V. 18]

$D \subset \mathbb{R}^N$ bounded Lipschitz domain

$$\left\{ u \in L^p(0,T;W_0^{1,p}(D)),\ \partial_t u \in L^{p'}(0,T;W^{-1,p'}(D)) \right\} \hookrightarrow C([0,T],L^2(D)).$$

Chain rule with $\langle \partial_t u, \Psi(t,x,u) \rangle$.

# III . Parabolic problem with $L^1$ data $\mu = \partial_t u - \text{div}[a(t,x,u,\nabla u)] - f$

$$f \in L^1(Q_T) + L^{p'}(0,T,V') \qquad u_0 \in L^1(D)$$

Dual-order assumption: $\quad f - \partial_t \psi - A(\psi) = g^+ - g^-,$

$$0 \leq g^\pm \in L^1(Q) + L^{p'}(0,T,W^{-1,p'}(D)).$$

**Definition (Entropy solution)**

$$u \in L^\infty\left(0,T,L^1(D)\right), \quad u \geq \psi, \quad \forall k > 0,\ T_k(u) \in L^p(0,T,W_0^{1,p}(D)).$$

$$V_1^p(0,T) = \{u \in L^p(0,T,W_0^{1,p}(D)),\ \partial_t u \in L^{p'}(0,T,W^{-1,p'}(D)) + L^1(Q)\}.$$

$\forall k > 0,\ v \in V_1^p(0,T) \cap L^\infty(Q),\ v \geq \psi \Rightarrow \qquad (\tilde{T}_k' = T_k)$

$$\int_D \tilde{T}_k(v(0) - u_0)dx + \int_0^T \langle \partial_t v - f, T_k(v-u)\rangle dt + \int_Q a(t,x,u,\nabla u)\nabla T_k(v-u)dxdt \geq 0$$

# III . Parabolic problem with $L^1$ data $\mu = \partial_t u - \operatorname{div}[a(t,x,u,\nabla u)] - f$

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Dual-order assumption: $\quad f - \partial_t \psi - A(\psi) = g^+ - g^-,$

$$0 \leq g^\pm \in L^1(Q) + L^{p'}(0,T,W^{-1,p'}(D)).$$

Exists: $\quad (g_n^\pm) \subset L^{p'}(0,T,W_0^{-1,p'}(D))^+$

$$g_n^\pm \to g^\pm \quad \text{in} \quad L^1(Q) + L^{p'}(0,T,W^{-1,p'}(D))$$

$$\Rightarrow \exists (u_n,\mu_n) \in \cdots \qquad \cdots 0 \leq \mu_n \leq g_n^+$$

Pb.:

inequality and not equality

$$L^1(Q) + L^{p'}(0,T,W^{-1,p'}(D)) \subsetneq \left(L^\infty(Q) \cap L^p(0,T,W_0^{1,p}(D))\right)'$$

## Definition (Renormalized Lewy-Stampacchia's inequalities)

$F^i \in L^1(Q), \quad G^i \in L^{p'}(Q)^d \ \text{(i=1,2)} \quad u_0 \in L^1(D).$

$$F^1 - \operatorname{div}\left(G^1\right) \le \frac{\partial u}{\partial t} - \operatorname{div}(a(\cdot,u,\nabla u)) \le F^2 - \operatorname{div}\left(G^2\right) \ \text{ with } \ u(t=0) = u_0$$

means :

$$u \in L^\infty\left(0,T,L^1(D)\right), \quad \forall k>0,\ T_k(u) \in L^p(0,T,W_0^{1,p}(D)),$$

$$\lim_{n\to+\infty} \frac{1}{n}\int_{\{|u|<n\}} a(\cdot,u,\nabla u)\nabla u dxdt = 0$$

For all suitable $S$ and $\varphi$,

$$\int_Q F^1 S'(u)\varphi + G^1\cdot\nabla\left(S'(u)\varphi\right) dxdt$$

$$\le -\int_Q \frac{\partial\varphi}{\partial t}S(u)dxdt - \int_D \varphi(0,x)S\left(u_0\right)dx + \int_Q a(t,x,u,\nabla u)\nabla\left(S'(u)\varphi\right)dxdt$$

$$\le \int_Q F^2 S'(u)\varphi + G^2\cdot\nabla\left(S'(u)\varphi\right)dxdt.$$

## Theorem (Renormalized techniques [O. Guibé, Y. Tahraoui, G. V. 24])

*Compactness & stability for $F_n^i \to F^i$ in $L^1(Q)$, $G_n^i \to G^i$ in $L^{p'}(Q)$ and $u_{0,n} \to u_0$ in $L^1(Q)$.*

# Stochastic case. I. penalization $V = L^2(D) \cap W_0^{1,p}(D)$

$Au = -\text{div}[a(\omega, t, x, u, \nabla u)]$ monotone (+ coercive + growth cond.)

$f \in L^{p'}_{\mathcal{P}}(\Omega_T, V')$, $(\Omega_T = \Omega \times (0, T)$, $\mathcal{P}$ for predictable)

$$\forall \epsilon > 0, \quad \exists u_\epsilon \in L^2\big(\Omega, C([0,T], L^2(D))\big) \,\cap\, L^p_{\mathcal{P}}(\Omega_T, V), \qquad u(0) = u_0,$$

$$\partial_t \left[ u_\epsilon - \int_0^\cdot \underbrace{G(\cdot, \max(u_\epsilon, \psi))}_{\widetilde{G}(\cdot, u)} \, dW \right] \underbrace{-\text{div}\left[ a(\cdot, u_\epsilon, \nabla u_\epsilon) \right]}_{A(u_\epsilon)} - \frac{1}{\epsilon} \left[ \left( u_\epsilon - \psi \right)^- \right]^{q-1} = f$$

$q = \min(2, p)$

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Itô energy:

$$\mathbb{E}\|u_\epsilon\|^2_{C([0,T],L^2)} + \mathbb{E}\|u_\epsilon\|^p_{L^p(0,T,V)} + \tfrac{1}{\epsilon}\mathbb{E}\|\big(u_\epsilon - \psi\big)^-\|^q_{L^q(Q)} \leq C$$

$$u_\epsilon \rightharpoonup u \geq \psi \qquad \widetilde{A}u_\epsilon \rightharpoonup \chi$$

$$L^p_{\mathcal{P}}(\Omega_T, V) \ \& \ L^2_w(\Omega, L^\infty(0,T,L^2)) - * \qquad L^{p'}_{\mathcal{P}}(\Omega_T, V')$$

# I. penalization $du_\epsilon + [Au_\epsilon - \frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{q-1}]dt = fdt + \widetilde{G}(u_\epsilon)dW$

**Dual-order assumption :**

$$\partial_t\Big(\psi - \int_0^\cdot G(\psi)dW\Big) + A\psi - f = h^+ - h^-, \quad h^\pm \geq 0 \quad \text{in } L^{p'}_{\mathcal{P}}(0,T,V')$$

i. $h^+$ regular : Itô formula to simulate $\|(u_\epsilon-\psi)^-\|^2_{L^2}$

$$\mu_\epsilon = -\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{q-1} \quad \text{bounded in} \quad L^{q'}_{\mathcal{P}}(\Omega_T, L^{q'}(D))$$

$$\Longrightarrow (u_\epsilon) \quad \text{Cauchy} \quad \text{in} \quad L^2(\Omega, C([0,T], L^2(D))$$

$$\Longrightarrow \chi = Au \quad \text{Minty.}$$

$$\exists u \in L^p_{\mathcal{P}}(\Omega_T, V) \cap L^2(\Omega, C([0,T], L^2)), \quad \mu \in L^{q'}_{\mathcal{P}}(\Omega_T, L^{q'}(D)), \quad u(0) = u_0,$$
$$\mu \geq 0, \quad u \geq \psi, \quad \mu(u-\psi) = 0, \quad du + [Au - \mu]dt = fdt + G(u)dW.$$

# II. Lewy-Stampacchia

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Similarly

$$\exists v \in L^p_{\mathcal{P}}(\Omega_T, V) \cap L^2(\Omega, C([0,T], L^2)), \quad \lambda \in L^{q'}_{\mathcal{P}}(\Omega_T, L^{q'}(D)), \quad v(0) = u_0,$$
$$\lambda \leq 0, \quad v \leq u, \quad \lambda(v-u) = 0, \quad dv + [Av - \lambda]dt = (f + h^+)dt + G(v)dW.$$

# II. Lewy-Stampacchia

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Uniqueness method: $\quad v \geq \psi, \quad u = v,$

$$0 \leq \partial_t\left(u - \int_0^{\cdot} G(u)dW\right) + Au - f \leq h^+.$$

$h^+$ regular

# II. Lewy-Stampacchia

i. $0 \leq h_n^+ \in L^{q'}_{\mathcal{P}}(\Omega_T, L^{q'}(D))$ $\quad \exists (u_n, \mu_n) \in \cdots ,$

$$0 \leq \partial_t \Big( u_n - \int_0^\cdot G(u_n) dW \Big) + A u_n - f_n = \mu_n \leq h_n^+.$$

ii. $0 \leq h_n^+ \to h^+$ in $L^{p'}_{\mathcal{P}}(\Omega_T, V')$

# II. Lewy-Stampacchia

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ii. $0 \le h_n^+ \to h^+$ in $L^{p'}_{\mathcal{P}}(\Omega_T, V')$

$$0 \le \mu_n \le h_n^+ \quad \text{bounded in} \quad L^{p'}_{\mathcal{P}}(\Omega_T, V')$$

$$\Longrightarrow (u_n) \quad \text{Cauchy} \quad \text{in} \quad L^2(\Omega, C([0,T], L^2(D))$$

$$\Longrightarrow A u_n \rightharpoonup A u \quad \text{Minty.}$$

... the result holds [Y. Tahraoui, G. V. 21].

Rmk. Need of an equation ($i.e.$ $\mu$) to apply Itô's formula

# The pseudomonotone case: $Au = -\mathrm{Div}[a(\cdot, u, \nabla u)]$

## SPDE with constraints and a pseudomonotone operator

• multiplicative noise :
Niklas Sapountzoglou, Yassine Tahraoui, GV & Aleksandra Zimmermann (submitted)

$\max(1, \frac{2d}{d+2}) < p < +\infty$ and $\lambda \in \mathbb{R} \mapsto a(\cdot, \lambda, \vec{\xi})$ Lipschitz.

Compactness methods in stochastic: Prokhorov and Skorokhod

Gyöngy-Krylov

• additive noise :
Niklas Sapountzoglou, Yassine Tahraoui, GV & Aleksandra Zimmermann (in preparation)

Parabolic capacity et diffuse measures.

# The hyperbolic case

## Hyperbolic SPDE with constraints [I. Biswas, Y. Tahraoui, G. V. 23]

$$du - \operatorname{div} f(t,x,u) + \Lambda = g(t,x,u) + G(t,u)dW, \quad 0 \leq u \leq \psi$$

with suitable assumptions.

## Definition (Entropy obstacle formulation)

A solution is $(u, \Lambda_1, \Lambda_2) \in L^2_{\mathcal{P}}(\Omega_T, L^2(\mathbb{R}^d))^3$, s.t. $\Lambda = \Lambda_1 - \Lambda_2$,

$$0 \leq u \leq \psi, \quad 0 \leq \Lambda_1, \Lambda_2, \quad \Lambda_1(u-\psi) = 0, \quad \Lambda_2 u = 0,$$

$$\int_{Q_T} \Big[\eta(u-k)\,\partial_t\varphi - \int_k^u \eta'(\tau-k)\partial_u f(\cdot,\tau)\,d\tau \nabla\varphi + [g(\cdot,u) - \Lambda]\varphi\eta'(u-k)\Big]\,dx\,dt$$
$$+ \sum_{\ell\geq 1}\int_{Q_T} \sigma_\ell(\cdot,u)\eta'(u-k)\varphi\,d\beta_\ell(t)\,dx + \frac{1}{2}\int_{Q_T}\sum_{\ell\geq 1}\sigma_\ell(\cdot,u)^2\eta''(u-k)\varphi\,dx\,dt$$
$$+ \int_{\mathbb{R}^d}\eta(u_0-k)\varphi(0)\,dx \geq 0.$$

$k \in \mathbb{R}$, $(\eta, f)$ regular convex entropy pair, $\varphi \in D^+([0,T[\times\mathbb{R}^d)$

# The hyperbolic case

## Hyperbolic SPDE with constraints [I. Biswas, Y. Tahraoui, G. V. 23]

$$du - \mathrm{div} f(t,x,u) + \Lambda = g(t,x,u) + G(t,u)dW, \quad 0 \leq u \leq \psi$$

with suitable assumptions. $\Lambda = \Lambda_1 - \Lambda_2$

## Theorem (Lewy-Stampacchia's inequalities)

$\exists!(u,\Lambda)$ *solution s.t.*

$$0 \leq \Lambda_1 \leq \Big( -\partial_t \big(\psi - \int_0^\cdot G(\cdot,\psi)dW\big) + divf(\cdot,\psi) + g(\cdot,\psi)\Big)^+$$

$$0 \leq \Lambda_2 \leq g(\cdot,0)^- \qquad \text{in } L^2(\Omega \times Q_T)$$

# The hyperbolic case

## Hyperbolic SPDE with constraints

[I. Biswas, Y. Tahraoui, G. V. 23]

$$du - \operatorname{div} f(t,x,u) + \Lambda = g(t,x,u) + G(t,u)dW, \quad 0 \leq u \leq \psi$$

with suitable assumptions. $\Lambda = \Lambda_1 - \Lambda_2$

Uniqueness:
Kruzkhov's doubling variable (importance of having Lagrange multipliers),
Kato's inequality (properties of the $\Lambda s$).

Existence:
Parabolic obstacle problem with artificial viscosity and Lagrange multiplier (importance of having an equation for Itô's formula),
a priori estimates thanks to parabolic Lewy-Stampacchia's inequalities,
Young-measures.