Well-Posedness of Parabolic Equations in Time-Weighted Spaces

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Motivation (Chemotaxis)

- *Strong solutions* to

$$
\begin{aligned}
&\partial_t u = \Delta u - \mathrm{div}(u\nabla v)\,, && (t,x) \in (0,\infty)\times\Omega\,, && u(0,x) = u^0(x)\,,\\
&\partial_t v = \Delta v + u - v\,, && (t,x) \in (0,\infty)\times\Omega\,, && v(0,x) = v^0(x)\,,\\
&\partial_\nu u = \partial_\nu v = 0\,, && (t,x) \in (0,\infty)\times\partial\Omega\,,
\end{aligned}
$$

typically require certain regularity for initial value u^0, e.g.

$$u^0 \in W_p^1(\Omega)\,.$$

$\rightarrow$ *Global strong* solutions then require W_p^1-bound on u.

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  typically require certain regularity for initial value u^0, e.g.

$$u^0 \in W_p^1(\Omega)\,.$$

  $\to$ *Global strong* solutions then require W_p^1-bound on u.

- Alternatively, for $u^0 \in L_p(\Omega)$ one may use parabolic regularity ensuring

$$u(t)\in W_p^1(\Omega)\,,\quad t>0\,,$$

  and a time-weight to control the singularity at $t=0$:

$$\lim_{t\to 0^+} t^{1/2}\|u(t)\|_{W_p^1} = 0\,.$$

  $\to$ *Global strong* solutions then require only L_p-bound on u.

- Note: $\mathrm{div}(u\nabla v)$ not defined (in L_p) at $u(0)=u^0\in L_p(\Omega)$.

Outline

I. Abstract Semilinear Cauchy Problems (with B. Matioc (Regensburg)):

$$u' = Au + f(u)\,, \quad t > 0\,, \qquad u(0) = u^0 \notin \mathrm{dom}(f)$$

$\rightarrow$ Weaker a priori estimates imply global existence.

II. Coagulation-Fragmentation Equations with Size Diffusion (with Ph. Laurençot (Chambéry)):

$$\partial_t \phi(t,x) = \partial_x^2 \phi(t,x) + \mathcal{F}(\phi)(x) + \mathcal{K}(\phi,\phi)(x)\,, \quad t > 0\,, \qquad \phi(0) = \phi^0$$

$\rightarrow$ Global existence in L_1.

Abstract Semilinear Cauchy Problems

Abstract Semilinear Problems: Setting and Notation

- Fix E_0, E_1: $\mathbb{K}$-Banach spaces with $E_1 \overset{d}{\hookrightarrow} E_0$.

- For $\theta \in (0,1)$, let $E_\theta := (E_0, E_1)_\theta$ be any (real, complex, continuous) interpolation space with

$$E_1 \overset{d}{\hookrightarrow} E_\theta \overset{d}{\hookrightarrow} E_\vartheta \overset{d}{\hookrightarrow} E_0\,, \qquad 0 \le \vartheta \le \theta \le 1\,.$$

- Consider the semilinear parabolic problem

$$u' = Au + f(u)\,, \quad t > 0\,, \qquad u(0) = u^0\,,$$

where

$$A \in \mathcal{H}(E_1, E_0)\,, \qquad f \in C^{1-}(E_\xi, E_0)\,, \qquad u^0 \in E_\alpha\,.$$

That is, $A \in \mathcal{L}(E_1, E_0)$ generates an analytic semigroup $(e^{tA})_{t\ge 0} \subset \mathcal{L}(E_0)$.

- Example: $\Delta_N \in \mathcal{H}(W^2_{p,N}(\Omega), L_p(\Omega))$, $p \in (1,\infty)$,

$$\implies \quad W^{2\theta}_{p,N}(\Omega) \doteq (L_p(\Omega), W^2_{p,N}(\Omega))_{\theta,p}, \quad 2\theta \ne 1, 1+1/p$$

Key Properties of Analytic Semigroups

- Any (reasonable) solution to

$$u' = Au + f(u)\,, \quad t \in (0, T]\,, \qquad u(0) = u^0\,,$$

with $A \in \mathcal{H}(E_1, E_0)$ solves the fixed point equation (variation-of-constant-formula)

$$u(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A} f(u(s))\,\mathrm{d}s\,, \quad t \in [0, T]\,.$$

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- Recall that

$$E_1 \hookrightarrow E_\theta \hookrightarrow E_\vartheta \hookrightarrow E_0\,, \qquad 0 \le \vartheta \le \theta \le 1\,.$$

Parabolic regularizing effects mean that

$$e^{tA} : E_0 \to E_1 \quad \text{for } t > 0$$

and can be expressed as ($M \ge 1$, $\omega \in \mathbb{R}$)

$$\|e^{tA}\|_{\mathcal{L}(E_\theta)} + t^{\vartheta-\theta}\|e^{tA}\|_{\mathcal{L}(E_\vartheta, E_\theta)} \le Me^{\omega t}\,, \qquad t \ge 0\,, \quad 0 \le \vartheta \le \theta \le 1\,.$$

The "Standard" Case $u^0 \in \mathrm{dom}(f)$

- Suppose that

$$f \in C^{1-}(E_\xi, E_0)\,, \qquad u^0 \in E_\alpha\,, \qquad 0 \le \xi < \alpha < 1\,. \tag{1}$$

Note that $u^0 \in E_\alpha \hookrightarrow E_\xi$, i.e. $f(u^0)$ defined.

Theorem (Da Prato-Grisvard '75, Sinestrari-Vernole '77, Amann '84,...)

Suppose (1). *Then*

$$u' = Au + f(u)\,, \quad t \in (0,T]\,, \qquad u(0) = u^0\,,$$

has a unique maximal strong solution

$$u(\cdot; u^0) \in C([0,t^+), E_\alpha) \cap C^1((0,t^+), E_0) \cap C((0,t^+)), E_1)\,.$$

The map $(t, u^0) \mapsto u(t; u^0)$ *is a semiflow on* E_α. *If* $t^+ < \infty$, *then*

$$\lim_{t \nearrow t^+} \|u(t; u^0)\|_{E_\alpha} = \infty\,.$$

Proof: Banach's fixed point theorem. □

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- Q: If $\alpha < \xi$, do we still get a semiflow?
Then $E_\alpha \not\subset E_\xi$ and $t \mapsto f(u(t))$ has a singularity at $t = 0$ $\to$ time-weighted spaces

Time-Weighted Spaces

- Let $T > 0$, $\sigma \in \mathbb{R}$, and E Banach space. Then the time-weighted space

$$C_\sigma((0,T],E) := \left\{ u \in C((0,T],E) \ : \ \lim_{t\to 0} t^\sigma \|u(t)\|_E = 0 \right\}$$

  equipped with the norm

$$\|u\|_{C_\sigma((0,T],E)} := \sup\left\{ t^\sigma \|u(t)\|_E \ : \ t \in (0,T] \right\}$$

  is a Banach space.

- Note that

$$C_\sigma((0,T],E) \hookrightarrow C_\nu((0,T],E)\,, \quad \sigma \le \nu\,.$$

- The rhs $f = f(u)$ needs to be Lipschitz and “behave well” on time-weighted spaces.

The Semilinearity on Time-Weighted Spaces

- Assume $f : E_\xi \to E_0$ is Lipschitz in the sense that, for some $q \geq 1$ (and all $R > 0$),

$$\|f(w) - f(v)\|_{E_0} \leq c(R)\big(1 + \|w\|_{E_\xi}^{q-1} + \|v\|_{E_\xi}^{q-1}\big)\|w - v\|_{E_\xi}, \qquad w, v \in E_\xi \cap \mathbb{B}_{E_\alpha}(0, R).$$

  In particular,

$$\|f(v)\|_{E_0} \sim \|v\|_{E_\xi}^{q}, \quad \|v\|_\xi \to \infty .$$

- Then, since

$$\|u(t)\|_{E_\xi} \leq t^{-\sigma}||u||_{C_\sigma((0,T],E_\xi)}, \quad t \in (0, T],$$

  the Nemytskii operator

$$\big(u \mapsto f(u)\big) : C_\sigma((0, T], E_\xi) \cap C([0, T], E_\alpha) \to C_{q\sigma}((0, T], E_0)$$

  is Lipschitz continuous.

- In applications: q is usually given (e.g. $q = 2$ if f is bilinear), but there is some flexibility w.r.t. the functional analytic setting.

Fixed Point Problem

- Let $0 \leq \alpha < \xi < 1$, hence $E_\xi \hookrightarrow E_\alpha$. Consider $f : E_\xi \to E_0$ and $u^0 \in E_\alpha$.
- We try to find $u \in C_\sigma((0,T], E_\xi) \cap C([0,T], E_\alpha)$ solving

$$u(t) \stackrel{!}{=} e^{tA} u^0 + \int_0^t e^{(t-s)A} f(u(s))\, \mathrm{d}s \,, \quad t \in [0,T] \,.$$

Note: if $u^0 \in E_\alpha$, then

$$\|e^{tA} u^0\|_{E_\xi} \leq c t^{\alpha - \xi} \|u^0\|_{E_\alpha} \,, \quad t > 0 \,,$$

that is, $\sigma = \xi - \alpha$.

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- Since, for $u \in C_\sigma((0,T], E_\xi)$,

$$\|e^{(t-s)A}\|_{\mathcal{L}(E_0,E_\xi)} \sim (t-s)^{-\xi}\,, \qquad \|f(u(s))\|_{E_0} \sim s^{-q\sigma}\,,$$

we have

$$\left\| \int_0^t e^{(t-s)A} f(u(s))\,\mathrm{d}s \right\|_{E_\xi} \sim t^{1-\xi-q\sigma} \overset{!}{\sim} t^{-\sigma}\,.$$

This requires at least that $1 - \xi - q\sigma \geq -\sigma$, that is,

$$q(\xi - \alpha) \overset{!}{\leq} 1 - \alpha\,.$$

Main Theorem

- Let $f : E_\xi \to E_0$ be Lipschitz with $(q \geq 1)$

$$\|f(w)-f(v)\|_{E_0} \leq c(R)\big(1+\|w\|_{E_\xi}^{q-1}+\|v\|_{E_\xi}^{q-1}\big)\|w-v\|_{E_\xi}\,, \qquad w,v \in E_\xi \cap \mathbb{B}_{E_\alpha}(0,R)\,, \quad (2)$$

and

$$0 \leq \alpha < \xi < 1\,, \qquad q(\xi-\alpha) \leq 1-\alpha\,. \quad (3)$$

Theorem (Matioc-W. '23)

Let $A \in \mathcal{H}(E_1, E_0)$ and suppose (2) and (3). Then, for each $u^0 \in E_\alpha$, the problem

$$u' = Au + f(u)\,, \quad t > 0\,, \qquad u(0) = u^0\,,$$

has a unique maximal strong solution

$$u(\cdot;u^0) \in C_{\xi-\alpha}((0,t^+),E_\xi) \cap C([0,t^+),E_\alpha) \cap C^1((0,t^+),E_0) \cap C((0,t^+)),E_1)\,.$$

The map $(t,u^0) \mapsto u(t;u^0)$ is a semiflow on E_α. If $t^+ < \infty$, then

$$\limsup_{t\to t^+} \|u(t;u^0)\|_{E_\alpha} = \infty\,.$$

Remarks

- *Weaker a-priori estimates yield global existence for strong solutions*.

- The critical case

$$q(\xi - \alpha) = 1 - \alpha\,, \quad \text{i.e.} \quad \alpha = (q\xi - 1)/(q - 1) < 1\,,$$

  is more difficult. In certain applications to pdes, E_α then corresponds to critical spaces with scaling invariance.

- A similar result is true for quasilinear parabolic equations

$$u' = A(u)u + f(u)\,, \quad t > 0\,, \qquad u(0) = u^0\,,$$

  with $u^0 \in \mathrm{dom}(A(\cdot)) \setminus \mathrm{dom}(f)$ (again, this is considerably more technical and uses evolution operators $U_{A(u)}(t,s)$).

- Related results were proven in the framework of maximal L_p-regularity (LeCrone-Prüß-Wilke '14, Prüß-Wilke '17, Hytönen-van Nerven-Veraar-Weis '23) or continuous maximal regularity (LeCrone-Simonett '20).

Application to Chemotaxis

For $\Omega \subset \mathbb{R}^n$ open, bounded, smooth consider

$$\begin{aligned} \partial_t u &= \Delta u - \mathrm{div}(u\nabla v)\,, \qquad t > 0\,, \quad x \in \Omega\,, \\ \partial_t v &= \Delta v + u - v\,, \qquad t > 0\,, \quad x \in \Omega\,, \end{aligned}$$

subject to Neumann boundary conditions $\partial_\nu u = \partial_\nu v = 0$ on $\partial\Omega$ and

$$(u,v)(0,\cdot) = (u^0, v^0)\,.$$

Proposition

For $\max\{1, n/2\} < p < \infty$ *and* $(u^0, v^0) \in L_p^+(\Omega) \times W_{2p}^{1,+}(\Omega)$ *there is a unique maximal strong solution*

$$(u,v) \in C\big([0,t^+), L_p(\Omega) \times W_{2p}^1(\Omega)\big)\,.$$

If $t^+ < \infty$, *then*

$$\limsup_{t\to t^+} \|u(t)\|_{L_p} = \infty\,.$$

Proof: $E_0 := W_p^{-2\varepsilon}(\Omega) \times W_{2p}^{1-2\varepsilon}(\Omega)$ and $E_1 := W_p^{1-2\varepsilon}(\Omega) \times W_{2p,N}^{3-2\varepsilon}(\Omega)$ with $\varepsilon > 0$. □

- Biler '98: Weak solutions.

Part II

Coagulation-Fragmentation Equations with Size Diffusion

Coagulation-Fragmentation with Size Diffusion

- Applications: growth of microtubules, plankton cells, ice crystals, ...
- Literature: Melzak '57, Aizenman & Bak '79, Ferkinghoff-Borg et al. '03,...
- $\phi = \phi(t,x) \geq 0$ distribution function of particles of size $x \in (0,\infty)$ and time $t \geq 0$.
- Mechanisms: random fluctuation: $D\partial_x^2$

  fragmentation: $\{x\} \quad \stackrel{a(x)b(y,x)}{\longrightarrow} \quad \{y\} + \{...\} + \{...\}$

  coagulation: $\{x\} + \{y\} \quad \stackrel{k(x,y)}{\longrightarrow} \quad \{x+y\}$

- Standard assumptions: a, b, k are non-negative, measurable, and

$$D = 1\,, \qquad \int_0^x y b(y,x)\,\mathrm{d}y = x\,, \qquad k(x,y) = k(y,x)\,.$$

The Model (Ferkinghoff-Borg et al. '03, '06)

$$\partial_t \phi(t,x) = \partial_x^2 \phi(t,x) + \mathcal{F}\big(\phi(t,\cdot)\big)(x) + \mathcal{K}\big(\phi(t,\cdot),\phi(t,\cdot)\big)(x)\,,$$
$$\phi(t,0) = 0\,, \qquad \phi(0,x) = \phi^0(x)\,,$$

for $(t,x) \in (0,\infty)^2$, where the (linear) fragmentation operator is

$$\mathcal{F}(\phi)(x) := -a(x)\phi(x) + \int_x^\infty a(y)b(x,y)\phi(y)\,\mathrm{d}y$$

and the (bilinear) coagulation operator is

$$\mathcal{K}(\phi,\psi)(x) := \frac{1}{2}\int_0^x k(y,x-y)\phi(y)\psi(x-y)\,\mathrm{d}y - \phi(x)\int_0^\infty k(x,y)\psi(y)\,\mathrm{d}y\,.$$

- The total mass is (formally) conserved:

$$\int_0^\infty x\phi(t,x)\,\mathrm{d}x = \int_0^\infty x\phi^0(x)\,\mathrm{d}x\,.$$

- The phase space is (for technical reasons: $m > 1$) $E_0 := L_1((0,\infty),(x+x^m)\mathrm{d}x)$.

The Linear Part (Diffusion & Fragmentation)

- The linear part is of the form

$$A\phi := \partial_x^2\phi + \mathcal{F}(\phi) = \partial_x^2\phi - a\phi + B\phi\,,$$

where $B \geq 0$ is the nonlocal operator

$$(B\phi)(x) := \int_x^\infty a(y)b(x,y)\phi(y)\,\mathrm{d}y\,.$$

- Assume that ($\delta \in (0,1)$)

$$a \in L^+_{\infty,loc}([0,\infty))\,, \qquad (1-\delta)x^2 \geq \int_0^x y^2 b(y,x)\,\mathrm{d}y\,.$$

- Example: $a(x) = a_0 x^\kappa$, $b(y,x) = (\zeta+2)y^\zeta x^{-(1+\zeta)}$, $-1 < \zeta \leq 0 \leq \kappa$.

- With $E_0 = L_1((0,\infty),(x+x^m)\mathrm{d}x)$ set

$$E_1 := \{\phi \in E_0\ :\ \partial_x^2\phi \in E_0\,,\ a\phi \in E_0\,,\ \phi(0)=0\}\,, \qquad \|\phi\|_{E_1} := \|\phi\|_{E_0} + \|\partial_x^2\phi\|_{E_0} + \|a\phi\|_{E_0}\,.$$

Generator

Proposition

We have

$$A = \partial_x^2 - a + B \in \mathcal{H}_+(E_1, E_0)\,,$$

that is, A generates a positive analytic semigroup $(e^{tA})_{t\geq 0}$ on E_0. Moreover,

$$\int_0^\infty x(e^{tA}\phi)(x)\,\mathrm{d}x = \int_0^\infty x\phi(x)\,\mathrm{d}x\,, \quad t \geq 0\,.$$

Proof: ∂_x^2 : positive contraction semigroup on $E_0 = L_1((0,\infty),(x+x^m)\mathrm{d}x)$
$\partial_x^2 - a$: absorption semigroup (using $a \geq 0$)
$\partial_x^2 - a + B$: Miyadera perturbation. □

- Further properties of A available (under additional assumptions).

Interpolation Spaces

Define the complex interpolation spaces

$$E_\xi := [E_0, E_1]_\xi\,, \quad \xi \in [0,1]\,,$$

between

$$E_0 = L_1((0,\infty),(x+x^m)\mathrm{d}x)\,, \qquad E_1 = \{\phi \in E_0\ :\ \partial_x^2\phi \in E_0\,,\ a\phi \in E_0\,,\ \phi(0)=0\}\,.$$

Lemma

Let $0 \le \xi_0 < \xi < 1$ *and set*

$$\ell_\xi(x) := \begin{cases} x^{1-2\xi_0}\,, & x \in (0,1)\,,\\ (1+a(x))^\xi x^m\,, & x > 1\,. \end{cases}$$

Then

$$E_\xi \hookrightarrow L_1\big((0,\infty),\ell_\xi(x)\mathrm{d}x\big) =: Y_\xi\,.$$

Proof: One can show that $E_1 \hookrightarrow L_1\big((0,\infty),x^r\mathrm{d}x\big) =: X_r$ for $r \in (-1,1]$.

Hence $E_\xi \hookrightarrow [X_1, X_{r-1}]_\xi = X_{1+\xi(r-1)}$ with $r := 1 - 2\xi_0/\xi$. □

The Bilinear Part (Coagulation)

- For the bilinear coagulation operator

$$\mathcal{K}(\phi, \psi)(x) = \frac{1}{2} \int_0^x k(y, x-y)\phi(y)\psi(x-y)\,\mathrm{d}y - \phi(x) \int_0^\infty k(x,y)\psi(y)\,\mathrm{d}y$$

assume for some $k_* > 0$ that

$$0 \le k(x,y) = k(y,x) \le k_* \frac{\ell_\xi(x)(y+y^m) + \ell_\xi(y)(x+x^m)}{x+y+(x+y)^m}\,, \qquad (x,y) \in (0,\infty)^2\,, \quad (4)$$

Proposition

Assume (4) *and let* $Y_\xi = L_1\big((0,\infty), \ell_\xi(x)\mathrm{d}x\big)$ *with* $\xi \in (0,1)$. *Then* $\mathcal{K} : Y_\xi \times Y_\xi \to E_0$ *is bilinear and*

$$\int_0^\infty x\mathcal{K}(\phi,\phi)(x)\,\mathrm{d}x = 0\,.$$

In particular, $f(\phi) := \mathcal{K}(\phi,\phi)$ *satisfies the Lipschitz condition on* E_ξ *with* $q = 2$.

Well-Posedness

Theorem (Laurençot-W. '22)

Given any non-negative $\phi^0 \in E_\alpha$ *with* $\alpha \in [0,1)$ *and* $0 < 2\xi \leq 1+\alpha$, *the coagulation-fragmentation equation with size diffusion has a unique, maximal, non-negative, mass-conserving solution*

$$\phi(\cdot;\phi^0) \in C_{\xi-\alpha}\big((0,t^+),E_\xi\big) \cap C\big([0,t^+),E_\alpha\big) \cap C\big((0,t^+),E_1\big) \cap C^1\big((0,t^+),E_0\big) .$$

If $t^+ < \infty$, *then*

$$\limsup_{t\to t^+} \|\phi(t;\phi^0)\|_{E_\alpha} = \infty .$$

- Note: $\|\cdot\|_{E_\alpha}$ only explicitly known for $\alpha = 0$!
- If $0 < \xi \leq 1/2$, then $\alpha = 0$ is possible and a priori estimates in

$$E_0 = L_1((0,\infty),(x+x^m)\mathrm{d}x)$$

imply global existence.

Global Existence of Mass-Conserving Solutions

Corollary

If $0 < \xi \leq 1/2$ *and*

$$k(x,y) \leq k_0 \frac{xy}{x+y} \left[(1+a(x))^{\xi} + (1+a(y))^{\xi} \right] , \quad (x,y) \in (1,\infty)^2 ,$$

then, for any non-negative $\phi^0 \in E_0$,

$$\sup_{t \in [0,t^+) \cap [0,T]} \|\phi(t;\phi^0)\|_{E_0} < \infty , \quad T > 0 ,$$

that is, $t^+ = \infty$.

- $\alpha = 0$ only possible in time-weighted spaces.
- Neither "classical" theory (no a priori estimates known in E_ξ) nor maximal regularity theory (in natural phase space L_1) seem to work in order to derive global existence.

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