What is a solution of a PDE? Trial lecture August 2017 1

Consider the Cauchy problem

$$(SCL)\quad \begin{cases} \partial_t u + \operatorname{div}(F(u)) = 0 & \text{in } \mathbb{R} \times (0,T) \\ u(x,0) = u_0(x) & \text{on } \mathbb{R} \end{cases}$$

$F = (F_1, F_2, \dots, F_N) : \mathbb{R} \to \mathbb{R}^N$

where $\operatorname{div}(F(u)) = \sum_{i=1}^{N} \partial_{x_i}(F_i(u))$ and $u_0 : \mathbb{R}^N \to \mathbb{R}$ are some functions to be determined

- <u>Classical solutions</u>

<u>Def</u>0: We say that u is a classical solution of (SCL) if it satisfies the problem pointwise everywhere.

Let us now make the choice $F(u) = \frac{1}{2}u^2$. Then (SCL) reduces to the inviscid Burgers' equation: $\partial_t u + (\frac{1}{2}u^2)_x = 0$. Denote it by (SCL^B)

We clearly see that a classical solution must be C^1 in space and time $\left((\frac{1}{2}u^2)_x = u u_x \right)$.

How does the C^1-solutions of (SCL^B) look like?

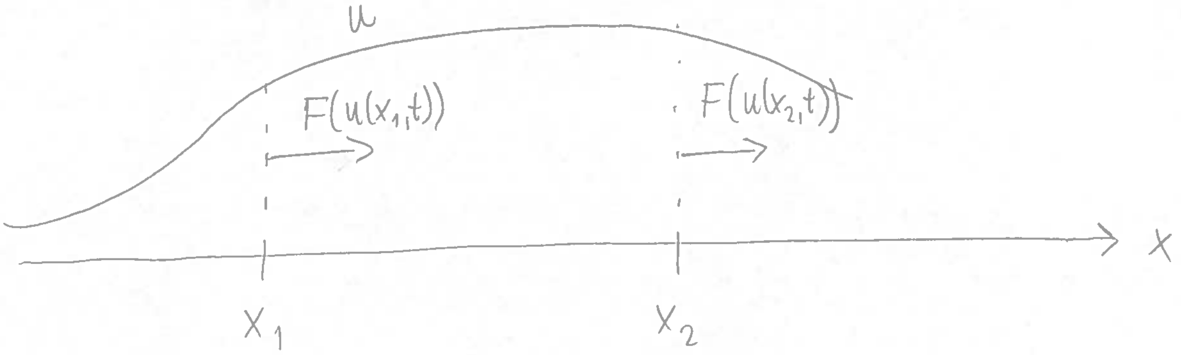
Classical solutions

• Why do we study (SCL) ?

If u, F are regular enough, we can integrate (SCL) in x to get

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = F(u(x_1,t)) - F(u(x_2,t))$$

We see that the quantity measured by u is conserved in the sense that the total amount of u can only change (in time) due to the inflow at the point x_1 and the outflow at the point x_2. F is thus called the flux function.



Example

u can describe the density of cars on a road (where the cars are points on a line), and

$F(u) = v(u)\,u$ where v is the speed of the cars.

$\frac{\text{km}}{\text{h}}$ $\frac{\text{**}}{\text{km}}$

Why do we study (SCL)?

classical

• What do we want from a solution?

Even though (SCL) appears to be a simple equation it is not completely clear that we can solve it explicitly. In mathematics we mostly care about the properties of the solution. The properties should reflect the real world scenario (say cars on a road). In our 1D example, it should for instance be true that if we start with 100 cars and they move along the 1D road, we should be able to count 100 cars later on (remember that (SCL) do not add/remove any cars to our scenario). In mathematics we call this property "conservation of mass".

Hadamard's well-posedness

Mathematical models of physical phenomena should have properties such that

- a solution exists
- the solution is unique
- the solution's behaviour changes continuously with the initial conditions.

• Method of characteristics

The idea is to consider the solution of (SCL^B) as the surface $\{(t, x, u(x,t)) \mid (t,x) \in \mathbb{R}^2\} \subset \mathbb{R}^3$.
Let Γ be a curve in $\mathbb{R}^3$ parameterized by $(t(\eta), x(\eta), z(\eta))$. We want to construct a surface $S \subset \mathbb{R}^3$ parameterized by $(t, x, u(x,t))$ such that $u = u(x,t)$ satisfies $1 u_t + u u_x = 0$ and $\Gamma \subset S$. That is, we solve

$$\frac{\partial t}{\partial \xi} = 1, \quad \frac{\partial x}{\partial \xi} = u, \quad \frac{\partial z}{\partial \xi} = 0$$

with

$$t(\xi_0, \eta) = t(\eta), \quad x(\xi_0, \eta) = x(\eta), \quad z(\xi_0, \eta) = z(\eta).$$

By inverting $x = x(\xi, \eta)$, $t = t(\xi, \eta)$, we get $\xi = \xi(x,t)$ and $\eta = \eta(x,t)$. Then

$$u(x,t) = z(\xi(x,t), \eta(x,t))$$

satisfies (SCL^B) and the condition $\Gamma \subset S$.

$$\left(\frac{du}{d\xi} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial \xi} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} = 0 \right)$$

$$\left.\begin{aligned}\frac{\partial t}{\partial \xi} &= 1\\ \frac{\partial x}{\partial \xi} &= u\\ \frac{\partial z}{\partial \xi} &= 0\end{aligned}\right\}\quad \left.\begin{aligned} t &= \xi + t_0\\ x &= z\xi + x_0\\ z &= z_0\end{aligned}\right\}\quad x = x_0 + z_0(t - t_0)$$

The initial condition is $u(x,0) = u_0(x)$ which gives

$t(0,\eta) = 0$, $x(0,\eta) = \eta$, $z(0,\eta) = u_0(\eta)$.

Hence,

$$t_0 = 0, \quad x_0 = \eta, \quad z_0 = u_0(\eta)$$

We get

$$t(\xi,\eta) = \xi(x,t)$$
$$x(\xi,\eta) = u_0(\eta(x,t))\,\xi(x,t) + \eta(x,t)$$

When is it possible to invert this relation to get $\xi = \xi(x,t)$, $\eta = \eta(x,t)$? According to the implicit function theorem, we compute

$$J = \begin{bmatrix} \frac{\partial t}{\partial \xi} & \frac{\partial t}{\partial \eta} \\ \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ u_0 & u_0'(\eta)\xi + 1 \end{bmatrix}$$

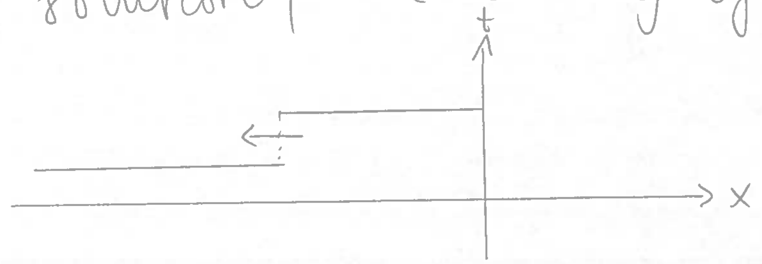
$$\Rightarrow \det J = u_0'(\eta)\xi + 1 = u_0'(\eta)t + 1$$

The determinant has then to be not zero (6)
if a solution should exist for some open set of (x,t) The implicit function thm. As $t \geq 0$, we need to enforce $u_0' > 0$ for this to happen. If not, say there exists $\tilde{x}$ such that $u_0'(\tilde{x}) < 0$, then $1 + t^* u_0'(\tilde{x}) = 0$ when $t^* = -\frac{1}{u_0'(\tilde{x})}$.

Conclusion: If we want smooth solutions, they will only exist for a finite time depending on the initial data. Exist until the characteristics meet.

<u>Example</u>

Let us go back to our car example. We know that when a traffic light changes from green to red not all cars stop driving at the same time. There is a shock or discontinuity travelling backwards, that is, our solution, the density of cars is discontinuous.



- Distributional solution

Let us assume that u is a classical solution of (SCL). Multiplying the equation by a very nice function φ, we get after integrating and integrating by parts that

$$\int_0^\infty \int_{-\infty}^\infty (u\,\varphi_t + f(u)\,\varphi_x)\,dx\,dt = 0 \quad \forall\, \varphi \in C_c^\infty$$

Def 1: Let $u_0 \in L^1_{loc}$ $F \in W^{1,\infty}$ Then $u \in L^1_{loc}$ is called a distributional solution of (SCL) if

(i) $\partial_t u + \partial_x(F(u)) = 0$ in $\mathcal{D}'(\mathbb{R} \times (0,T))$

(ii) $\operatorname*{ess\,lim}_{t \to 0^+} \int u(x,t)\,\varphi(x,t)\,dx = \int u_0(x)\,\varphi(x,t)\,dx \quad \forall\, \varphi \in C_c^\infty$

Note that if u is regular enough, a distributional solution u is a classical solution.

Definition 1 (i) implies the rather well-known Rankine-Hugoniot condition:

$$s'(t)(u_l - u_r) = F_l - F_r$$

, $s(t)$ the shock curve, the subscripts l and r denotes left-and right-side of the shock.

This condition restricts the class of admissible shocks. That is, when characteristics collide, we can continue the solution using the Rankine-Hugoniot condition. ⑧

However, consider (SCL^B) with the initial data

$$u_0(x) = \begin{cases} 1, & x < 0 \\ -1, & x > 0 \end{cases}$$

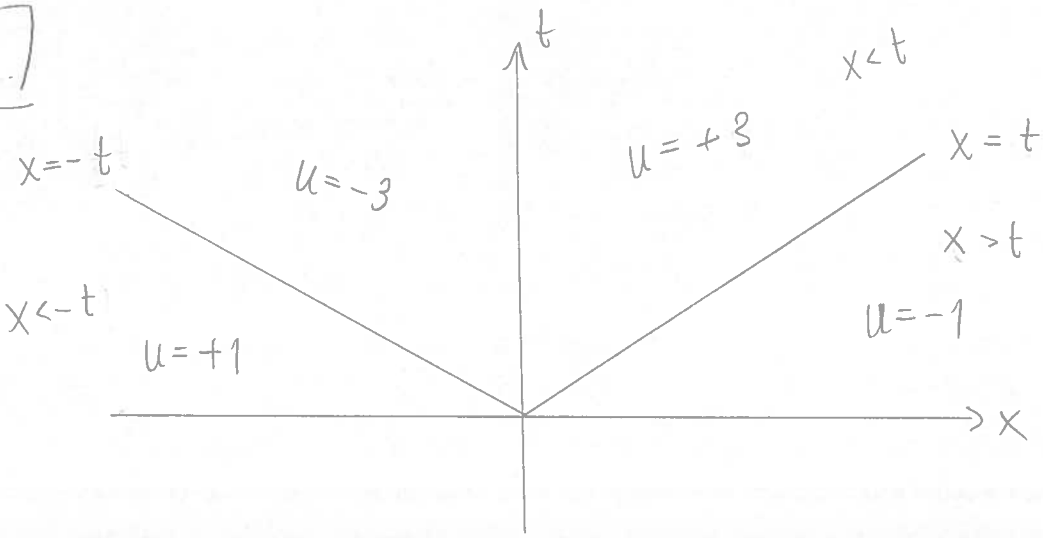
For each $\alpha \geq 1$,

$$u(x,t) = \begin{cases} +1, & 2x < (1-\alpha)t \\ -\alpha, & (1-\alpha)t < 2x < 0 \\ +\alpha, & 0 < 2x < (\alpha-1)t \\ -1, & (\alpha-1)t < 2x \end{cases}$$

is the distributional solution.

$$\text{R-H}: \quad x'(t) = \frac{\frac{1}{2}u_\ell^2 - \frac{1}{2}u_r^2}{u_\ell - u_r} = \frac{1}{2}(u_\ell + u_r)$$

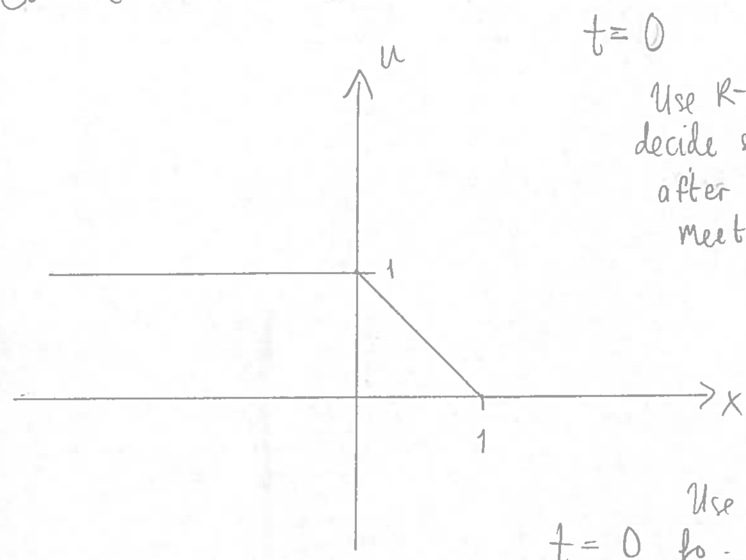
$\boxed{\alpha = 3.}$



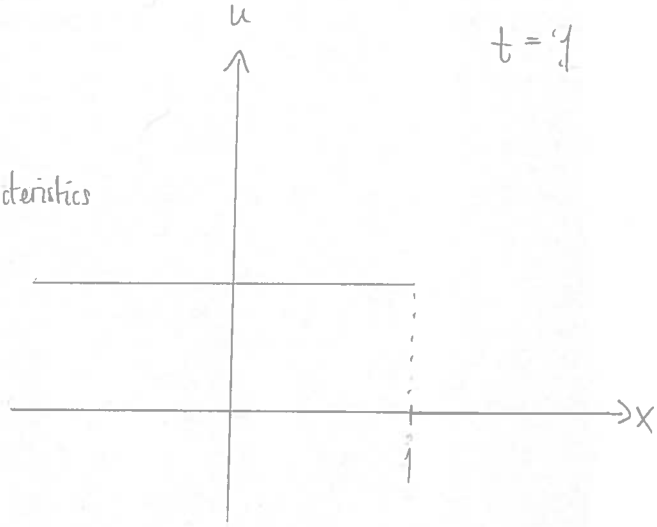
- <u>Entropy solutions</u>

Even though the concept of distributional solutions allowed solutions with low regularity, the concept simply allowed for too many solutions.

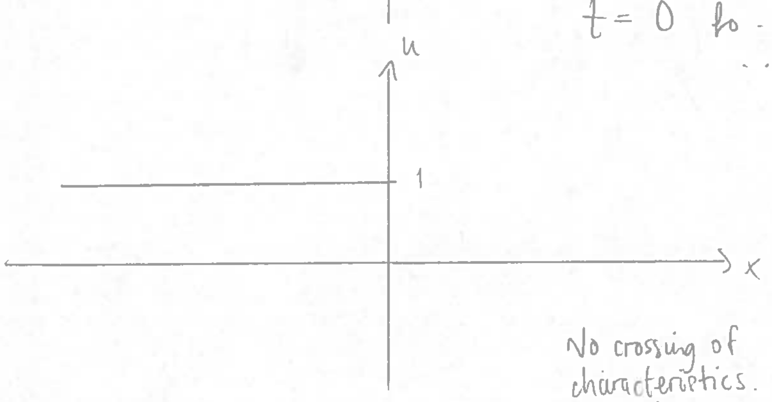
Consider (SCL^B):

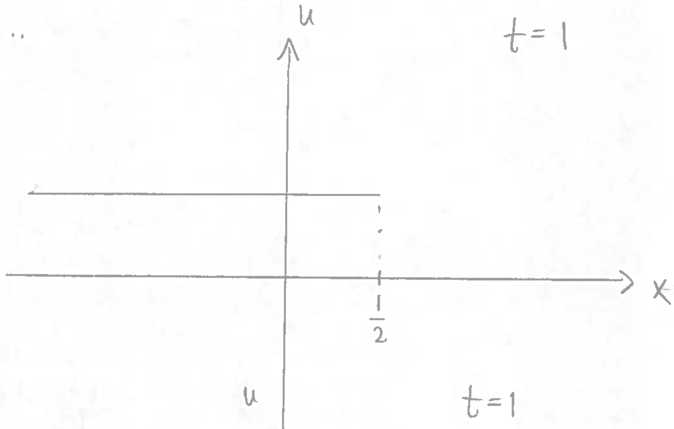


Use R-H to decide soln after characteristics meet

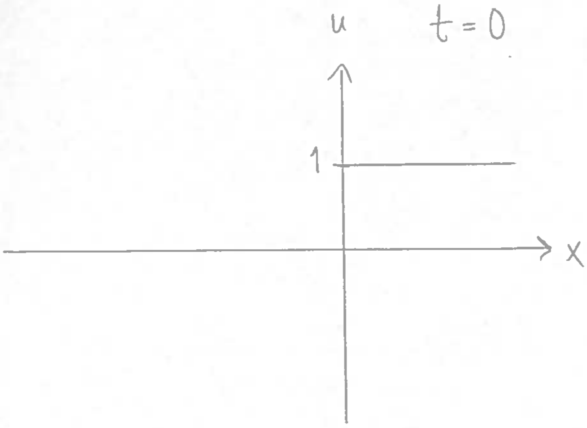


Use R-H to ....

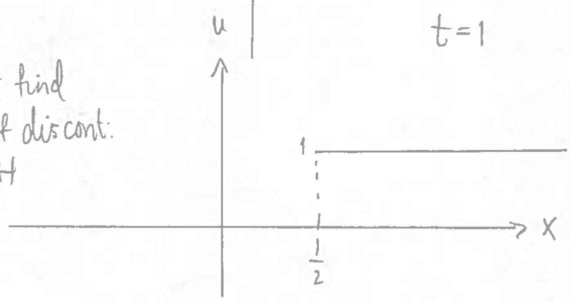




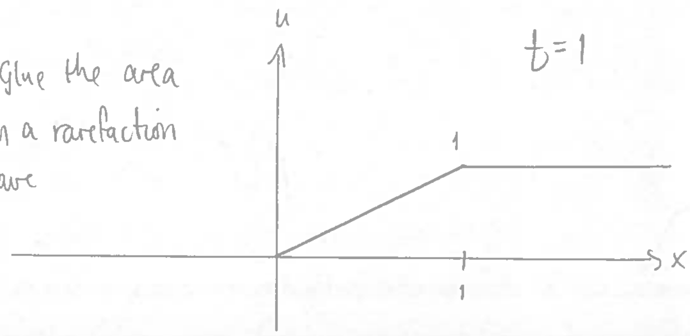
No crossing of characteristics. Area with no info.



1. Just find curve of discont. by RH



2. Glue the area with a rarefaction wave



From the characteristics, we note that the speed of the solution is

$$\frac{dx}{dt} = u$$

That is, bigger value of u gives bigger speed. In the last example, we expect the part of the solution to the right to move faster. We thus do not want situation 1. above.

We need a condition to rule out such an unphysical solution.

Def 2: Let $F \in W^{1,\infty}_{loc}$, $u_0 \in L^\infty$. Then $u \in L^\infty$ is called an entropy solution of (SCL) if

(i) $\partial_t(\eta(u)) + \partial_x(q(u)) \leq 0$ in $\mathcal{D}'_+(\mathbb{R}^1 \times (0,T))$

(ii) $\operatorname{ess\,lim}_{t \to 0^+} \int \eta(u)\, \varphi(x,t)\, dx = \int \eta(u_0)\, \varphi(x,0)\, dx \quad \forall\, 0 \leq \varphi \in C_c^\infty$

Remark: • η is any convex function and $q' = F'\eta'$.

- Enough to take $\eta(u) = |u-k|$, $q(u) = \operatorname{sign}(u-k)(F(u) - F(k))$ for all $k \in \mathbb{R}$.
- Can assume $F \in W^{1,\infty}$ since solutions are bounded, and also that $F(0) = 0$ by adding constants to the equation.

Theorem (Kružkov, 1970)

Assume $F \in W^{1,\infty}_{loc}$ and $u_0 \in L^\infty$. Then there exists a unique entropy solution $u \in L^\infty(\mathbb{R} \times (0,T)) \cap C([0,T]; L^1_{loc}(\mathbb{R}))$ satisfying, for $R>0$ and $L_F := \operatorname{ess\,sup} |F'|$,

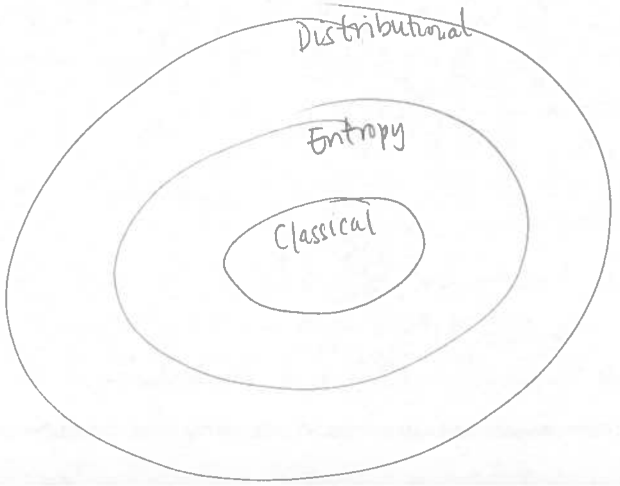
$$\int_{|x|<R} |u(x,t) - v(x,t)|\, dx \leq \int_{|x|<R+L_F t} |u_0(x) - v_0(x)|\, dx$$

If, in addition, $u_0 \in L^\infty \cap L^1$, and hence, $u \in L^\infty(\mathbb{R} \times (0,T)) \cap C([0,T]; L^1(\mathbb{R}))$. Then

$$\| u(\cdot,t) - v(\cdot,t) \|_{L^1} \leq \| u_0 - v_0 \|_{L^1} .$$

Remark: • The boundedness is needed in the last part to make sure that $\eta(u), q(u) \in L^1_{loc}$ in Definition 2 (i).

We note that entropy solutions are distributional solutions (choose $k = \pm \|u\|_{L^\infty}$), and any classical solution $-$ in $L^\infty \cap C(L^1_{loc})$ is an entropy solution (multiply by $\eta'(u)$ and integrate).



Proof: Let us do a formal proof of the L^1-contraction. 11A
By Definition 2(i) and since it is enough to take $\eta(u) = |u-k|$, $q(u) = \mathrm{sign}(u-k)(F(u)-F(k))$ for all $k \in \mathbb{R}$, we get

$$\partial_t(|u-k|) + \partial_x\big(\mathrm{sign}(u-k)(F(u)-F(k))\big) \leq 0$$
$$\partial_s(|v-k|) + \partial_y\big(\mathrm{sign}(v-k)(F(v)-F(k))\big) \leq 0.$$

Let $k = v(y,s)$ in the equation for u, $k = u(x,t)$ in the equation for v, and use that $|v-u| = |u-v|$, $\mathrm{sign}(v-u) = -\mathrm{sign}(u-v)$ to obtain

$$\partial_t(|u-v|) + \partial_x\big(\mathrm{sign}(u-v)(F(u)-F(v))\big) \leq 0$$
$$\partial_s(|u-v|) + \partial_y\big(\mathrm{sign}(u-v)(F(u)-F(v))\big) \leq 0$$

Multiply each of the above inequalities by $\frac{1}{2}\delta_{y=x}\delta_{s=t}$, add them, and integrate over x, y and s to get

$$0 \geq \iiint (\partial_t + \partial_s)(|u-v|)\tfrac{1}{2}\delta_{y=x}\delta_{s=t}$$
$$+ (\partial_x + \partial_s)\big(\mathrm{sign}(u-v)(F(u)-F(v))\big)\tfrac{1}{2}\delta_{y=x}\delta_{s=t}\,dx\,dy\,ds$$
$$= \int \frac{\partial_t + \partial_t}{2}(|u-v|) + \frac{\partial_x + \partial_x}{2}\big(\mathrm{sign}(u-v)(F(u)-F(v))\big)\,dx$$
$$= \frac{d}{dt}\int |u-v|\,dx + \int \partial_x\big(\mathrm{sign}(u-v)(F(u)-F(v))\big)\,dx$$

11B

Since u,v is e.g. in $C^1_b \cap L^1$ (that is, Barbălat gives $\lim_{|x|\to\infty}(u,v)(x,t)=0$), we have

$$\int \partial_x \big(\operatorname{sign}(u-v)\,(F(u)-F(v))\big)\,dx$$

$$= \operatorname{sign}\big(u(\infty,t)-v(\infty,t)\big)\big(F(u(\infty,t))-F(v(\infty,t))\big)$$

$$- \operatorname{sign}\big(u(-\infty,t)-v(-\infty,t)\big)\big(F(u(-\infty,t))-F(v(-\infty,t))\big)$$

$$= 0$$

Indeed recall that $\operatorname{sign}(0)=0$ or we can even use that $F(0)-F(0)=0$.

(We also note that for the same reasons

$$\int \partial_x |F(u)-F(v)|\,dx = 0$$

$$\int \partial_x \big(F(u)-F(v)\big)\,dx = 0 \quad)$$

$\blacksquare$

- Kinetic solutions

Is it possible to produce a theory such that $u_0 \in L^1$ still makes sure that $\eta(u), q(u) \in L^1_{loc}$ in Definition 2(i)?

1. It is possible to answer this question using nonlinear semigroup theory. Then one can construct a unique mild solution u of (SCL) for any $u_0 \in L^1$. If u_0 is in addition bounded, the mild solution coincides with the entropy solution. See e.g. Crandall, 1972 and Crandall & Liggett, 1971.
2. Another way of answering is through renormalized solutions. Then there exists a unique renormalized solution u of (SCL) for any $u_0 \in L^1$. If u_0 is in addition bounded, the renormalized solution coincides with the entropy solution. See e.g. Bénilan & Carrillo & Wittbold, 2000. Note that they also prove that the renormalized solution is always the unique mild solution, and thus, characterizing the mild solution.

We are going to discuss a third option: kinetic solutions.

The Boltzmann equation for dilute gases is written as

$$(B_\varepsilon) \quad \partial_t f_\varepsilon + \xi\, \partial_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon, f_\varepsilon)$$

where the constant ε represents the mean free path, $f_\varepsilon(x,t,\xi)$ is the density of particles which at time t and position x moves with velocity ξ, and $Q(f_\varepsilon, f_\varepsilon)$ is Boltzmann's quadratic collisonal operator. The question is what can we model with (B) at a macroscopic level, that is, when $\varepsilon \to 0^+$. The hard part is to characterize the right-hand side. A slightly modified and simplified (cf. BGK model of Boltzmann's equantion) model is proposed for (SCL):

$$(B'_\varepsilon) \quad \partial_t f_\varepsilon + F'(\xi)\, \partial_x f_\varepsilon = \frac{1}{\varepsilon}\left(\chi(\xi; u_\varepsilon) - f_\varepsilon \right)$$

This equation tells us that particles are transported by $F'(\xi) D_x \cdot$ and that their collisions are governed by the nonlinear kernel $\frac{1}{\varepsilon}(\chi(\xi; u_\varepsilon) - f_\varepsilon)$.

Here,

$$u_\varepsilon(x,t) = \int_{\mathbb{R}} f_\varepsilon(x,t,\xi)\, d\xi$$

denotes the local density of particles at (x,t), and the "equilibrium function" $\chi(\xi; u_\varepsilon(x,t))$ is defined by

$$\chi(\xi; u_\varepsilon) = \begin{cases} 1 & 0<\xi<u_\varepsilon \\ -1 & u_\varepsilon<\xi<0 \\ 0 & (u_\varepsilon-\xi)\xi \le 0 \end{cases}$$

In the limit as $\varepsilon \to 0^+$, we see that $f_\varepsilon \to \chi(\xi; u)$ to make sure that the right-hand side of (B') converges. Hence, we obtain

$$(B') \quad \partial_t \chi(\xi; u) + F'(\xi)\, \partial_x \chi(\xi; u) = \partial_\xi m(x,t,\xi)$$

where m is a nonnegative measure: the kinetic defect measure. ~~Formally, $u := \lim_{\varepsilon\to 0^+} u_\varepsilon$ is the entropy solution of (SCL). See Perthame & Tadmor, 1990 for more details~~

~~Hints: f_ε solving (B_ε')~~ $\overset{\cdot\, \mathrm{sign}(f_\varepsilon - \chi_k)}{\underset{\int_\xi d\xi}{\Longrightarrow}} \int_{\mathbb{R}} \partial_t |f_\varepsilon - \chi_k|\, d\xi + \int_{\mathbb{R}} F'(\xi)\, \partial_x |f_\varepsilon - \chi_k|\, d\xi \le 0$

$$\partial_t \eta(u) \qquad \partial_x q(u)$$

Let us first note that, formally,

$\overset{\lim_{\varepsilon\to 0} u_\varepsilon(x,t)}{\overset{\shortparallel}{u}}(x,t) = \int_{\mathbb{R}} \chi(\xi;u)\,d\xi$ is an entropy solution of (SCL): Multiply (B') by $\eta'(\xi)$ and integrate in ξ to get

$$\int_{\mathbb{R}} \partial_t \chi(\xi;u)\,\eta'(\xi)\,d\xi = \partial_t \int_{\mathbb{R}} \chi(\xi;u)\,\eta'(\xi)\,d\xi$$
$$= \partial_t\left(\eta(u)-\eta(0)\right) = \partial_t \eta(u)$$

$$\int_{\mathbb{R}} F'(\xi)\,\partial_x \chi(\xi;u)\,\eta'(\xi)\,d\xi = \partial_x \int_{\mathbb{R}} \chi(\xi;u)\,F'(\xi)\,\eta'(\xi)\,d\xi$$
$$= \partial_x \int_{\mathbb{R}} \chi(\xi;u)\,q'(\xi)\,d\xi$$
$$= \partial_x\left(q(u)-q(0)\right) = \partial_x q(u)$$

$$\text{*)}\quad \int_{\mathbb{R}} \partial_\xi m\,\eta'(\xi)\,d\xi = \int_{\mathbb{R}} \partial_\xi\left(m\,\eta'(\xi)\right)d\xi - \int_{\mathbb{R}} \eta''(\xi)\,m\,d\xi$$
$$\overset{\text{"}m\in L^\infty_0\text{"}}{=} -\int_{\mathbb{R}} \underbrace{\eta''(\xi)}_{\geq 0}\,\underbrace{m}_{\geq 0}\,d\xi \leq 0$$

$$\Rightarrow \partial_t \eta(u) + \partial_x q(u) \leq 0$$

Note that have also added meaning to m in the sense that

$$\partial_t \eta(u) + \partial_x q(u) = -\int_{\mathbb{R}} \eta''(\xi)\,m\,d\xi \quad (\text{in } \mathcal{D}')$$

Second, remember the rigourous derivation of Definition 2: For any $\eta \in C^2(\mathbb{R})$ multiply

$$\partial_t u^\varepsilon + \partial_x F(u^\varepsilon) = \varepsilon \partial^2_{xx} u^\varepsilon$$

by $\eta'(u^\varepsilon)$ to get

$$\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) = \varepsilon \partial^2_{xx} \eta(u^\varepsilon) - \underbrace{\varepsilon \eta''(u^\varepsilon)|\partial_x u^\varepsilon|^2}_{=: m_\varepsilon^{\eta''}}$$

Let

$$m_\varepsilon^{\eta''}(x,t) = \int_{\mathbb{R}} \eta''(\xi)\, m_\varepsilon(x,t,\xi)\, d\xi$$

with

$$m_\varepsilon(x,t,\xi) = \delta(\xi - u^\varepsilon)\, \varepsilon |\partial_x u^\varepsilon|^2$$

Taking the limit as $\varepsilon \to 0^+$ in the above, it is possible to show that there exists a kinetic defect measure $m^{\eta''}(x,t)$ satisfying

$$m^{\eta''}(x,t) = \int_{\mathbb{R}} \eta''(\xi)\, m(x,t,\xi)\, d\xi$$

with $m(x,t,\xi)$ being a nonnegative measure, and such that

$$\partial_t \eta(u) + \partial_x q(u) = -\int_{\mathbb{R}} \eta''(\xi)\, m\, d\xi \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times (0,T))$$

(Every nonnegative distr. is equal to a nonnegative Radon measure.)

Third, given an entropy definition, we can also get the kinetic equation by simply using that

$$\eta(u)-\eta(0) = \int_{\mathbb{R}} \eta'(\xi)\, \chi(\xi;u)\, d\xi$$

$$q(u)-q(0) = \int_{\mathbb{R}} F'(\xi)\, \eta'(\xi)\, \chi(\xi;u)\, d\xi .$$

In fact,

$$\partial_t(\eta(u)-\eta(0)) + \partial_x(q(u)-q(0)) = -\int_{\mathbb{R}} \eta''(\xi)\, m\, d\xi$$

gives

$$\int \eta'(\xi)\left(\partial_t \chi(\xi;u) + F'(\xi)\,\partial_x \chi(\xi;u) - \partial_\xi m\right) d\xi = 0 .$$

We are now ready to give our definition of a kinetic solution.

Def 3: A function $f = f(x,t,\xi)$ in $L^\infty_t(0,\infty; L^1_{x,\xi}(\mathbb{R}^{N+1}))$ is called a generalized kinetic solution of (SCL) if (18)

(i) $$\begin{cases} \partial_t f + F'(\xi)\partial_x f = \partial_\xi m \\ f(x,0,\xi) = \chi(\xi, u_0(x)) \quad , u_0 \in L^1 \end{cases}$$

holds in $\mathcal{D}'(\mathbb{R}^N \times [0,\infty) \times \mathbb{R})$ for some measure $m \geq 0$; and

(ii) for some function $\mu(\xi)$ and measure $\nu \geq 0$ we have

$$\int_0^\infty \int_{\mathbb{R}^N} m(x,t,\xi)\,dx\,dt \leq \mu(\xi) \in L^\infty_0(\mathbb{R})$$

$$|f(x,t,\xi)| = \operatorname{sgn}(\xi)\, f(x,t,\xi) \leq 1$$

$$\partial_\xi f = \delta(\xi) - \nu(x,t,\xi)$$

Remark: • For a (bounded) entropy solution u, the function $f(x,t,\xi) = \chi(\xi; u(x,t))$ is a generalized kinetic solution, that is, $u = \int \chi \, d\xi$ is a kinetic solution of (SCL). Conversely, if u is bounded kinetic solution of (SCL), then u is an entropy solution.

Thus, the kinetic solution concept extends (in L^1) the one for entropy solutions, and they coincide for bounded solutions.

- It is possible, through a limit (and approximation) procedure for (B') to show existence of the kinetic solution $u \in C((0,\infty; L^1(\mathbb{R}^N))$ under the assumptions $u_0 \in L^1$, $F \in W^{1,\infty}_{loc}$.
- Definition 3(ii) is needed to properly characterize limits of sequences $\chi(\xi; u_n(x,t))$. Moreover, the condition on the measure m is natural in the sense that kinetic solutions are an extension of entropy solutions: Recall that

$$\partial_t \eta(u) + \partial_x q(u) = -\int_{\mathbb{R}} \eta''(\xi) m(x,t,\xi) d\xi \quad \text{in } \mathcal{D}'$$

We obtain

$$\frac{d}{dt} \int_{\mathbb{R}^N} \eta(u)\, dx = -\int_{\mathbb{R}^N}\int_{\mathbb{R}} \eta''(\xi) m(x,t,\xi) d\xi$$

by considering cut-off functions $\varphi_R(x) = \varphi(\frac{x}{R})$ as test functions. Since η is nonnegative, we get

$$\int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}} \eta''(\xi) m(x,t,\xi) d\xi dx dt \leq \int_{\mathbb{R}^N} \eta(u_0)\, dx$$

$$\vee\!|$$

$$\int_0^\infty \int_{\mathbb{R}^N} \int_{\mathbb{R}} \eta''(\xi) m(x,t,\xi) d\xi dx dt \qquad \forall \text{ convex } \eta \text{ with } \eta(0)=0$$

Now, let $\xi_0 \geq 0$ and $\eta(\xi) := (\xi - \xi_0)^+$
(with $\eta''(\xi) = \delta_{\xi=\xi_0}$), then

$$\int_0^T \int_{\mathbb{R}^N} m(x,t,\xi_0)\,dx\,dt \leq \int_{\mathbb{R}^N} (u_0 - \xi_0)^+\,dx$$

To continue, let $\xi_0 < 0$ and $\eta(\xi) := (\xi - \xi_0)^-$, then

$$\int_0^T \int_{\mathbb{R}^N} m(x,t,\xi_0)\,dx\,dt \leq \int_{\mathbb{R}^N} (u_0 - \xi_0)^-\,dx$$

$$\Longrightarrow \int_0^T \int_{\mathbb{R}^N} m(x,t,\xi)\,dx\,dt \leq \mu(\xi)$$

where

$$\mu(\xi) := \mathbb{1}_{\{\xi \geq 0\}} \| (u_0 - \xi)^+ \|_{L^1} + \mathbb{1}_{\{\xi < 0\}} \| (u_0 - \xi)^- \|_{L^1}$$
$$\leq \| u_0 \|_{L^1} < \infty$$

Indeed,

$$\int_{\mathbb{R}^N} \mathbb{1}_{\{\xi \geq 0\}} (u_0 - \xi)^+\,dx + \int_{\mathbb{R}^N} \mathbb{1}_{\{\xi < 0\}} (u_0 - \xi)^-\,dx$$

$$= \int_{u_0 \geq \xi \geq 0} (u_0 - \xi)\,dx + \int_{0 > \xi \geq u_0} (\xi - u_0)\,dx$$

$$\leq \int_{u_0 \geq 0} u_0\,dx + \int_{0 > u_0} u_0\,dx = \| u_0 \|_{L^1}$$

$$\overset{LDCT}{\Longrightarrow} \mu \in L^\infty_0(\mathbb{R})$$

$\sup_\xi \left|\int \varphi \, dm\right| < \infty \quad \forall \varphi \in C_0$ $\quad m \in C_0(w\text{-}M^1)$ (21)

Theorem (Perthame, 1998)

Assume $F \in W^{1,\infty}_{loc}$, $u_0 \in L^1$. Let $f = f(x,t,\xi)$ be a generalized kinetic solution of (SCL). Then we have $f(x,t,\xi) = \chi(\xi; u(x,t))$ a.e., and $u = u(x,t)$ is a kinetic solution of (SCL). Moreover,

(i) $f(x,t,\xi) \longrightarrow \chi(\xi; u_0)$ in $L^1(\mathbb{R}^{N+1}_{x,\xi})$ as $t \to 0^+$; and

$$\cancel{\int_0^t \int_{\mathbb{R}^N} m(s,x,\xi)\, dx\, ds \to 0 \quad \forall \xi \text{ as } t \to 0^+}$$

$$\cancel{\text{"} m(x,t,\xi = u(x,t)) = 0 \text{"}}$$

(ii) the kinetic solutions u, v corresponding to u_0, v_0 respectively satisfy

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1} \leq \|u_0 - v_0\| \quad \text{a.e. } t \geq 0$$

Remark: • The statement $f(x,t,\xi) \xrightarrow{t \to 0^+} \underset{= f(x,t=0,\xi)}{\chi(\xi; u_0)}$ in $L^1(\mathbb{R}^{N+1}_{x,\xi})$ is the same as saying $u \in C([0,\infty); L^1(\mathbb{R}^N))$.

Proof: • Step 1: $f(x,t,\xi) = \chi(\xi; u(x,t))$. We will not prove this fact.

• Step 2: $f(x,t,\xi) \xrightarrow{t\to 0^+} \chi(\xi; u_0(x))$ in $L^1(\mathbb{R}^{N+1}_{x,\xi})$. We will not prove this fact.

• Step 3: $\|u(\cdot,t) - v(\cdot,t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}$. Let us do a formal proof. To simplify, write $\chi_u(\xi) := \chi(\xi; u)$. We will need several properties of this function. Remember that $u(x,t) = \int_{\mathbb{R}} \chi_u(\xi)\, d\xi$. Indeed, arguing for $\xi > 0$, we get

$$\int_{\mathbb{R}} \chi_u(\xi)\, d\xi = \int 1_{\{0<\xi<u\}}\, d\xi = \int_0^u d\xi = u.$$

Moreover, we note that

$$\chi_u = \begin{cases} 1, & 0<\xi<u \\ -1, & u<\xi<0 \\ 0, & (u-\xi)\xi \leq 0 \end{cases} \qquad \chi_v = \begin{cases} 1, & 0<\xi<v \\ -1, & v<\xi<0 \\ 0, & (v-\xi)\xi \leq 0 \end{cases}$$

$$\chi_u - \chi_v = \begin{cases} 1, & 0<\xi<u \ \& \ (v-\xi)\xi \leq 0,\ (u-\xi)\xi \leq 0 \ \& \ v<\xi<0 \ \Rightarrow v<\xi<u \\ -1, & u<\xi<0 \ \& \ (v-\xi)\xi \leq 0,\ (u-\xi)\xi \leq 0 \ \& \ 0<\xi<v \ \Rightarrow u<\xi<v \\ 0, & 0<\xi<u \ \& \ 0<\xi<v,\ u<\xi<0 \ \& \ v<\xi<0 \ \Rightarrow \text{otherwise} \\ & (u-\xi)\xi \leq 0 \ \& \ (v-\xi)\xi \leq 0 \end{cases}$$

Hence,

$$\int_{\mathbb{R}} |\chi_u - \chi_v| \, d\xi = \begin{cases} \int_{\mathbb{R}} 1_{\{v<\xi<u\}} \, d\xi = u - v \\ \int_{\mathbb{R}} 1_{\{u<\xi<v\}} \, d\xi = v - u \end{cases} = |u - v|$$

and

$$\frac{d}{dt} \int_{\mathbb{R}^N} |u - v| \, dx \leq 0 \iff \frac{d}{dt} \int_{\mathbb{R}^N} \int_{\mathbb{R}} |\chi_u - \chi_v| \, d\xi \, dx \leq 0$$

Remember that χ_u, χ_v solve linear equations with constant coefficients. Let us use this fact and the special structure of χ_u, χ_v in a clever way. For real numbers a, b, we have

$$(a-b)^2 = (a-b)(a-b) = a^2 + b^2 - 2ab,$$

or equivalently,

$$|a-b|^2 = |a|^2 + |b|^2 - 2ab.$$

Now, replacing a, b by χ_u, χ_v give

$$|\chi_u - \chi_v|^2 = |\chi_u|^2 + |\chi_v|^2 - 2\chi_u \chi_v$$

The special structure of $\chi_u, \chi_v, \chi_u - \chi_v$ gives

$$|\chi_u - \chi_v|^2 = |\chi_u - \chi_v| \,, \quad |\chi_u|^2 = |\chi_u| \,, \quad |\chi_v|^2 = |\chi_v|$$

and we end up with

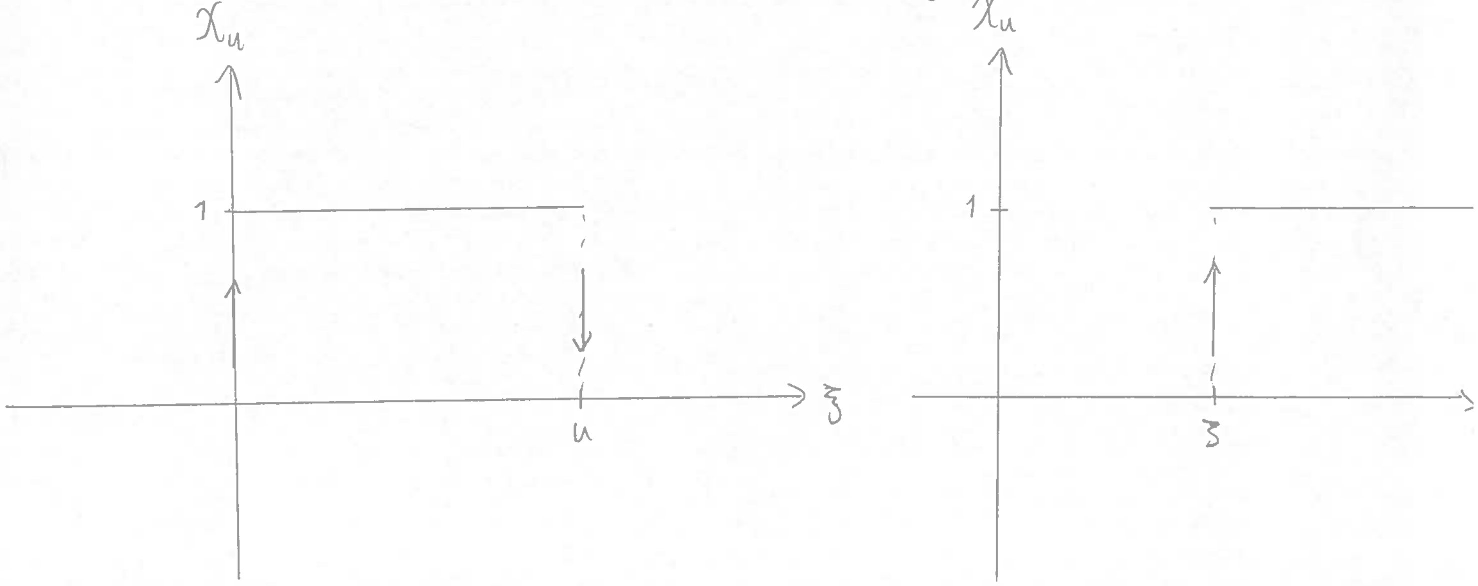
$$|\chi_u - \chi_v| = |\chi_u| + |\chi_v| - 2\chi_u \chi_v \,.$$

Throughout the following calculations, we need

$$\partial_\xi \chi_u(\xi) \quad \text{and} \quad \partial_u \chi_u(\xi) \,.$$

Let us argue for $\xi > 0$ (the argument is similar for $\xi < 0$):

$$\chi_u(\xi) \overset{\xi>0}{=} \mathbb{1}_{\{0<\xi<u\}}$$



$$\Rightarrow \quad \partial_\xi \chi_u = 1 \cdot \delta_{\xi=0} - 1 \cdot \delta_{\xi=u} \,, \quad \partial_u \chi_u = 1 \cdot \delta_{u=\xi}$$

To sum up,

$$\frac{d}{dt}\int_{\mathbb{R}^N}|u-v|\,dx \le 0 \iff \frac{d}{dt}\int_{\mathbb{R}^N}\int_{\mathbb{R}}|\chi_u-\chi_v|\,d\xi\,dx \le 0$$

$$\iff \frac{d}{dt}\int_{\mathbb{R}^N}\int_{\mathbb{R}}|\chi_u|+|\chi_v|-2\chi_u\chi_v\,d\xi\,dx \le 0$$

Multiply the kinetic equation in Definition 3(i) for χ_u, χ_v ✓
by $\mathrm{sign}(\xi)$ to get

$$\partial_t(\chi_u)\,\mathrm{sign}(\xi) + F'(\xi)\,\partial_x(\chi_u)\,\mathrm{sign}(\xi) = \partial_\xi(m)\,\mathrm{sign}(\xi)$$
$$\partial_t(\chi_v)\,\mathrm{sign}(\xi) + F'(\xi)\,\partial_x(\chi_v)\,\mathrm{sign}(\xi) = \partial_\xi(n)\,\mathrm{sign}(\xi)$$

Integrate in ξ and x to obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}}\partial_t(\chi_u)\,\mathrm{sign}(\xi)\,d\xi\,dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}}\partial_t(|\chi_u|)\,d\xi\,dx$$
$$= \frac{d}{dt}\int_{\mathbb{R}^N}\int_{\mathbb{R}}|\chi_u|\,d\xi\,dx$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}} F'(\xi)\,\partial_x(\chi_u)\,\mathrm{sign}(\xi)\,d\xi\,dx$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}} F'(\xi)\,\partial_u(\chi_u)\,\partial_x(u)\,\mathrm{sign}(\xi)\,d\xi\,dx$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}} F'(\xi)\,\delta_{u=\xi}\,\partial_x(u)\,\mathrm{sign}(\xi)\,d\xi\,dx$$
$$= \int_{\mathbb{R}^N} F'(u)\,\partial_x(u)\,\mathrm{sign}(u)\,dx$$
$$= \int_{\mathbb{R}^N}\partial_x(|F(u)|)\,dx = |F(u(\infty,t))| - |F(u(-\infty,t))|$$
$$\overset{u\in C_0^1\cap L^1}{=} |F(0)| - |F(0)| = 0$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}} \partial_\xi(m)\, \mathrm{sign}(\xi)\, d\xi\, dx \overset{(*)}{=} -\int_{\mathbb{R}^N}\int_{\mathbb{R}} 2\delta_{\xi=0}\, m\, d\xi\, dx$$

$$= -2\int_{\mathbb{R}^N} m(x,t,\xi=0)\, dx$$

That is,

$$\frac{d}{dt}\int_{\mathbb{R}^{N+1}} |\chi_u|\, d\xi\, dx = -2\int_{\mathbb{R}^N} m(x,t,0)\, dx$$

$$\frac{d}{dt}\int_{\mathbb{R}^{N+1}} |\chi_v|\, d\xi\, dx = -2\int_{\mathbb{R}^N} n(x,t,0)\, dx$$

Now, for the product we multiply the equation for χ_u by χ_v and the equation for χ_v by χ_u to obtain

$$\partial_t(\chi_u)\chi_v + F'(\xi)\partial_x(\chi_u)\chi_v = \partial_\xi(m)\chi_v$$

$$\partial_t(\chi_v)\chi_u + F'(\xi)\partial_x(\chi_v)\chi_u = \partial_\xi(n)\chi_u$$

Adding them gives

$$\partial_t(\chi_u\chi_v) + F'(\xi)\partial_x(\chi_u\chi_v) = \partial_\xi(m)\chi_v + \partial_\xi(n)\chi_u.$$

Note that we really use linearity here!! Integrating in ξ and x, using $\partial_x(\chi_u\chi_v) = \delta_{\xi=u}\partial_x u\,\chi_v + \delta_{\xi=v}\partial_x u\,\chi_u$, and $\int \partial_x(F(u)+F(v))\,dx = 0$ yield

$$\frac{d}{dt}\int_{\mathbb{R}^N}\int_{\mathbb{R}} \chi_u \chi_v \, d\xi \, dx$$

$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}} \partial_\xi(m)\chi_v \, d\xi dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}} \partial_\xi(n)\chi_u \, d\xi dx$$

$$\overset{\text{"}m,n\in L_0^\infty\text{"}}{=} -\int_{\mathbb{R}^N}\int_{\mathbb{R}} m \, \partial_\xi \chi_v \, d\xi dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}} n \, \partial_\xi \chi_u \, d\xi dx$$

$$= -\int_{\mathbb{R}^N}\int_{\mathbb{R}} m\left(\delta_{\xi=0} - \delta_{\xi=v}\right) d\xi dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}} n\left(\delta_{\xi=0} - \delta_{\xi=u}\right) d\xi dx$$

$$= -\int_{\mathbb{R}^N} m(x,t,\xi=0)\,dx + \int_{\mathbb{R}^N} m(x,t,\xi=v)\,dx$$

$$- \int_{\mathbb{R}^N} n(x,t,\xi=0)\,dx + \int_{\mathbb{R}^N} n(x,t,\xi=u)\,dx$$

$$\overset{m,n\geq 0}{\geq} -\int_{\mathbb{R}^N} m(x,t,\xi=0)\,dx - \int_{\mathbb{R}^N} n(x,t,\xi=0)\,dx$$

$$\Rightarrow \frac{d}{dt}\int_{\mathbb{R}^{N+1}} |\chi_u| + |\chi_v| - 2\chi_u\chi_v \, d\xi dx$$

$$\leq -2\int_{\mathbb{R}^N} m(x,t,0)\,dx - 2\int_{\mathbb{R}^N} n(x,t,0)\,dx$$

$$+ 2\left(\int_{\mathbb{R}^N} m(x,t,0)\,dx + \int_{\mathbb{R}^N} n(x,t,0)\,dx\right)$$

$$= 0$$

$\blacksquare$