On nonlocal (and local) equations of porous medium type

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25 May 2018

In collaboration with
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A talk given at
2018 EWM-EMS Summer School
Nonlocal interactions in Partial Differential Equations and Geometry

Let $Q_T := \mathbb{R}^N \times (0, T)$. We consider the following Cauchy problem:

$$\text{(GPME)} \quad \begin{cases} \partial_t u - \left(\operatorname{tr}(\sigma\sigma^T D^2\varphi(u)) + \mathcal{L}^\mu[\varphi(u)]\right) = 0 & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

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Main well-posedness results:

- Uniqueness for $u_0 \in L^\infty$ with $u - u_0 \in L^1$.
- Existence for $u_0 \in L^1 \cap L^\infty$.
- Convergent numerical schemes for $u_0 \in L^1 \cap L^\infty$.

The diffusion operator

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$$\mathcal{L}^\mu[\psi](x) := \int_{\mathbb{R}^N\setminus\{0\}} \big(\psi(x+z)-\psi(x)\big)\,\mathrm{d}\mu(z).$$

The most common example is the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha\in(0,2)$.

Selective summary of previous results

Local case: $\partial_t u = \Delta u$, $\partial_t u = \Delta u^m$, $\partial_t u = \Delta\varphi(u)$.

J. L. Vázquez. *The porous medium equation. Mathematical theory*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

Nonlocal case: $\partial_t u = \mathcal{L}^\mu[\varphi(u)]$.

- Well-posedness when $\mathcal{L}^\mu = -(-\Delta)^{\frac{\alpha}{2}}$:

Many people: Vázquez, de Pablo, Quirós, Rodríguez, Brändle, Bonforte, Stan, del Teso, Muratori, Grillo, Punzo, ...

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- Well-posedness for other $\mathcal{L}^\mu$:

Nonsingular operators

F. Andreu-Vaillo, J. Mazón, J. D. Rossi, and J. J. Toledo-Melero. *Nonlocal diffusion problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

Fractional Laplace like operators (with some x-dependence)

A. de Pablo, F. Quirós, and A. Rodríguez. Nonlocal filtration equations with rough kernels. *Nonlinear Anal.*, 137:402–425, 2016.

Selective summary of previous results

Previous results (mostly) rely on:

- The porous medium nonlinearity $\varphi(u) = u^m$ with $m > 1$.
- A very restrictive class of Lévy operators.
- The use of L^1-energy solutions.

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In our case:

- Uniqueness is hard to prove because of a very weak solution concept (however, existence is then easier).
- The result we obtain is kind of different since we work in L^∞.
- We can handle very weak assumptions on φ and $\mathcal{L}^\mu$.

Assumptions

Unless otherwise stated we always assume that

(A_φ) $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing,

and

(A_μ) $\mu \geq 0$ is a symmetric Radon measure on $\mathbb{R}^N \setminus \{0\}$ satisfying

$$\int_{|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} 1 \,\mathrm{d}\mu(z) < \infty.$$

Assumptions

The assumption

$$\varphi : \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing,}$$

includes nonlinearities of the following kind

- the porous medium,
- fast diffusion, and
- (one-phase) Stefan problem.

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  - for the function J with $\int_{\mathbb{R}^d} J(z)\,\mathrm{d}z = 1$, $\mathcal{L}^{J\,\mathrm{d}z}[\psi] = J * \psi - \psi$;

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  - Fourier multipliers $\mathcal{F}(\mathcal{L}^\mu[\psi]) = -s_{\mathcal{L}^\mu} \mathcal{F}(\psi)$.

Simplification

To simplify, we consider $\sigma \equiv 0$, that is,

$$\mathrm{tr}(\sigma\sigma^T D^2\varphi(u)) + \mathcal{L}^\mu[\varphi(u)] = \mathcal{L}^\mu[\varphi(u)].$$

However, the same approach will work for the full operator.

Distributional solution

Definition

Under the assumptions (A_φ), (A_μ), and $u_0 \in L^\infty(\mathbb{R}^N)$, $u \in L^\infty(Q_T)$ is a **distributional solution** of (GPME) if

$$0 = \int_0^T \int_{\mathbb{R}^N} \Big(u(x,t)\partial_t\psi(x,t) + \varphi(u(x,t))\mathcal{L}^\mu[\psi(\cdot,t)](x) \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}^N} u_0(x)\psi(x,0)\,\mathrm{d}x$$

for all $\psi \in C_c^\infty(\mathbb{R}^N \times [0,T))$.

Uniqueness

Theorem (Preuniqueness, [del Teso&JE&Jakobsen, 2017])

Assume (A_φ) *and* (A_μ). *Let* $u(x,t)$ *and* $\hat{u}(x,t)$ *satisfy*

$$u, \hat{u} \in L^\infty(Q_T),$$

$$u - \hat{u} \in L^1(Q_T),$$

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = \partial_t \hat{u} - \mathcal{L}^\mu[\varphi(\hat{u})] \qquad \text{in} \qquad \mathcal{D}'(Q_T),$$

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_{\mathbb{R}^N} (u(x,t) - \hat{u}(x,t))\psi(x,t)\,\mathrm{d}x = 0 \quad \forall \psi \in C_c^\infty(\mathbb{R}^N \times [0,T)).$$

Then $u = \hat{u}$ *a.e. in* Q_T.

Uniqueness

Corollary (Uniqueness, [del Teso&JE&Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^\infty(\mathbb{R}^N)$. *Then there is at most one distributional solution* u *of* (GPME) *such that* $u \in L^\infty(Q_T)$ *and* $u - u_0 \in L^1(Q_T)$.

Proof: Assume there are two solutions u and $\hat{u}$ with the same initial data u_0. Then all assumptions of Theorem Preuniqueness obviously hold ($\|u - \hat{u}\|_{L^1} \leq \|u - u_0\|_{L^1} + \|\hat{u} - u_0\|_{L^1} < \infty$), and $u = \hat{u}$ a.e. □

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Uniqueness holds for $u_0 \notin L^1$, for example $u_0(x) = c + \phi(x)$ for $c \in \mathbb{R}$ and $\phi \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. However, periodic u_0's are not included because of the assumption $u - u_0 \in L^1$.

Existence

Theorem (Existence, [del Teso, JE, Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then there exists a unique distributional solution* u *of* (GPME) *satisfying*

$$u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N)).$$

Proof: By convergence of numerical solution (as we will see later). □

The proof of Theorem Preuniqueness

Based on a proof by Brézis and Crandall.

H. Brézis and M. G. Crandall. Uniqueness of solutions of the initial-value problem for $u_t - \Delta\varphi(u) = 0$. *J. Math. Pures Appl. (9)*, 58(2):153–163, 1979.

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1. Define $U := u - \hat{u}$ and $\Phi := \varphi(u) - \varphi(\hat{u})$, then U solves

$$\begin{cases} \partial_t U - \mathcal{L}^\mu[\Phi] = 0 & \text{in} \quad Q_T \\ U(x,0) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases}$$

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Note that $U \in L^1 \cap L^\infty$ and $\Phi \in L^\infty$.

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2. Consider

$$\varepsilon v_\varepsilon - \mathcal{L}^\mu[v_\varepsilon] = g \qquad \text{in} \qquad \mathbb{R}^N,$$

   and define $B_\varepsilon^\mu[g] := v_\varepsilon$, that is, $B_\varepsilon^\mu = (\varepsilon I - \mathcal{L}^\mu)^{-1}$ is the resolvent of $\mathcal{L}^\mu$.

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5. Show that $h_\varepsilon \to 0$ as $\varepsilon \to 0^+$.

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1. $\varepsilon B^\mu_\varepsilon[U] \to 0$ implies $h_\varepsilon \to 0$ as $\varepsilon \to 0^+$.
2. Enough to prove that $\varepsilon B^\mu_\varepsilon[\gamma] \to 0$ for all $\gamma \in C_\mathrm{c}^\infty(\mathbb{R}^N)$. Note that $\Gamma_\varepsilon := \varepsilon B^\mu_\varepsilon[\gamma]$ solves

$$\varepsilon\Gamma_\varepsilon - \mathcal{L}^\mu[\Gamma_\varepsilon] = \varepsilon\gamma \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^N).$$

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4. (Liouville) If $\operatorname{supp}\mu \neq \emptyset$, $\Gamma \in C_0$, and $\mathcal{L}^\mu[\Gamma] = 0$ in $\mathcal{D}'$, then $\Gamma \equiv 0$.

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Note that a general Liouville result do not hold for $\mathcal{L}^\mu$: Take $\mu(z) = \delta_{2\pi}(z) + \delta_{-2\pi}(z)$, then $\mathcal{L}^\mu[\cos](x) = 0$, but this function is not constant.

Extensions and related results

We can also consider the following Cauchy problem:

(x-GPME) $$\begin{cases} \partial_t u - A^{\lambda}[\varphi(u)] = 0 & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

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where

- $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and
- A^λ is a x-dependent generalization of $\mathcal{L}^\mu$.

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We can also consider the following Cauchy problem:

(x-GPME)
$$\begin{cases} \partial_t u - A^{\lambda}[\varphi(u)] = 0 & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

where

- $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and
- A^{λ} is a x-dependent generalization of $\mathcal{L}^{\mu}$.

Main results:

- Uniqueness in $L^1 \cap L^{\infty}$.
- Energy solutions $\iff$ distributional solutions with finite energy.

Important observations

$$\Delta_h\psi(x) := \frac{\psi(x+he_i)+\psi(x-he_i)-2\psi(x)}{h^2}$$
$$= \int_{\mathbb{R}^N} \big(\psi(x+z)-\psi(x)\big)\,\mathrm{d}\nu_h(z) =: \mathcal{L}^{\nu_h}[\psi](x)$$

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where

$$\nu_h(z) := \frac{1}{h^2}\sum_{i=1}^{N} \delta_{he_i}(z)+\delta_{-he_i}(z)$$

satisfies $\nu_h(\mathbb{R}^N) < \infty$.

Important observations

By now, there exist several spatial discretizations of $\mathcal{L}^\mu$ (e.g. quadrature and spectral methods).

Y. Huang and A. Oberman. Finite difference methods for fractional Laplacians. Preprint, arXiv:1611.00164v1 [math.NA], 2016.

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Our contribution is to note and exploit that (some of) the discretizations of $\mathcal{L}^\mu$ is again a Lévy operator.

Numerical schemes for (GPME)

Recall that our Cauchy problem was given as

$$\text{(GPME)} \quad \begin{cases} \partial_t u - \big(\operatorname{tr}(\sigma\sigma^T D^2\varphi(u)) + \mathcal{L}^\mu[\varphi(u)]\big) = 0 & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

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Our numerical scheme can then take the following form

$$\text{(NumGPME)} \quad \begin{cases} \frac{U_\beta^j - U_\beta^{j-1}}{\Delta t} = G_{\Delta x}(U_\beta^j, U_\beta^{j-1}) & \text{in} \quad \Delta x\mathbb{Z}^N \times \Delta t\mathbb{N}, \\ \text{``}U_\beta^0 = u_0\text{''} & \text{in} \quad \Delta x\mathbb{Z}^N. \end{cases}$$

In our most general case, we have that

$$G_{\Delta x}(U_\beta^j, U_\beta^{j-1}) := \mathcal{L}^{\nu_{1,\Delta x}}[\varphi_1^{\Delta x}(U_\beta^j)] + \mathcal{L}^{\nu_{2,\Delta x}}[\varphi_2^{\Delta x}(U_\beta^{j-1})]$$

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Thus our framework includes

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- combinations of the above.

Note that by our previous observations, we are, in fact, able to approximate local operators of the form

$$\mathrm{tr}(\sigma\sigma^T D^2 \cdot).$$

The scheme defined by (NumGPME) is

- monotone,
- (conservative if the φ's involved are Lipschitz)
- L^p-stable, and
- consistent.

Convergence of the numerical schemes

The scheme defined by (NumGPME) is

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- L^p-stable, and
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Theorem (Convergence, [del Teso&JE&Jakobsen, 2018])

For the interpolant U, we have

$$U \to u \qquad \textit{in} \qquad C([0,T];L^1_{\text{loc}}(\mathbb{R}^N)) \qquad \textit{as} \qquad \Delta x, \Delta t \to 0^+$$

where $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T];L^1_{\text{loc}}(\mathbb{R}^N))$ is a distributional solution of (GPME).

Note that we only assume $u_0 \in L^1 \cap L^\infty$.

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Thank you for your attention!