Nonlocal (and local) nonlinear diffusion equations. Background, analysis, and numerical approximation

Jørgen Endal
URL: http://folk.ntnu.no/jorgeen

Department of mathematical sciences//Departamento de Matemáticas
NTNU, Norway//UAM, Spain

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In collaboration with
F. del Teso and E. R. Jakobsen

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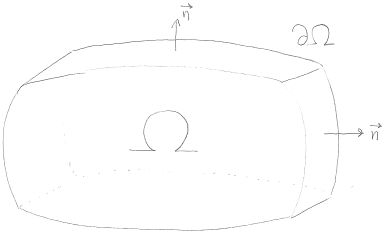
Goal: Numerical simulations

The goal of this presentation is to obtain mathematically rigorous numerical simulations for diffusion equations.

In the context of finite-difference approximations for equations in $\mathbb{R}^N \times (0, T)$.

Diffusion is the act of "spreading out" – the movement from areas of high concentration to areas of low concentration.

How do we model this phenomena?



Let u be some heat density inside a region Ω. The rate of change of the total quantity within Ω equals the negative of the net flux through $\partial\Omega$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u \,\mathrm{d}x = -\int_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = -\int_\Omega \mathrm{div}\mathbf{F}\,\mathrm{d}V,$$

or

$$\partial_t u = -\mathrm{div}\mathbf{F}.$$

Introduction: Mathematical modelling

In many situations, $\mathbf{F} \sim Du$, but in the opposite direction (the flow is from high to low consetration):

$$\mathbf{F} = -a(u)Du,$$

and we get

$$\partial_t u = \mathrm{div}(a(u)Du).$$

- **Case 1:** $a(u) = 1$. We obtain the heat equation

$$\partial_t u = \Delta[u]$$

- **Case 2:** $a(u) = u^{m-1}$. We obtain the porous medium equation

$$\partial_t u = \Delta[u^m]$$

J. L. Vázquez. *The porous medium equation. Mathematical theory*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

Introduction: Special case when $m = 6$

It is possible to use

$$\begin{cases} \partial_t u = \Delta[u^6] & \text{in} \quad \mathbb{R}^N \times (0,T), \\ u(x,0) = M\delta_0 & \text{on} \quad \mathbb{R}^N, \end{cases}$$

to describe the propagation of heat immediately after a nuclear explosion.

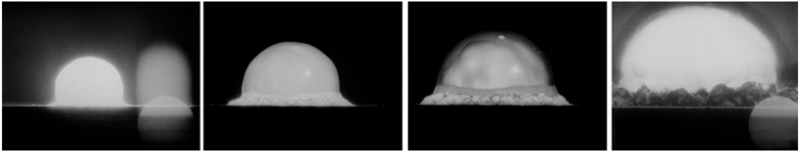
The solution (Barenblatt-solution) will actually be given as

$$t^{-\gamma_1} \max\left\{0, C - k|x|^2 t^{-2\gamma_2}\right\}^{\frac{1}{5}}.$$

See video:
https://www.youtube.com/watch?v=Q3ezhvCzWCM

G. I. Barenblatt. *Scaling, self-similarity, and intermediate asymptotics*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1996.

Introduction: Special case when $m = 6$



Local linear diffusion

Let us consider

$$\text{(HE)} \qquad \begin{cases} \partial_t u = \Delta[u] & \text{in} \quad \mathbb{R}^N \times (0, T), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N. \end{cases}$$

In fact, the solution is given by (use the Fourier transform)

$$u(x,t) = [K(\cdot,t) * u_0](x) = \int_{\mathbb{R}^N} K(x-y,t)u_0(y)\,\mathrm{d}y$$

where

$$K(z,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|z|^2}{4t}} > 0 \qquad \text{with} \qquad \int K(z,t)\,\mathrm{d}z = 1.$$

If $u_0 > 0$ (on a set) then $u > 0$ (everywhere), that is, some heat is distributed to the whole space immediately.

Local linear diffusion

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Immediate consequences are:

- (Mass conservation) $\int u = \int u_0$.

Why:

$$\int_{\mathbb{R}^N} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y,t)u_0(y)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_{\mathbb{R}^N} K(x,t)\,\mathrm{d}x \int_{\mathbb{R}^N} u_0(y)\,\mathrm{d}y.$$

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Immediate consequences are:

- (Mass conservation) $\int u = \int u_0$.
- (L^1-bound) $\|u(\cdot,t)\|_{L^1} \leq \|u_0\|_{L^1}$.

Why:

$$\int_{\mathbb{R}^N} |u(x,t)| \,\mathrm{d}x = \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} K(x-y,t) u_0(y) \,\mathrm{d}y \right| \mathrm{d}x$$

$$\leq \int_{\mathbb{R}^N} K(x,t) \,\mathrm{d}x \int_{\mathbb{R}^N} |u_0(y)| \,\mathrm{d}y.$$

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Why:

$$|u(x,t)| = \left| \int_{\mathbb{R}^N} K(x-y,t) u_0(y) \,\mathrm{d}y \right|$$

$$\leq \|u_0\|_{L^\infty} \int_{\mathbb{R}^N} K(x,t) \,\mathrm{d}x.$$

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- (L^1–L^∞-smoothing) $\|u(\cdot,t)\|_{L^\infty} \leq Ct^{-\frac{N}{2}}\|u_0\|_{L^1}$.

Why:

$$|u(x,t)| = \left| \int_{\mathbb{R}^N} K(x-y,t) u_0(y) \, \mathrm{d}y \right|$$

$$\leq \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} |u_0(y)| \, \mathrm{d}y.$$

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Immediate consequences are:

- (Mass/heat conservation) $\int u = \int u_0$.
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- (L^1–L^∞-smoothing) $\|u(\cdot,t)\|_{L^\infty} \leq Ct^{-\frac{N}{2}}\|u_0\|_{L^1}$.
- (L^1-contraction) For two solutions u, v,
  $\|u(\cdot,t) - v(\cdot,t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}$.
- (Comparison) For two solutions u, v, $u_0 \leq v_0 \implies u \leq v$.

Theorem

Assume $u_0 \in L^1 \cap L^\infty$. Then there exists a unique solution $u \in L^1 \cap L^\infty$ of (HE).

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Local nonlinear diffusion

Choose $m > 1$, and consider

$$\text{(PME)} \qquad \begin{cases} \partial_t u = \Delta[u^m] & \text{in} \quad \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N. \end{cases}$$

Why do we make life harder than it needs to be?

- We lose the linear structure.
  - $u - v, u + v, \partial_t u, \partial_{x_i} u$, *etc* are no longer immediate solutions.
  - There is no convolution formula for the solution anymore.
- We gain a more accurate behaviour.
  - Solutions will have finite speed of propagation: Heat will spend some time spreading.
  - As we saw, some applications require nonlinear.

But:

- We are able to prove that (PME) enjoys similar properties as (HE): L^1-contraction, comparison, L^1- and L^∞-bounds, L^1–L^∞-smoothing, and conservation of mass.
- We thus obtain similar existence and uniqueness results.

Thoughts

- Which other equations will behave in a similar way?
- How general can we make the nonlinearity $u \mapsto u^m$ and the operator Δ?
- Why are the mentioned properties so important?

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Properties of the operator

Definition

Given a **linear** operator $\mathcal{L} : C_b^2(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$, we say that

- $\mathcal{L}$ satisfies the **global comparison principle** if given a global maximum (resp. minimum) x_0 of ψ, we have that $\mathcal{L}[\psi](x_0) \leq 0$ (resp. ≥ 0).
- $\mathcal{L}$ is **translation invariant** if

$$\mathcal{L}[\psi(\cdot + y)](x) = \mathcal{L}[\psi](x + y) \qquad \text{for all} \qquad x, y \in \mathbb{R}^N.$$

Note that the Laplacian satisfies both conditions: It is linear, has a "sign" at extremal points, and is x-independent.

Which other operators have these properties?

Characterization of operators

Theorem

A ***linear*** *operator which is* ***translation invariant*** *and satisfies the* ***global comparison principle*** *is of the form* $\mathcal{L} = \mathcal{L}^{\sigma,b} + \mathcal{L}^{\mu}$ *where*

$$\mathcal{L}^{\sigma,b}[\psi(x)] := \mathrm{tr}(\sigma\sigma^T D^2\psi(x)) + b \cdot D\psi(x)$$

$$\mathcal{L}^{\mu}[\psi(x)] := \int_{|z|>0} \left(\psi(x+z) - \psi(x) - z \cdot D\psi(x)\mathbf{1}_{|z|\leq 1}\right) \mathrm{d}\mu(z)$$

Here, $\sigma \in \mathbb{R}^{N\times p}$, $b \in \mathbb{R}^N$ *and* $\mu \geq 0$ *is a Radon measure satisfying*

$$\int \min\{|z|^2, 1\} \, \mathrm{d}\mu(z) < \infty.$$

P. Courrège. Sur la forme intégro-différentielle des opérateurs de C_k^∞ dans C satisfaisant au principe du maximum. *Séminaire Brelot-Choquet-Deny. Théorie du Potentiel*, 10(1):1–38, 1965–1966.

We slightly reduce the class of possible operators by remembering that Δ is **self-adjoint**:

$$\int \Delta[f]g = \int f\Delta[g].$$

Why: Integrate by parts twice.

Characterization of operators

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We end up with

$$\mathcal{L}[\psi](x) = \mathrm{tr}(\sigma\sigma^T D^2\psi(x)) + \text{P.V.}\int_{|z|>0} \big(\psi(x+z) - \psi(x)\big)\,\mathrm{d}\mu(z),$$

where $\mathcal{L} : W^{2,p} \to L^p$ with $p \in [1,\infty]$.

Note that

$$\mathcal{L}[\psi] = \Delta[\psi]$$

when $\mu \equiv 0$ and $\sigma\sigma^T = I$.

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Note that

$$\mathcal{L}[\psi] = \Delta[\psi]$$

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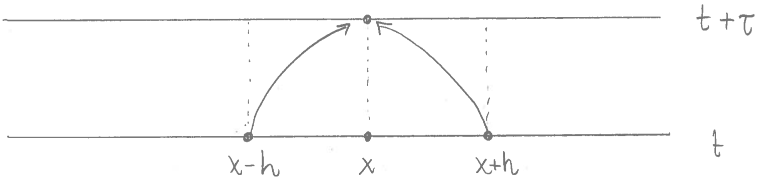
Local vs. nonlocal diffusion

Let $u(x,t)$ be the probability for a particle to be at discrete $x \in h\mathbb{Z}$, $t \in \Delta t\mathbb{N} \cap [0,T]$.

Assume that we are only allowed to jump one point either to the left or to the right, each with probability $\frac{1}{2}$.

The probability of being at point x at time $t + \Delta t$ is then

$$u(x, t + \Delta t) = \frac{1}{2}u(x + h, t) + \frac{1}{2}u(x - h, t).$$



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Choose (the scaling) $\Delta t = \frac{1}{2}h^2$ and divide by it to obtain

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = \frac{u(x+h,t) + u(x-h,t) - 2u(x,t)}{h^2}.$$

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Assume that we are only allowed to jump one point either to the left or to the right, each with probability $\frac{1}{2}$.

As $\Delta t, h \to 0^+$,

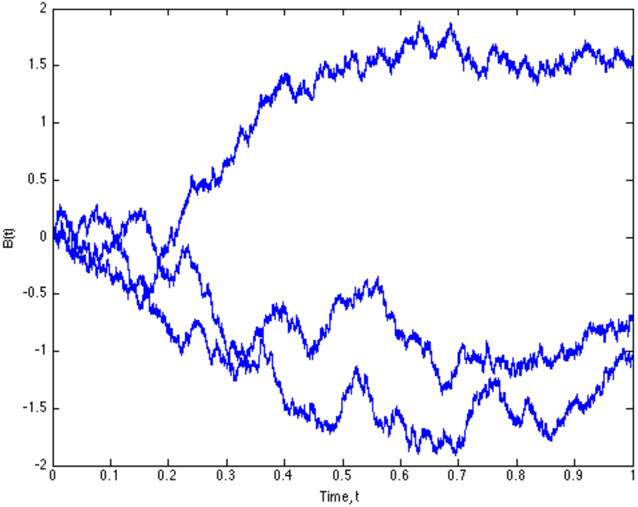
$$\partial_t u = \Delta u \qquad \text{in} \qquad \mathbb{R} \times (0,T),$$

that is, u is a solution of the heat equation.

A. Einstein. Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Annalen der Physik* (in German), 322(8): 549–560, 1905.

Local vs. nonlocal diffusion

Probability: u is the density of Brownian particles.



Local vs. nonlocal diffusion

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

We choose a density $K : \mathbb{R} \to [0, \infty)$ up to normalization factors as

$$K(y) = \begin{cases} \frac{1}{|y|^{1+\alpha}} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

for $\alpha \in (0, 2)$. It satisfies

(i) $K(y) = K(-y)$

(ii) $\sum_{k \in \mathbb{Z}} K(k) = 1$.

As before, the probability of being at point x at time $t + \Delta t$ is

$$u(x, t + \Delta t) = \sum_{k \in \mathbb{Z}} K(k) u(x + hk, t).$$

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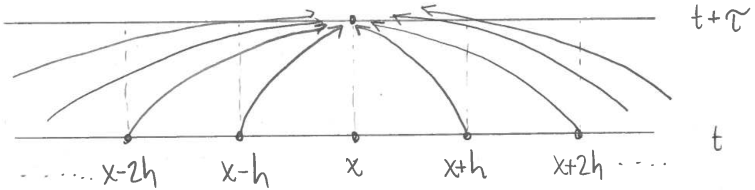

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for $\alpha \in (0, 2)$. It satisfies

(i) $K(y) = K(-y)$

(ii) $\sum_{k \in \mathbb{Z}} K(k) = 1.$

Then, for the choice (of scaling) $\Delta t = h^\alpha$,

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \sum_{\mathbb{Z} \ni \beta \neq 0} \big(u(x + h\beta, t) - u(x, t)\big) K(h\beta) h.$$

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As $\Delta t, h \to 0^+$,

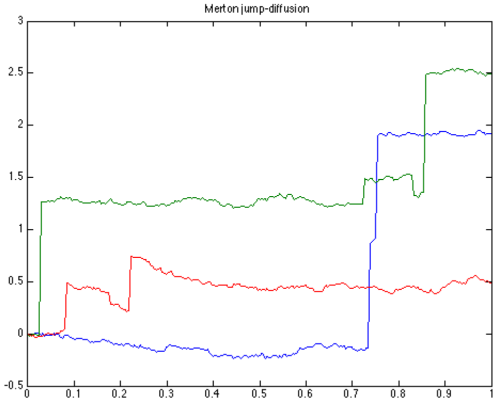
$$\begin{aligned}\partial_t u &= \text{P.V.} \int_{|z|>0} \big(u(x+z,t) - u(x,t)\big) \frac{c_{1,\alpha}}{|z|^{1+\alpha}} \,\mathrm{d}z \\ &= -(-\Delta)^{\frac{\alpha}{2}} u \qquad \text{in} \qquad \mathbb{R} \times (0,T)\end{aligned}$$

where $c_{1,\alpha} > 0$ and $-(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ is the fractional Laplacian. We thus observe that u is a solution of the fractional heat equation.

E. Valdinoci. From the long jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl. SeMA*, (49):33–44, 2009.

Local vs. nonlocal diffusion

Probability: u is the density of Lévy particles.



Picture due to A. Meucci (2009).

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Nonlocal nonlinear diffusion

Let $Q_T := \mathbb{R}^N \times (0, T)$. We consider the following Cauchy problem:

$$\text{(GPME)} \qquad \begin{cases} \partial_t u = \mathcal{L}[\varphi(u)] & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

where

- $$\begin{aligned} \mathcal{L}[\psi] &= \mathcal{L}^\sigma[\psi] + \mathcal{L}^\mu[\psi] \\ &= \text{ local } + \text{ nonlocal } \quad \text{(self-adjoint)} \end{aligned}$$
- $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and
- u_0 some rough initial data.

Main results:

- Uniqueness for $u_0 \in L^\infty$ with $u - u_0 \in L^1$.
- Convergent numerical schemes in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ for $u_0 \in L^1 \cap L^\infty$.

Assumptions

The assumption

(A_φ) $\qquad \varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing,

includes nonlinearities of the following kind

- the porous medium $\varphi(u) = u^m$ with $m > 1$,
- fast diffusion $\varphi(u) = u^m$ with $0 < m < 1$, and
- (one-phase) Stefan problem $\varphi(u) = \max\{0, u - c\}$ with $c > 0$.

The assumption

(A$_\mu$) $\mu \geq 0$ is a symmetric Radon measure on $\mathbb{R}^N \setminus \{0\}$ satisfying

$$\int_{|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} 1 \,\mathrm{d}\mu(z) < \infty.$$

ensures that our $\mathcal{L}^\mu$ includes important examples:

- the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$;
- relativistic Schrödinger type operators $m^\alpha I - (m^2 I - \Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ and $m > 0$;
- for the measure ν with $\nu(\mathbb{R}^N) < \infty$, $\mathcal{L}^\nu[\psi](x) = \int_{\mathbb{R}^N} \big(\psi(x+z) - \psi(x)\big) \,\mathrm{d}\nu(z)$;
- for the function J with $\int_{\mathbb{R}^d} J(z) \,\mathrm{d}z = 1$, $\mathcal{L}^{J\,\mathrm{d}z}[\psi] = J * \psi - \psi$;
- Fourier multipliers $\mathcal{F}(\mathcal{L}^\mu[\psi]) = -s_{\mathcal{L}^\mu} \mathcal{F}(\psi)$.

Local case: $\partial_t u = \Delta u$, $\partial_t u = \Delta u^m$, $\partial_t u = \Delta\varphi(u)$.

- Well-posedness:

J. L. Vázquez. *The porous medium equation. Mathematical theory*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

- Numerical results:

Risebro, Karlsen, Bürger, DiBendedetto, Droniou, Eymard, Gallouet, Ebmeyer,. . .

Nonlocal case: $\partial_t u = \mathcal{L}^\mu[\varphi(u)]$.

- Well-posedness when $\mathcal{L}^\mu = -(-\Delta)^{\frac{\alpha}{2}}$:

Many people: Vázquez, de Pablo, Quirós, Rodríguez, Brändle, Bonforte, Stan, del Teso, Muratori, Grillo, Punzo, ...

- Well-posedness for other $\mathcal{L}^\mu$:

Nonsingular operators

F. Andreu-Vaillo, J. Mazón, J. D. Rossi, and J. J. Toledo-Melero. *Nonlocal diffusion problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

Fractional Laplace like operators (with some x-dependence)

A. de Pablo, F. Quirós, and A. Rodríguez. Nonlocal filtration equations with rough kernels. *Nonlinear Anal.*, 137:402–425, 2016.

- Well-posedness for related $\mathcal{L}^\mu$:

G. Karch, M. Kassmann, and M. Krupski. A framework for non-local, non-linear initial value problems. arXiv, 2018.

Nonlocal case: $\partial_t u = \mathcal{L}^\mu[\varphi(u)]$.

- Numerical results:

Discretizations of the singular integral:

E. R. Jakobsen, K. H. Karlsen, and C. La Chioma. Error estimates for approximate solutions to Bellman equations associated with controlled jump-diffusions. *Numer. Math.*, 110(2):221–255, 2008.

J. Droniou. A numerical method for fractal conservation laws. *Math. Comp.*, 79(269):95–124, 2010.

S. Cifani and E. R. Jakobsen. Entropy solution theory for fractional degenerate convection-diffusion equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(3):413–441, 2011.

Y. Huang and A. Oberman. Numerical methods for the fractional Laplacian: a finite difference–quadrature approach. *SIAM J. Numer. Anal.*, 52(6):3056–3084, 2014.

Powers of the discrete Laplacian:

O. Ciaurri, L. Roncal, P. R. Stinga, J. L. Torrea, and J. L. Varona. Nonlocal discrete diffusion equations and the fractional discrete Laplacian, regularity and applications. *Adv. Math.*, 330:688–738, 2018.

Bounded domain:

N. Cusimano, F. del Teso, L. Gerardo-Giorda, and G. Pagnini. Discretizations of the spectral fractional Laplacian on general domains with Dirichlet, Neumann, and Robin boundary conditions. *SIAM J. Numer. Anal.*, 56(3):1243–1272, 2018.

Thoughts

- Which other equations will behave in a similar way?
- How general can we make the nonlinearity $u \mapsto u^m$ and the operator Δ?
- Why are the mentioned properties so important?

Thoughts

- Which other equations will behave in a similar way?
- How general can we make the nonlinearity $u \mapsto u^m$ and the operator Δ?
- Why are the mentioned properties so important?

Goal: Numerical simulations

The goal of this presentation is to obtain mathematically rigorous numerical simulations.

So, what do we need?

- **UNIQUENESS**: Connected with convergence. **Any** approximation converges to the same actual solution.
- **PROPERTIES/COMPACTNESS**: We need to identify an abstract space in which we cannot escape. The properties of the numerical scheme will help us do so.
- **CONVERGENCE**: Connected with uniqueness. As the grid gets finer, we are sure that the numerical solution becomes a more and more accurate approximation of the actual solution. Note that we can be certain of this without knowing the actual solution.

Uniqueness

Concept of solution

Let us reconsider

$$\text{(HE)} \qquad \begin{cases} \partial_t u = \Delta[u] & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N. \end{cases}$$

When does this equation actually make sense?

Well, at least when $u \in C^1([0,T]; C^2(\mathbb{R}^N))$ because then

$$\partial_t u = \Delta[u] \qquad \text{for all} \qquad (x,t) \in Q_T$$

and

$$u(x,0) = u_0(x) \qquad \text{for all} \qquad x \in \mathbb{R}^N.$$

We call such a solution a pointwise solution.

And yes, this is (in general) very restrictive.

Concept of solution

Nature is in fact way more rough. Typically, $0 \leq u_0 \in L^1$ because it represents a density of some sort. Then we expect $0 \leq u \in L^1$.

But: How do we differentiate u with respect to time and twice with respect to space?

Note that even if solutions of the heat equation will become C^∞, the solutions of the porous medium equation is not more than C^γ for some $\gamma \in (0,1)$ (however, C^∞ where $u > 0$).

Concept of solution: distributional

Definition

u is a **distributional solution/very weak** of (GPME) if

$$0 = \int_0^T \int_{\mathbb{R}^N} \Big(u(x,t)\partial_t \psi(x,t) + \varphi(u(x,t))\mathcal{L}[\psi(\cdot,t)](x) \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}^N} u_0(x)\psi(x,0)\,\mathrm{d}x$$

for all $\psi \in C_c^\infty(\mathbb{R}^N \times [0,T))$.

- Positive: We require very little of u.
- Negative: The more general the solution concept, the more difficult it is to prove uniqueness.

Uniqueness

Theorem (Uniqueness, [del Teso&JE&Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^\infty(\mathbb{R}^N)$. *Then there is at most one distributional solution* u *of* (GPME) *such that* $u \in L^\infty(Q_T)$ *and* $u - u_0 \in L^1(Q_T)$.

Corollary (Uniqueness, [del Teso&JE&Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$. *Then there is at most one distributional solution* $u \in L^1 \cap L^\infty(\mathbb{R}^N)$ *of* (GPME).

Properties

Finite-difference discretizations: local

Again we return to

$$
\text{(HE)} \qquad \begin{cases} \partial_t u = \Delta[u] & \text{in} \quad \mathbb{R} \times (0, T), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}. \end{cases}
$$

Recall what we did with the random walk (with $\frac{\Delta t}{h^2} = \frac{1}{2}$):

$$
\frac{U_h(x, t + \Delta t) - U_h(x, t)}{\Delta t} = \frac{U_h(x + h, t) + U_h(x - h, t) - 2U_h(x, t)}{h^2}.
$$

You probably recognize the left-hand side ($\approx \partial_t u$) as

$$
u(x, t + \Delta t) = u(x, t) + \Delta t \partial_t u(x, t) + O(\Delta t^2),
$$

and the right-hand side ($\approx \partial^2_{xx} u$) as

$$
u(x + h, t) = u(x, t) + h\partial_x u(x, t) + \frac{h^2}{2}\partial^2_{xx} u(x, t) + O(h^3)
$$

$$
u(x - h, t) = u(x, t) - h\partial_x u(x, t) + \frac{h^2}{2}\partial^2_{xx} u(x, t) + O(h^3).
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Finite-difference discretizations: local

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You probably recognize the left-hand side ($\approx \partial_t\psi$) as

$$\psi(x,t+\Delta t) = \psi(x,t) + \Delta t\partial_t\psi(x,t) + O(\Delta t^2),$$

and the right-hand side ($\approx \partial_{xx}^2\psi$) as

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Written in a different way:

$$\left\| \partial_t \psi - \frac{\psi(x,t+\Delta t) - \psi(x,t)}{\Delta t} \right\|_{L^1(\mathbb{R}^N)} = O(\Delta t^2)$$

and

$$\left\| \partial_{xx}^2 \psi - \frac{\psi(x+h,t) + \psi(x-h,t) - 2\psi(x,t)}{h^2} \right\|_{L^1(\mathbb{R}^N)} = O(h^3).$$

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$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\partial_t u \qquad\qquad\qquad\qquad \partial^2_{xx} u$$

Note that we have implicitly assumed that $U_h \to u$ when $h \to 0^+$!

Again we return to

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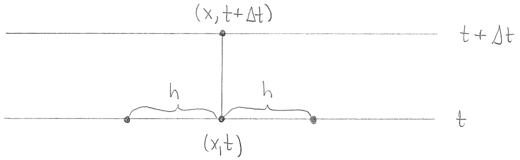
Recall what we did with the random walk (with $\frac{\Delta t}{h^2} \leq \frac{1}{2}$):

Explicit method:

$$U_h(x, t+\Delta t) = U_h(x,t) + \frac{\Delta t}{h^2}\big(U_h(x+h,t) + U_h(x-h,t) - 2U_h(x,t)\big).$$



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Lax equivalence theorem: Consistent finite-difference methods of a linear equation are convergent **iff** they are stable (at least CFL).

Finite-difference discretizations: local

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$$\frac{\Delta t}{h^2}\begin{bmatrix} (\frac{h^2}{\Delta t}-2) & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & (\frac{h^2}{\Delta t}-2) & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & (\frac{h^2}{\Delta t}-2) & 1 & 0 & \cdots & 0 \\ & & & \ddots & & & \\ 0 & \cdots & \cdots & 0 & 1 & (\frac{h^2}{\Delta t}-2) & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & (\frac{h^2}{\Delta t}-2) \end{bmatrix}\begin{bmatrix} U^0_{-m} \\ U^0_{-m+1} \\ U^0_{-m+2} \\ \vdots \\ U^0_{m-1} \\ U^0_{m} \end{bmatrix} = \begin{bmatrix} U^1_{-m} \\ U^1_{-m+1} \\ U^1_{-m+2} \\ \vdots \\ U^1_{m-1} \\ U^1_{m} \end{bmatrix}$$

Comment: • Outside $[-M,M]$, we put $U_h = 0$.
• Sparse matrix, easy to "build".

Finite-difference discretizations: local

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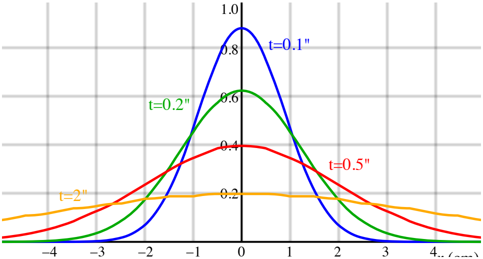

Figure due to Wikipedia.

Finite-difference discretizations: nonlocal

Let us for simplicity study

$$\text{(FHE)} \qquad \begin{cases} \partial_t u = \mathcal{L}^\mu[u] & \text{in} \quad \mathbb{R}\times(0,T), \\ u(x,0)=u_0(x) & \text{on} \quad \mathbb{R}. \end{cases}$$

Let us try to deduce that

$$\begin{aligned} \mathcal{L}^h[\psi](x) &:= \sum_{\mathbb{Z}\ni\beta\neq 0} \big(\psi(x+h\beta)-\psi(x)\big)\omega_{\beta,h} \\ &\approx \text{P.V.}\int_{|z|>0} \big(\psi(x+z)-\psi(x)\big)\,\mathrm{d}\mu(z) = \mathcal{L}^\mu[\psi] \end{aligned}$$

where $\omega_\beta = \omega_{-\beta} \geq 0$. Recall what we did with the long-jump random walk.

In a similar way,

$$\Delta_h[\psi](x) := \big(\psi(x-h)-\psi(x)\big)\frac{1}{h^2} + \big(\psi(x+h)-\psi(x)\big)\frac{1}{h^2} \approx \Delta[\psi](x).$$

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Finite-difference discretizations: nonlocal

Roughly speaking, we can separate between 3 cases for the nonlocal operator $\mathcal{L}^\mu$:

- Singluar part:
  $\int_{0<|z|\leq r} \big(\psi(x+z)-\psi(x)\big)\,\mathrm{d}\mu(z)$
- Nonsingluar, middle part:
  $\int_{r<|z|\leq R} \big(\psi(x+z)-\psi(x)\big)\,\mathrm{d}\mu(z)$
- Nonsingular, tail part:
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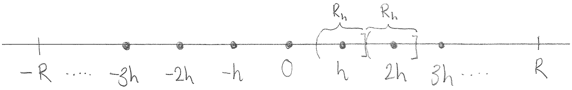
Finite-difference discretizations: nonlocal

Let us use the grid

$$\mathcal{G}_h := \{h\beta \,:\, \beta \in \mathbb{Z}\} \qquad \text{and} \qquad R_h := h\Big(-\frac{1}{2},\frac{1}{2}\Big]$$



Let $\{p_\beta^k\}_\beta$ be an interpolation basis of order k for the uniform-in-space spatial grid $\mathcal{G}_h$, and let the interpolant of a function ψ be $I_h^k[\psi](z) := \sum_{\mathbb{Z}\ni\beta\neq 0} \psi(h\beta)p_\beta^k(z)$. Then (with $r = h$)

$$\mathcal{L}^h[\psi](x) = \int_{|z|>h} I_h^k\big[\psi(x+\cdot) - \psi(x)\big](z)\,\mathrm{d}\mu(z).$$

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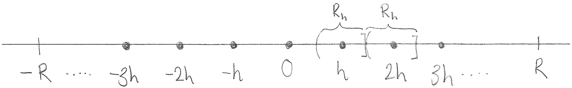

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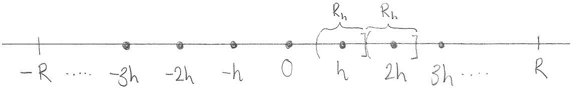
Monotone ($\int_{|z|>h} p_\beta^k(z)\,\mathrm{d}\mu(z) \geq 0$) when $k = 0, 1$.
Better monotonicity if μ abs. cont. and regular (Newton-Cotes).

Finite-difference discretizations: nonlocal



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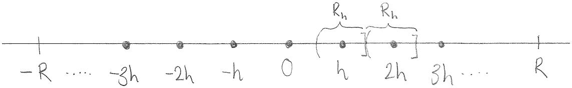

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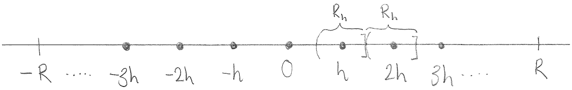
$k = 0$ (midpoint rule/constant interpolation basis):

$$\int_{h\beta+R_h} \big(\psi(x+z)-\psi(x)\big)\,\mathrm{d}\mu(z) \approx \big(\psi(x+h\beta)-\psi(x)\big)\mu(h\beta+R_h)$$

Finite-difference discretizations: nonlocal



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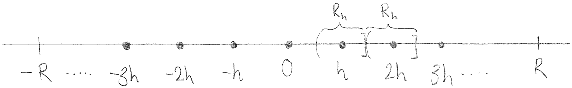

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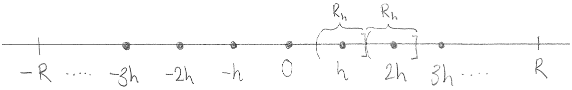
$$\|\mathcal{L}^\mu[\psi] - \mathcal{L}^h[\psi]\|_{L^1(\mathbb{R}^N)} \to 0 \qquad \text{as} \qquad h \to 0^+.$$

Finite-difference discretizations: nonlocal



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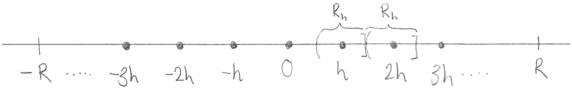
$$\| -(-\Delta)^{\frac{\alpha}{2}}[\psi] - \mathcal{L}^h[\psi]\|_{L^1(\mathbb{R}^N)} = O(h + h^{2-\alpha}).$$

Finite-difference discretizations: nonlocal



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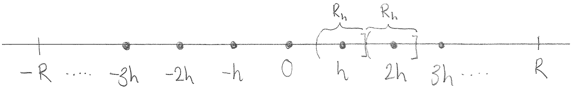
Finite-difference discretizations: nonlocal

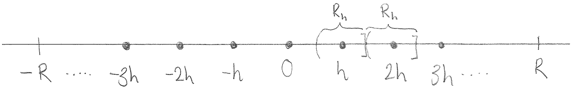


Let $\{p_\beta^k\}_\beta$ be an interpolation basis of order k for the uniform-in-space spatial grid $\mathcal{G}_h$, and let the interpolant of a function ψ be $I_h[\psi](z) := \sum_{\beta\neq 0} \psi(h\beta)p_\beta^k(z)$. Then (with $r = h$)

$$\mathcal{L}^h[\psi](x) = \sum_{\mathbb{Z}\ni\beta\neq 0} \big(\psi(x+h\beta)-\psi(x)\big)\omega_{\beta,h}.$$

$k = 0$ (midpoint rule/constant interpolation basis):

$$\| -(-\Delta)^{\frac{\alpha}{2}}[\psi] - \mathcal{L}^h[\psi]\|_{L^1(\mathbb{R}^N)} = O(h + h^{2-\alpha}).$$



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$$\text{“}\Delta_h[\psi](x) \subset \mathcal{L}^h[\psi](x)\text{”} = \sum_{\mathbb{Z}\ni\beta\neq 0} \big(\psi(x+h\beta)-\psi(x)\big)\omega_{\beta,h}.$$

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Let us return to

$$
\text{(FHE)} \qquad \begin{cases} \partial_t u = \mathcal{L}^{\mu}[u] & \text{in} \quad \mathbb{R} \times (0, T), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}. \end{cases}
$$

Explicit method:

$$
U_h(x, t+\Delta t) = U_h(x,t) + \Delta t \sum_{\mathbb{Z} \ni \beta \neq 0} \big(U_h(x+h\beta, t) - U_h(x,t) \big) \omega_{\beta,h}.
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Explicit method (with midpoint rule):

$$U_h(x, t+\Delta t) = U_h(x,t) + \frac{\Delta t}{h^\alpha} \sum_{\mathbb{Z} \ni \beta \neq 0} \big(U_h(x+h\beta, t) - U_h(x,t)\big)\, C_\beta,$$

where

$$C_{-\beta} = C_\beta = \frac{c_{1,\alpha}}{\alpha}\Big((\beta - \frac{1}{2})^{-\alpha} - (\beta + \frac{1}{2})^{-\alpha}\Big) \qquad \text{when} \qquad \beta \geq 1.$$

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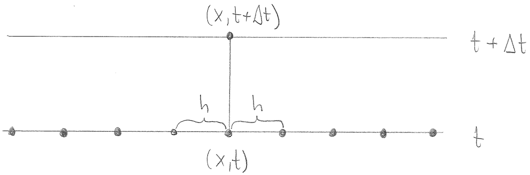

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$$[U^1_{-m}, U^1_{-m+1}, U^1_{-m+2}, \cdots, U^1_{m-1}, U^1_m]^T =$$

$$\frac{\Delta t}{h^\alpha}\begin{bmatrix} (\frac{h^\alpha}{\Delta t}-C) & C_1 & C_2 & \cdots & \cdots & \cdots & C_{2m} \\ C_1 & (\frac{h^\alpha}{\Delta t}-C) & C_1 & C_2 & \cdots & \cdots & C_{2m-1} \\ C_2 & C_1 & (\frac{h^\alpha}{\Delta t}-C) & C_1 & C_2 & \cdots & C_{2m-2} \\ & & & \ddots & & & \\ C_{2m-1} & \cdots & \cdots & C_2 & C_1 & (\frac{h^\alpha}{\Delta t}-C) & C_1 \\ C_{2m} & \cdots & \cdots & \cdots & C_2 & C_1 & (\frac{h^\alpha}{\Delta t}-C) \end{bmatrix}\begin{bmatrix} U^0_{-m} \\ U^0_{-m+1} \\ U^0_{-m+2} \\ \vdots \\ U^0_{m-1} \\ U^0_m \end{bmatrix}$$

Comment: • Outside $[-M,M]$, we put $U_h = 0$ AND outside $[-2M,2M]$, we put $C_\beta = 0$.

• Dense matrix, hard to "build".

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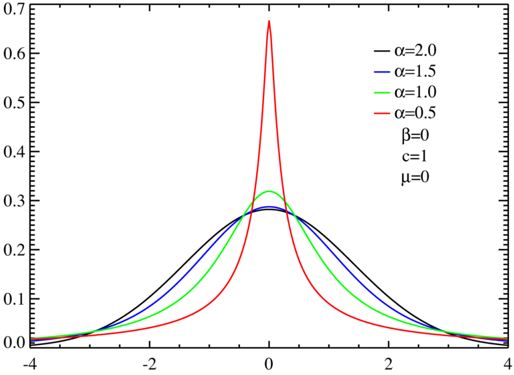

Figure due to Wikipedia.

Numerical schemes for (GPME)

Recall that our Cauchy problem was given as

$$\text{(GPME)} \qquad \begin{cases} \partial_t u = \mathcal{L}[\varphi(u)] & \text{in} \quad Q_T = \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N. \end{cases}$$

Corresponding numerical scheme (NM):

$$\begin{cases} \frac{U_\beta^j - U_\beta^{j-1}}{\Delta t} = \mathcal{L}^{\nu_{h,1}}[\varphi(U_\beta^j)] + \mathcal{L}^{\nu_{h,2}}[\varphi^h(U_\beta^{j-1})] & \text{in} \quad h\mathbb{Z}^N \times \Delta t\mathbb{N}, \\ \text{``}U_\beta^0 = u_0\text{''} & \text{in} \quad h\mathbb{Z}^N, \end{cases}$$

where

$$\mathcal{L}^{\nu_{h,1}} + \mathcal{L}^{\nu_{h,2}} \approx \mathcal{L} = \mathcal{L}^\sigma + \mathcal{L}^\mu$$

$$\varphi^h \approx \varphi$$

Convergence

Convergence of the numerical schemes

Theorem (Convergence, [del Teso&JE&Jakobsen, 2018])

For the interpolant U_h, we have

$$U_h \to u \qquad \text{in} \qquad C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N)) \qquad \text{as} \qquad h \to 0^+$$

where $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ is a distributional solution of (GPME).

Note that we only assume $u_0 \in L^1 \cap L^\infty$.

Advantage using general nonlocal framework

Keep in mind the following formula:

$$\mathcal{L}^h[\psi](x) = \sum_{\mathbb{Z}\ni\beta\neq0} \big(\psi(x+h\beta) - \psi(x)\big)\omega_{\beta,h}.$$

Now, note that

$$\sum_{\mathbb{Z}\ni\beta\neq0} \big(\psi(x+h\beta) - \psi(x)\big)\omega_{\beta,h} = \int_{|z|>0} \big(\psi(x+z) - \psi(x)\big)\,\mathrm{d}\nu_h(z)$$

where $\mathrm{d}\nu_h(z) = \sum_{\mathbb{Z}\ni\beta\neq0} \omega_{\beta,h}\,\mathrm{d}\delta_{h\beta}(z)$.

This includes the **local** discretization by simply choosing

$$\omega_{\beta,h} = \begin{cases} \frac{1}{h^2} & \text{when } \beta = \{-1,1\}, \\ 0 & \text{otherwise.} \end{cases}$$

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Moreover, the discretizations of **local** and **nonlocal** operators are **nonlocal** operators!!

Proof of convergence

1. Since the operator and the nonlinearity are x-**independent**, the numerical scheme can be written, for $x \in \mathbb{R}^N$, as
$$U^j(x) - \Delta t \mathcal{L}^{\nu_{h,1}}[\varphi(U^j)](x) = U^{j-1}(x) + \Delta t \mathcal{L}^{\nu_{h,2}}[\varphi^h(U^{j-1})](x).$$
2. At every time step, we have a combination of explicit and implicit steps:
$$\text{(EP)} \qquad w - \Delta t \mathcal{L}^{\nu_{h,1}}[\varphi(w)] = f \qquad \text{on} \qquad \mathbb{R}^N,$$
where $U^j = w = T_{\text{imp}}[f]$ and
$$f(x) = T_{\text{exp}}[U^{j-1}](x) = U^{j-1}(x) + \Delta t \mathcal{L}^{\nu_{h,2}}[\varphi^h(U^{j-1})](x).$$
3. Well-posedness of (NM) $\Longleftrightarrow$ Well-posedness of (EP) and properties of T_{exp}.
4. To study T_{exp}, the CFL-condition comes naturally
$$\Delta t L_{\varphi^h} \nu_{h,2}(\mathbb{R}^N) \leq 1 \qquad \text{"time derivative} \sim \text{spatial derivatives"}$$

Proof of convergence

5. Both operators T_{imp} and T_{exp} are "well-posed" in $L^1 \cap L^\infty$ and enjoy
   - comparison principle;
   - L^1-contraction; and
   - L^1/L^∞-bounds.
6. All properties then carries over to the numerical scheme (NM).
7. In particular, we have for the interpolant U_h

$$\sup_h \|U_h(\cdot + \xi, t) - U_h(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq \lambda(|\xi|)$$

$$\sup_h \|U_h(\cdot, t) - U_h(\cdot, s)\|_{L^1(K)} \leq \lambda(|t - s|).$$

8. An application of the Arzelà-Ascoli and Kolmogorov-Riesz compactness theorems then gives the desired compactness and convergence in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$. Check that the limit of the numerical solution is indeed a distributional solution.
9. And then all the properties carries over to distributional solutions of (GPME).

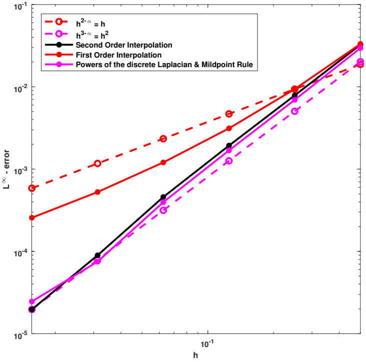
Numerical simulations

Numerical simulations

Main difference between local and nonlocal:

the computational domain is different from the actual domain.

Error plot for the fractional heat equation with $\alpha = 1$

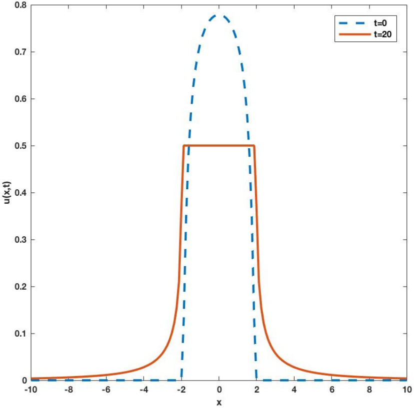


Comments: • We see that it converges, but we also KNOW that it does!

• We do the simulations with "classical" solutions, so we basically test the consistency error of the operator.

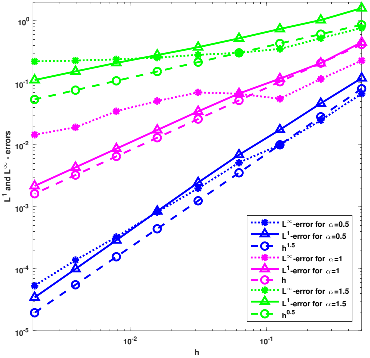
• The MpR behaves better in practise $O(h^2)$ than in theory $O(h)$.

The fractional (one-phase) Stefan problem with $\alpha = 1$: plot



Comments: • $\varphi(u) = \max\{0, u - 0.5\}$.

• $\varphi(u)$ is only Lipschitz even if u is smooth!



Comments: • Recall that "Error"$\sim h + h^{2-\alpha}$.

• Since pointwise values did not make sense, the error is more stable in L^1.

Numerical simulations: 2D (one-phase) Stefan problem

• 2D (one-phase) Stefan problem with $\varphi(u) = \max\{0, u - 1\}$.
Explicit method. $\mathcal{L} = ((\frac{1}{2}, \frac{47}{100}) \cdot D)^2 + (-\partial^2_{xx})^{\frac{1}{4}}$.

References

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Thank you for your attention!