Nonlocal (and local) nonlinear diffusion equations. Background, analysis, and numerical approximation

Jørgen Endal
URL: http://folk.ntnu.no/jorgeen

Department of mathematical sciences
NTNU, Norway

18 June 2019

In collaboration with
F. del Teso and E. R. Jakobsen

A talk given at
MAFELAP 2019, Brunel University London


References

F. del Teso, JE, E. R. Jakobsen. Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type. *Adv. Math.*, 305:78–143, 2017.

F. del Teso, JE, E. R. Jakobsen. On distributional solutions of local and nonlocal problems of porous medium type. *C. R. Acad. Sci. Paris, Ser. I*, 355(11):1154–1160, 2017.

F. del Teso, JE, E. R. Jakobsen. Robust numerical methods for nonlocal (and local) equations of porous medium type. Part I: Theory. To appear in *SIAM J. Numer. Anal.*, 2019.

F. del Teso, JE, E. R. Jakobsen. Robust numerical methods for nonlocal (and local) equations of porous medium type. Part II: Schemes and experiments. *SIAM J. Numer. Anal.*, 56(6):3611–3647, 2018.

Goal: Numerical simulations

The goal of this presentation is to study monotone finite-difference approximations of diffusion equations in $\mathbb{R}^N \times (0, T)$.

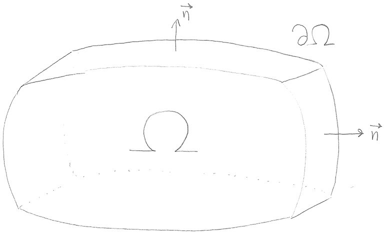
Goal: Numerical simulations

The goal of this presentation is to study monotone finite-difference approximations of diffusion equations in $\mathbb{R}^N \times (0, T)$.

Introduction: Mathematical modelling

Diffusion is the act of "spreading out" – the movement from areas of high concentration to areas of low concentration.

How do we model this phenomena?



Let u be some heat density inside a region Ω. The rate of change of the total quantity within Ω equals the negative of the net flux through $\partial\Omega$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} u \,\mathrm{d}x = -\int_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = -\int_{\Omega} \mathrm{div}\mathbf{F}\,\mathrm{d}V,$$

or

$$\partial_t u = -\mathrm{div}\mathbf{F},$$

where $\mathbf{F} = \mathbf{F}(u) := -a(u)Du$.

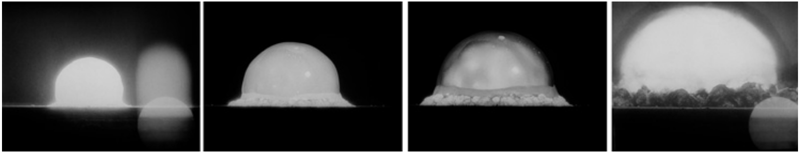
Introduction: Special case when $m = 6$

- $a(u) = u^{m-1}$.

It is possible to use

$$\begin{cases} \partial_t u = \Delta[u^6] & \text{in} \quad \mathbb{R}^N \times (0,T), \\ u(x,0) = M\delta_0 & \text{on} \quad \mathbb{R}^N, \end{cases}$$

to describe the propagation of heat immediately after a nuclear explosion.



G. I. Barenblatt. *Scaling, self-similarity, and intermediate asymptotics*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1996.

Nonlocal (local) nonlinear diffusion

Let $Q_T := \mathbb{R}^N \times (0, T)$. We consider the following Cauchy problem:

$$\text{(GPME)} \qquad \begin{cases} \partial_t u = \mathcal{L}[\varphi(u)] & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

where

- $$\begin{aligned} \mathcal{L}[\psi] &= \mathcal{L}^\sigma[\psi] + \mathcal{L}^\mu[\psi] \\ &= \text{local} \; + \; \text{nonlocal} \quad \text{(self-adjoint)} \end{aligned}$$
- $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and
- u_0 some rough initial data.

Main results:

- Uniqueness for $u_0 \in L^1 \cap L^\infty$.
- Convergent numerical schemes in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ for $u_0 \in L^1 \cap L^\infty$.

Assumptions

The assumption

(A_φ) $\qquad \varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing,

includes nonlinearities of the following kind

- linear,
- the porous medium $\varphi(u) = u^m$ with $m > 1$,
- fast diffusion $\varphi(u) = u^m$ with $0 < m < 1$, and
- (one-phase) Stefan problem $\varphi(u) = \max\{0, u - c\}$ with $c > 0$.

Assumptions

The assumption

(A_μ) $\mu \geq 0$ is a symmetric Radon measure on $\mathbb{R}^N \setminus \{0\}$ satisfying

$$\int_{|z|\leq 1} |z|^2 \, \mathrm{d}\mu(z) + \int_{|z|>1} 1 \, \mathrm{d}\mu(z) < \infty.$$

ensures that our $\mathcal{L}^\mu$ includes important examples:

- the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$;
- the anisotropic fractional Laplacian $-\sum_{i=1}^{N}(-\partial^2_{x_i x_i})^{\frac{\alpha_i}{2}}$ with $\alpha_i \in (0,2)$;
- relativistic Schrödinger type operators $m^\alpha I - (m^2 I - \Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ and $m > 0$;
- for the measure ν with $\nu(\mathbb{R}^N) < \infty$, $\mathcal{L}^\nu[\psi](x) = \int_{\mathbb{R}^N} \left(\psi(x+z) - \psi(x)\right) \mathrm{d}\nu(z)$;
- for the function J with $\int_{\mathbb{R}^d} J(z) \, \mathrm{d}z = 1$, $\mathcal{L}^{J \, \mathrm{d}z}[\psi] = J * \psi - \psi$;
- Fourier multipliers $\mathcal{F}(\mathcal{L}^\mu[\psi]) = -s_{\mathcal{L}^\mu} \mathcal{F}(\psi)$.

Theorem

*A **linear**, **self-adjoint** operator which is **translation invariant** and satisfies the **global comparison principle** is of the form* $\mathcal{L} = \mathcal{L}^\sigma + \mathcal{L}^\mu$ *where*

$$\mathcal{L}^\sigma[\psi(x)] := \mathrm{tr}(\sigma\sigma^T D^2\psi(x))$$

$$\mathcal{L}^\mu[\psi(x)] := \text{P.V.} \int_{|z|>0} \big(\psi(x+z) - \psi(x)\big)\,\mathrm{d}\mu(z)$$

Here, $\sigma \in \mathbb{R}^{N\times p}$ *and* $\mu \geq 0$ *is a symmetric Radon measure satisfying*

$$\int \min\{|z|^2, 1\}\,\mathrm{d}\mu(z) < \infty.$$

P. Courrège. Sur la forme intégro-différentielle des opérateurs de C_k^∞ dans C satisfaisant au principe du maximum. *Séminaire Brelot-Choquet-Deny. Théorie du Potentiel*, 10(1):1–38, 1965–1966.

Local case: $\partial_t u = \Delta u$, $\partial_t u = \Delta u^m$, $\partial_t u = \Delta\varphi(u)$.

- Well-posedness:

J. L. Vázquez. *The porous medium equation. Mathematical theory*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

- Numerical results:

Risebro, Karlsen, Bürger, DiBendedetto, Droniou, Eymard, Gallouet, Ebmeyer,. . .

Selective summary of previous results

Nonlocal case: $\partial_t u = \mathcal{L}^\mu[\varphi(u)]$.

- Well-posedness when $\mathcal{L}^\mu \sim -(-\Delta)^{\frac{\alpha}{2}}$:

Many people: Vázquez, de Pablo, Quirós, Rodríguez, Brändle, Bonforte, Stan, del Teso, Muratori, Grillo, Punzo, ...

- Numerical results:

Finite-difference discretizations of the singular integral:

E. R. Jakobsen, K. H. Karlsen, and C. La Chioma. Error estimates for approximate solutions to Bellman equations associated with controlled jump-diffusions. *Numer. Math.*, 110(2):221–255, 2008.

J. Droniou. A numerical method for fractal conservation laws. *Math. Comp.*, 79(269):95–124, 2010.

S. Cifani and E. R. Jakobsen. Entropy solution theory for fractional degenerate convection-diffusion equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(3):413–441, 2011.

Y. Huang and A. Oberman. Numerical methods for the fractional Laplacian: a finite difference–quadrature approach. *SIAM J. Numer. Anal.*, 52(6):3056–3084, 2014.

Powers of the discrete Laplacian:

O. Ciaurri, L. Roncal, P. R. Stinga, J. L. Torrea, and J. L. Varona. Nonlocal discrete diffusion equations and the fractional discrete Laplacian, regularity and applications. *Adv. Math.*, 330:688–738, 2018.

Uniqueness

Theorem (Uniqueness, [del Teso & JE & Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$. *Then there is at most one distributional/very weak solution* $u \in L^1 \cap L^\infty(Q_T)$ *of* (GPME).

Uniqueness

Theorem (Uniqueness, [del Teso & JE & Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$. *Then there is at most one distributional/very weak solution* $u \in L^1 \cap L^\infty(Q_T)$ *of* (GPME).

Numerical schemes for (GPME)

Recall that our Cauchy problem was given as

$$\text{(GPME)} \qquad \begin{cases} \partial_t u = \mathcal{L}[\varphi(u)] & \text{in} \quad Q_T = \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N. \end{cases}$$

Corresponding numerical scheme (NM):

$$\begin{cases} \frac{U_\beta^j - U_\beta^{j-1}}{\Delta t} = \mathcal{L}^{\nu_{h,1}}[\varphi(U_\beta^j)] + \mathcal{L}^{\nu_{h,2}}[\varphi^h(U_\beta^{j-1})] & \text{in} \quad h\mathbb{Z}^N \times \Delta t\mathbb{N}, \\ \text{``}U_\beta^0 = u_0\text{''} & \text{in} \quad h\mathbb{Z}^N, \end{cases}$$

where

$$\mathcal{L}^{\nu_{h,1}} + \mathcal{L}^{\nu_{h,2}} \approx \mathcal{L} = \mathcal{L}^\sigma + \mathcal{L}^\mu$$

$$\varphi^h \approx \varphi$$

Numerical schemes for (GPME)

Note that

- in the nonlinear case, we can no longer expect smooth solutions!

Our framework includes

- a mixture of implicit and explicit schemes (θ-methods);
- the possibility of discretizing the singluar and nonsingular parts of $\mathcal{L}^\mu$ in different ways; and
- combinations of the above.

Also:
Explicit methods only works for Lipschitz φ because of CFL. But, instead of doing implicit methods for "demanding" φ, we can do less costly explicit methods with approximating φ.

Convergence of the numerical schemes

Theorem (Convergence, [del Teso & JE & Jakobsen, 2018/2019])

For the interpolant U_h, we have

$$U_h \to u \qquad \text{in} \qquad C([0,T];L^1_{\text{loc}}(\mathbb{R}^N)) \qquad \text{as} \qquad h \to 0^+$$

where $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T];L^1_{\text{loc}}(\mathbb{R}^N))$ is a distributional solution of (GPME).

Note that we only assume $u_0 \in L^1 \cap L^\infty$.

Finite-difference discretizations: nonlocal

Let us for simplicity study

$$\text{(FHE)} \qquad \begin{cases} \partial_t u = \mathcal{L}^\mu[u] & \text{in} \quad \mathbb{R}\times(0,T), \\ u(x,0)=u_0(x) & \text{on} \quad \mathbb{R}. \end{cases}$$

Let us try to deduce that

$$\begin{aligned} \mathcal{L}^h[\psi](x) &:= \sum_{\mathbb{Z}\ni\beta\neq 0} \big(\psi(x+h\beta)-\psi(x)\big)\omega_{\beta,h} \\ &\approx \text{P.V.}\int_{|z|>0} \big(\psi(x+z)-\psi(x)\big)\,\mathrm{d}\mu(z) = \mathcal{L}^\mu[\psi] \end{aligned}$$

where $\omega_\beta = \omega_{-\beta} \geq 0$.

In a similar way,

$$\Delta_h[\psi](x) := \big(\psi(x-h)-\psi(x)\big)\frac{1}{h^2} + \big(\psi(x+h)-\psi(x)\big)\frac{1}{h^2} \approx \Delta[\psi](x).$$

Let us for simplicity study

$$\text{(FHE)} \qquad \begin{cases} \partial_t u = \mathcal{L}^\mu[u] & \text{in} \quad \mathbb{R}\times(0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}. \end{cases}$$

Let us try to deduce that

$$\begin{aligned}\mathcal{L}^h[\psi](x) := \sum_{\mathbb{Z}\ni\beta\neq 0} \big(\psi(x+h\beta)-\psi(x)\big)\omega_{\beta,h} \\ \approx \text{P.V.}\int_{|z|>0}\big(\psi(x+z)-\psi(x)\big)\,\mathrm{d}\mu(z) = \mathcal{L}^\mu[\psi]\end{aligned}$$

where $\omega_\beta = \omega_{-\beta} \geq 0$.

In a similar way,

$$\Delta_h[\psi](x) := \big(\psi(x-h)-\psi(x)\big)\frac{1}{h^2}+\big(\psi(x+h)-\psi(x)\big)\frac{1}{h^2} \approx \Delta[\psi](x).$$

Finite-difference discretizations: nonlocal

Let us for simplicity study

$$\text{(FHE)} \qquad \begin{cases} \partial_t u = \mathcal{L}^\mu[u] & \text{in} \quad \mathbb{R} \times (0, T), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}. \end{cases}$$

Let us try to deduce that

$$\mathcal{L}^h[\psi](x) := \sum_{\mathbb{Z} \ni \beta \neq 0} \big(\psi(x + h\beta) - \psi(x)\big)\omega_{\beta,h}$$
$$\approx \text{P.V.} \int_{|z|>0} \big(\psi(x+z) - \psi(x)\big) \,\mathrm{d}\mu(z) = \mathcal{L}^\mu[\psi]$$

where $\omega_\beta = \omega_{-\beta} \geq 0$. Recall what we did with the long-jump random walk.

In a similar way,

$$\mathcal{L}^h[\psi](x) \supset \Delta_h[\psi](x) := \sum_{\{-1,1\} \ni \beta \neq 0} \big(\psi(x+h\beta) - \psi(x)\big)\frac{1}{h^2} \approx \Delta[\psi](x).$$

Advantage using general nonlocal framework

Keep in mind the following formula:

$$\mathcal{L}^h[\psi](x) = \sum_{\mathbb{Z}\ni\beta\neq 0} \big(\psi(x+h\beta)-\psi(x)\big)\omega_{\beta,h}.$$

Now, note that

$$\sum_{\mathbb{Z}\ni\beta\neq 0} \big(\psi(x+h\beta)-\psi(x)\big)\omega_{\beta,h} = \int_{|z|>0} \big(\psi(x+z)-\psi(x)\big)\,\mathrm{d}\nu_h(z)$$

where $\mathrm{d}\nu_h(z) = \sum_{\mathbb{Z}\ni\beta\neq 0} \omega_{\beta,h}\,\mathrm{d}\delta_{h\beta}(z)$.

This includes the **local** discretization by simply choosing

$$\omega_{\beta,h} = \begin{cases} \frac{1}{h^2} & \text{when } \beta = \{-1,1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of convergence

1. Since the operator and the nonlinearity are x-**independent**, the numerical scheme can be written, for $x \in \mathbb{R}^N$, as

$$U^j(x) - \Delta t \mathcal{L}^{\nu_{h,1}}[\varphi(U^j)](x) = U^{j-1}(x) + \Delta t \mathcal{L}^{\nu_{h,2}}[\varphi^h(U^{j-1})](x).$$

2. At every time step, we have a combination of explicit and implicit steps:

$$\text{(EP)} \qquad w - \Delta t \mathcal{L}^{\nu_{h,1}}[\varphi(w)] = f \qquad \text{on} \qquad \mathbb{R}^N,$$

where $U^j = w = T_{\text{imp}}[f]$ and

$$f(x) = T_{\text{exp}}[U^{j-1}](x) = U^{j-1}(x) + \Delta t \mathcal{L}^{\nu_{h,2}}[\varphi^h(U^{j-1})](x).$$

3. Well-posedness of (NM) $\Longleftrightarrow$ Well-posedness of (EP) and properties of T_{exp}.
4. To study T_{exp}, the CFL-condition comes naturally

$$\Delta t L_{\varphi^h} \nu_{h,2}(\mathbb{R}^N) \leq 1 \qquad \text{“time derivative} \sim \text{spatial derivatives”}$$

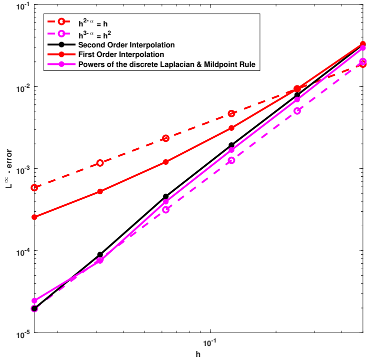
5. Both operators T_{imp} and T_{exp} are “well-posed” in $L^1 \cap L^\infty$ and enjoy
   - comparison principle;
   - L^1-contraction; and
   - L^1/L^∞-bounds.
6. All properties then carries over to the numerical scheme (NM).
7. In particular, we have for the interpolant U_h

$$\sup_h \|U_h(\cdot + \xi, t) - U_h(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq \lambda(|\xi|)$$

$$\sup_h \|U_h(\cdot, t) - U_h(\cdot, s)\|_{L^1(K)} \leq \lambda(|t - s|).$$

8. An application of the Arzelà-Ascoli and Kolmogorov-Riesz compactness theorems then gives the desired compactness and convergence in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$. Check that the limit of the numerical solution is indeed a distributional solution.
9. And then all the properties carries over to distributional solutions of (GPME).

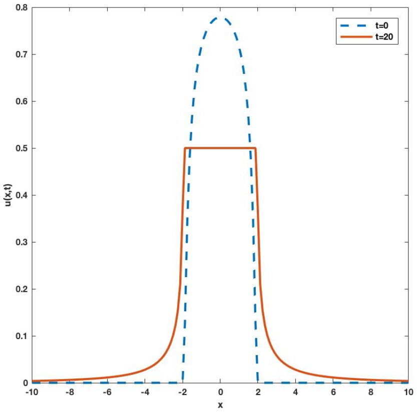
Error plot for the fractional heat equation with $\alpha = 1$



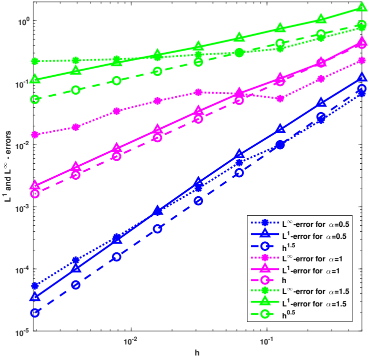
Comments: • We do the simulations with "classical" solutions, so we basically test the consistency error of the operator.

• The MpR behaves better in practise $O(h^2)$ than in theory $O(h)$.

The fractional (one-phase) Stefan problem with $\alpha = 1$: plot



Comments: • $\varphi(u) = \max\{0, u - 0.5\}$.
• $\varphi(u)$ is only Lipschitz even if u is smooth!



Comments: • Recall that "Error" $\sim h + h^{2-\alpha}$.

• Since pointwise values did not make sense, the error is more stable in L^1.

Numerical simulations: 2D (one-phase) Stefan problem

• 2D (one-phase) Stefan problem with $\varphi(u) = \max\{0, u-1\}$.
Explicit method. $\mathcal{L} = ((\frac{1}{2}, \frac{47}{100}) \cdot D)^2 + (-\partial^2_{xx})^{\frac{1}{4}}$.

Thank you for your attention!