Connections between L^1-solutions of Hamilton-Jacobi-Bellman equations and L^∞-solutions of convection-diffusion equations

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Main results

- "L^1-stability"/Contractions for HJB
- "L^∞-stability"/Weighted L^1-contraction for CDE
- Duality between HJB and CDE

N. Alibaud, JE, E. R. Jakobsen. Optimal and dual stability results for L^1 viscosity and L^∞ entropy solutions. arXiv, 2018.

How are HJB and CDE connected?

(HJ) $$\begin{cases} \partial_t \psi + H(\partial_x \psi) = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ \psi(\cdot,0) = \psi_0 & x \in \mathbb{R} \end{cases}$$

(SCL) $$\begin{cases} \partial_t u + \partial_x (H(u)) = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(\cdot,0) = u_0 & x \in \mathbb{R} \end{cases}$$

If u is the entropy solution of (SCL), then $\psi := \int^x u$ is the viscosity solution of (HJ) with $\psi_0 := \int^x u_0$. (Can be made rigorous in 1D.)

So, there is a connection, and in particular, information about u will give information about ψ.

How are HJB and CDE connected?

We will study the following Cauchy problems in $\mathbb{R}^N \times (0,\infty)$:

(HJB) $$\begin{cases} \partial_t \psi = \sup_{\xi \in \mathcal{E}} \left\{ b(\xi) \cdot D\psi + \operatorname{tr}\left(a(\xi) D^2 \psi\right) \right\} \\ \psi(\cdot, 0) = \psi_0 \end{cases}$$

(CDE) $$\begin{cases} \partial_t u + \operatorname{div} F(u) = \operatorname{div}\left(A(u) Du\right) \\ u(\cdot, 0) = u_0 \end{cases}$$

Why? And how are they related?

Note that if $\operatorname{div}\left(A(u)Du\right) = \Delta\varphi(u)$, then we replace $\operatorname{tr}\left(a(\xi)D^2\psi\right)$ by $a(\xi)\Delta\psi$.

How are HJB and CDE connected?

The Kato inequality for (CDE): For all $0 \leq \phi \in C_c^\infty$ and all $T \geq 0$,

$$\int_{\mathbb{R}^d} |u-v|(x,T)\phi(x,T)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0-v_0|(x)\phi(x,0)\,\mathrm{d}x$$
$$+ \iint_{\mathbb{R}^d\times(0,T)} \Big(|u-v|\partial_t\phi(x,t)$$
$$+ q(u,v)\cdot D\phi(x,t) + \mathrm{tr}\left(r(u,v)D^2\phi(x,t)\right) \Big)\,\mathrm{d}x\,\mathrm{d}t,$$

$$q_i(u,v) := \mathrm{sign}(u-v)\int_v^u F_i'(\xi)\,\mathrm{d}\xi,\ r_{ij}(u,v) := \mathrm{sign}(u-v)\int_v^u A_{ij}(\xi)\,\mathrm{d}\xi.$$

How are HJB and CDE connected?

The Kato inequality for (CDE): For all $0 \leq \phi \in C_c^\infty$ and all $T \geq 0$,

$$\int_{\mathbb{R}^d} |u - v|(x, T)\phi(x, T)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)\phi(x, 0)\,\mathrm{d}x$$

$$+ \iint_{\mathbb{R}^d \times (0,T)} \Big(|u - v|\partial_t \phi(x, t)$$

$$+ \operatorname{sign}(u - v) \int_{v(x,t)}^{u(x,t)} \left\{ F'(\xi) \cdot D\phi(x, t) + \operatorname{tr}\left(A(\xi)D^2\phi(x, t)\right)\right\} \mathrm{d}\xi \Big)\,\mathrm{d}x\,\mathrm{d}t,$$

$$q_i(u, v) := \operatorname{sign}(u-v) \int_v^u F_i'(\xi)\,\mathrm{d}\xi,\ \ r_{ij}(u, v) := \operatorname{sign}(u-v) \int_v^u A_{ij}(\xi)\,\mathrm{d}\xi.$$

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$$\int_{\mathbb{R}^d} |u-v|(x,T)\phi(x,T)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)\phi(x,0)\,\mathrm{d}x$$
$$+ \iint_{\mathbb{R}^d\times(0,T)} \Bigg(|u-v|\partial_t \phi(x,t)$$
$$+ |u-v| \operatorname*{ess\,sup}_{m\leq\xi\leq M} \left\{ F'(\xi)\cdot D\phi(x,t) + \mathrm{tr}\left(A(\xi)D^2\phi(x,t)\right)\right\} \Bigg)\,\mathrm{d}x\,\mathrm{d}t,$$

$$q_i(u,v) := \mathrm{sign}(u-v)\int_v^u F_i'(\xi)\,\mathrm{d}\xi,\ \ r_{ij}(u,v) := \mathrm{sign}(u-v)\int_v^u A_{ij}(\xi)\,\mathrm{d}\xi.$$

We thus have

$$\int_{\mathbb{R}^d} |u - v|(x, T)\phi(x, T)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)\phi(x, 0)\,\mathrm{d}x$$
$$+ \iint_{\mathbb{R}^d \times (0,T)} |u - v| \times$$
$$\times \left(\partial_t \phi + \operatorname*{ess\,sup}_{m \leq \xi \leq M} \left\{ F'(\xi) \cdot D\phi(x, t) + \operatorname{tr}\left(A(\xi) D^2 \phi(x, t) \right) \right\} \right) \mathrm{d}x\,\mathrm{d}t.$$

We recognize the backward version of the PDE in (HJB) with $\mathcal{E} = [m, M]$, $b = F'$, and $a = A$.

By approximation, we can take $\phi(x, t) = \psi(x, T - t)$ in the above.

BUT if u_0, v_0 are only bounded, then ψ needs to be integrable.

The Cauchy problems

We consider the following Cauchy problem in $\mathbb{R}^N \times (0, \infty)$:

$$
\text{(HJB)} \qquad \begin{cases} \partial_t \psi = \sup\limits_{\xi \in \mathcal{E}} \left\{ b(\xi) \cdot D\psi + \operatorname{tr}\left(a(\xi) D^2 \psi \right) \right\}, \\ \psi(\cdot, 0) = \psi_0, \end{cases}
$$

where $\psi_0 \in C_b(\mathbb{R}^N) \cap$ "$L^1(\mathbb{R}^N)$" and

$$
\text{(H1)} \qquad \begin{cases} \mathcal{E} \text{ is a nonempty set,} \\ b : \mathcal{E} \to \mathbb{R}^d \text{ bounded function,} \\ a = \sigma^a \left(\sigma^a\right)^T \text{ for some bounded } \sigma^a : \mathcal{E} \to \mathbb{R}^{d \times K}, \end{cases}
$$

with K being a fixed integer.

Background

The problem is often given as

$$\partial_t \psi = H(D\psi, D^2\psi) \quad \text{with} \quad H(p,X) = \sup_{\xi \in \mathcal{E}} \left\{ b(\xi) \cdot p + \operatorname{tr}\left(a(\xi)X\right) \right\}.$$

- It is a fully nonlinear equation in nondivergence form.
- The vector b and the matrix a may degenerate.
- Classical solutions may not exist, and a.e.-solutions may be nonunique.

- The works of Crandall, Lions, Evans, Ishii, Jensen,... suggest that viscosity solutions are indeed the right solution concept: existence, uniqueness and stability in C_b.
- Viscosity solutions are pointwise solutions, and the test function test the equation at local extremal points.

The Cauchy problems

We also consider the following Cauchy problem in $\mathbb{R}^N \times (0, \infty)$:

$$(\mathrm{CDE}) \qquad \begin{cases} \partial_t u + \operatorname{div} F(u) = \operatorname{div}\left(A(u)Du\right), \\ u(\cdot, 0) = u_0, \end{cases}$$

where $u_0 \in L^\infty(\mathbb{R}^N)$ and

$$(\mathrm{H2}) \qquad \begin{cases} F \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d), \\ A = \sigma^A \left(\sigma^A\right)^T \text{ with } \sigma^A \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{d \times K}). \end{cases}$$

Background

The problem was given as

$$\partial_t u + \operatorname{div} F(u) = \operatorname{div}\left(A(u)Du\right).$$

- It is an equation in divergence form.
- The vector F and the matrix A may degenerate, and we get a mixture of hyperbolic and parabolic equations. Moreover, the diffusion is anisotropic.
- Classical solutions may not exist, and distributional solutions may be nonunique.

- The works of Kružkov, Carrillo, Chen, Perthame,... suggest that entropy solutions are indeed the right solution concept: existence, uniqueness and stability in L^1.
- Entropy solutions are "signed" distributional solutions.

Previously known L^∞-stability for (CDE)

When $A(u) \equiv 0$ in (CDE), we have the classical result

$$\int_{\mathbb{R}^N} |u(x,t) - v(x,t)| \mathbf{1}_{B(x_0,R)}(x) \, \mathrm{d}x \le \int_{\mathbb{R}^N} |u_0(x) - v_0(x)| \mathbf{1}_{B(x_0,R+L_F t)}(x) \, \mathrm{d}x.$$

Note that $\mathbf{1}_{B(x_0,R+L_F t)}$ is a "supersolution" of

$$\begin{cases} \partial_t \psi = L_F |D\psi|, \\ \psi(\cdot,0) = \mathbf{1}_{B(x_0,R)}. \end{cases}$$

S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81(123):228–255, 1970.

Finally, finite speed of propagation is encoded in the estimate.

When $A(u) = \varphi'(u)I$ in (CDE), we have

$$\int_{\mathbb{R}^N} |u(x,t) - v(x,t)| \mathbf{1}_{B(x_0,\,R)} \,\mathrm{d}x \leq \int_{\mathbb{R}^N} |u_0(x) - v_0(x)| \Phi(\cdot, L_\varphi t) *_x \mathbf{1}_{B(x_0,\,R+1+L_F t)}(x) \,\mathrm{d}x.$$

Note that $\Phi(\cdot, L_\varphi t) *_x \mathbf{1}_{B(x_0,\,R+1+L_F t)}(x)$ is a "supersolution" of

$$\begin{cases} \partial_t \psi = L_F |D\psi| + L_\varphi (\Delta\psi)^+, \\ \psi(\cdot, 0) = \mathbf{1}_{B(x_0,\,R)}. \end{cases}$$

JE, E. R. Jakobsen. L^1 contraction for bounded (nonintegrable) solutions of degenerate parabolic equations. *SIAM J. Math. Anal.*, 46(6):3957–3982, 2014.

Note the finite infinite speed of propagation.

Previously known L^∞-stability for (CDE)

When $A(u)$ "general" in (CDE), we have

$$\int_{\mathbb{R}^N} |u(x,t)-v(x,t)|\mathrm{e}^{-|x|}\,\mathrm{d}x \\ \leq \mathrm{e}^{(L_F+L_A)t}\int_{\mathbb{R}^N} |u_0(x)-v_0(x)|\mathrm{e}^{-|x|}\,\mathrm{d}x.$$

Note that $\mathrm{e}^{(L_F+L_A)t}\mathrm{e}^{-|x|}$ is a "supersolution" of

$$\begin{cases}\partial_t\psi = L_F|D\psi| + L_A(\sup_{|h|=1} D^2\psi h\cdot h)^+,\\ \psi(\cdot,0)=\mathrm{e}^{-|\cdot|}.\end{cases}$$

G.-Q. Chen, E. DiBenedetto. Stability of entropy solutions to the Cauchy problem for a class of nonlinear hyperbolic-parabolic equations. *SIAM J. Math. Anal.*, 33(4):751–762, 2001.

H. Frid. Decay of Almost Periodic Solutions of Anisotropic Degenerate Parabolic-Hyperbolic Equations. In *Non-linear partial differential equations, mathematical physics, and stochastic analysis*, EMS Ser. Congr. Rep., pages 183–205. Eur. Math. Soc., Zürich, 2018.

L^1-results for (HJB)

A natural question appears:

Can we obtain

$$\|\psi(\cdot,t)-\hat{\psi}(\cdot,t)\|_{L^1}\leq\|\psi_0-\hat{\psi}_0\|_{L^1}$$

where $\psi,\hat{\psi}$ solve (HJB) with initial data $\psi_0,\hat{\psi}_0$?

Not really studied, and only some results for (HJ).

C.-T. Lin, E. Tadmor. L^1-stability and error estimates for approximate Hamilton-Jacobi solutions. *Numer. Math.*, 87(4):701–735, 2001.

What is possible? Initial guess

Consider the eikonal equation

$$\begin{cases} \partial_t \psi = C(|\partial_{x_1}\psi| + |\partial_{x_2}\psi| + \cdots + |\partial_{x_N}\psi|), \\ \psi(\cdot, 0) = \psi_0. \end{cases}$$

Control theory gives the following representation formula:

$$\psi(x,t) = \sup_{x+Ct[-1,1]^N} \psi_0 = \sup_{\overline{Q}_{Ct}(x)} \psi_0.$$

Moreover,

$$\int_{\mathbb{R}^N} \sup_{\overline{Q}_r(x)} \psi(\cdot, t)\,\mathrm{d}x = \int_{\mathbb{R}^N} \sup_{\overline{Q}_{r+Ct}(x)} \psi_0(x)\,\mathrm{d}x \leq \tilde{C}(t) \int_{\mathbb{R}^N} \sup_{\overline{Q}_r(x)} \psi_0\,\mathrm{d}x.$$

The Banach space L^∞_{int}

We consider the normed space

$$L^\infty_{\text{int}}(\mathbb{R}^N) := \{\psi \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \;:\; \|\psi\|_{L^\infty_{\text{int}}(\mathbb{R}^N)} < \infty\}$$

where

$$\|\psi\|_{L^\infty_{\text{int}}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \operatorname*{ess\,sup}_{\overline{Q}_1(x)} |\psi| \,\mathrm{d}x.$$

Theorem

- *L^∞_{int} is a Banach space.*
- *The space L^∞_{int} is continuously embedded into $L^1 \cap L^\infty$.*
- $\int_{\mathbb{R}^N} \operatorname{ess\,sup}_{\overline{Q}_{r+\varepsilon}(x)} |\psi| \,\mathrm{d}x \leq C_{r,\varepsilon} \int_{\mathbb{R}^N} \operatorname{ess\,sup}_{\overline{Q}_r(x)} |\psi| \,\mathrm{d}x.$

The same for second order equations?

Consider

$$\begin{cases} \partial_t \psi = (\partial_{xx}^2 \psi)^+, \\ \psi(\cdot, 0) = \psi_0. \end{cases}$$

For nonnegative solutions, we are able to obtain

$$\psi(\cdot, t) \in L^1 \qquad \Longleftrightarrow \qquad \psi_0 \in L^\infty_{\text{int}}.$$

It seems that L^∞_{int} is a good space for (HJB).

Largest subspace of L^1 stable by the equation (HJB)

Consider a space E such that

$$\begin{cases} E \text{ is a vector subspace of } C_b \cap L^1, \\ E \text{ is a normed space}, \\ E \text{ is continuously embedded into } L^1, \end{cases}$$

and the C_b-semigroup $G(t)$ associated with (HJB) such that

$$G(t) \text{ maps } E \text{ into itself and } G(t) : E \to E \text{ is continuous.}$$

Theorem (Best possible E, [Alibaud & JE & Jakobsen, 2018])

The space $C_b \cap L^\infty_{\text{int}}$ satisfies the above properties. Moreover, any other E satisfying the above properties is continuously embedded into $C_b \cap L^\infty_{\text{int}}$.

L^∞_{int}-stability for (HJB)

Theorem (L^∞_{int}-stability, [Alibaud & JE & Jakobsen, 2018])

Assume (H1). *Then*

$$\|\psi - \hat{\psi}\|_{L^\infty_{\text{int}}} \leq (1 + t|H|_{\text{conv}})^N (1 + \omega_N(t|H|_{\text{diff}}))\|\psi_0 - \hat{\psi}_0\|_{L^\infty_{\text{int}}}.$$

- The modulus of continuity $\omega_N(r)$ will typically be like $\sqrt{r}$.
- The seminorms $|H|_{\text{conv}}, |H|_{\text{diff}}$ measure nonlinearities in (HJB).
- $|||\psi - \hat{\psi}||| \leq \mathrm{e}^{t \max\{|H|_{\text{conv}}, |H|_{\text{diff}}\}} |||\psi_0 - \hat{\psi}_0|||$.

Important ingredient in the proofs, L^1-supersolution

L^∞-stability for (CDE)

Let $\mathcal{E} = [m, M]$, $b = F'$, and $a = A$ in (HJB).

Theorem (L^∞-stability, [Alibaud & JE & Jakobsen, 2018])

Assume (H2), u_0, v_0 *take values in* $[m, M]$, *and* $0 \leq \psi_0 \in BLSC$. *Then*

$$\int_{\mathbb{R}^N} |u(x,t) - v(x,t)| \psi_0(x) \,\mathrm{d}x \leq \int_{\mathbb{R}^N} |u_0(x) - v_0(x)| \psi(x,t) \,\mathrm{d}x.$$

- When we drop C_b, we might have nonunique solutions, and therefore we consider minimal solutions (unique by definition).
- Any other viscosity solution $\hat{\psi}$ of (HJB) will satisfy $\psi \leq \hat{\psi}$. Hence, it includes ALL previous results of this type.
- To make the right-hand side finite, we could require $u_0 - v_0 \in L^1$ or $\psi_0 \in L^\infty_{\text{int}}$.

A duality result

For the respective unique solutions u, ψ of (CDE),(HJB) define

$$S(t) : u_0 \in L^\infty(\mathbb{R}^d) \mapsto u(\cdot, t) \in L^\infty(\mathbb{R}^d) \quad \forall t \geq 0,$$

$$G_{m,M}(t) : \psi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d) \mapsto \psi(\cdot, t) \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d) \quad \forall t \geq 0.$$

Theorem (Semigroup duality [Alibaud & JE & Jakobsen, 2018])

Assume (H2)*, $m < M$, and consider the above semigroups. Then $\{G_{m,M}(t)\}_{t \geq 0}$ is the smallest strongly continuous semigroup on $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ satisfying, for all $t \geq 0$,*

$$\int_{\mathbb{R}^d} |S(t)u_0 - S(t)v_0| \psi_0 \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0| G_{m,M}(t)\psi_0 \, \mathrm{d}x,$$

for every $u_0, v_0 \in L^\infty(\mathbb{R}^d, [m, M])$, every $0 \leq \psi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$.

Given $S(t)$, then the above inequality characterizes $G_{m,M}(t)$.

A duality result, open problem

Theorem (Semigroup duality [Alibaud & JE & Jakobsen, 2018])

Assume (H2), $m < M$, *and consider the above semigroups. Then* $\{G_{m,M}(t)\}_{t\geq 0}$ *is the smallest strongly continuous semigroup on* $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ *satisfying, for all* $t \geq 0$,

$$\int_{\mathbb{R}^d} |S(t)u_0 - S(t)v_0|\psi_0 \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0| G_{m,M}(t)\psi_0 \, \mathrm{d}x,$$

for every $u_0, v_0 \in L^\infty(\mathbb{R}^d, [m, M])$, *every* $0 \leq \psi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$.

The dual question:

Given $G_{m,M}(t)$. Then $S(t)$ is a weak-$\star$ continuous semigroup on L^∞ satisfying the above inequality.

Is $S(t)$ the ONLY such semigroup satisfying such an inequality? If no, which ones do?

Thank you for your attention!