Connections between L^1-solutions of Hamilton-Jacobi-Bellman equations and L^∞-solutions of convection-diffusion equations

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Main results

- "L^1-stability"/Quasicontraction for HJB
- "L^∞-stability"/Weighted L^1-contraction for CDE
- Duality between HJB and CDE

N. Alibaud, JE, E. R. Jakobsen. Optimal and dual stability results for L^1 viscosity and L^∞ entropy solutions. arXiv, 2018.

How are HJB and CDE connected?

$$\text{(HJ)} \qquad \begin{cases} \partial_t U + H(\partial_x U) = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ U(\cdot,0) = U_0 & x \in \mathbb{R} \end{cases}$$

$$\text{(SCL)} \qquad \begin{cases} \partial_t u + \partial_x (H(u)) = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(\cdot,0) = u_0 & x \in \mathbb{R} \end{cases}$$

If u is the entropy solution of (SCL), then $U := \int^x u$ is the viscosity solution of (HJ) with $U_0 := \int^x u_0$. (Can be made rigorous in 1D.)

So, there is a connection, and in particular, information about u will give information about U.

How are HJB and CDE connected?

We will study the following Cauchy problems in $\mathbb{R}^N \times (0,\infty)$:

(HJB) $$\begin{cases} \partial_t \psi = \sup_{\xi \in \mathcal{E}} \left\{ b(\xi) \cdot D\psi + \mathrm{tr}\left(a(\xi) D^2 \psi\right) \right\} \\ \psi(\cdot, 0) = \psi_0 \end{cases}$$

(CDE) $$\begin{cases} \partial_t u + \mathrm{div} F(u) = \mathrm{div}\left(A(u) Du\right) \\ u(\cdot, 0) = u_0 \end{cases}$$

Why? And how are they related?

Note that if $\mathrm{div}\left(A(u)Du\right) = \Delta\varphi(u)$, then we replace $\mathrm{tr}\left(a(\xi)D^2\psi\right)$ by $a(\xi)\Delta\psi$.

How are HJB and CDE connected?

The Kato inequality for (CDE): For all $0 \leq \phi \in C_c^\infty$ and all $T \geq 0$,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |u-v|(x,T)\phi(x,T)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0-v_0|(x)\phi(x,0)\,\mathrm{d}x \\
\text{(KI)} \quad &+ \iint_{\mathbb{R}^d\times(0,T)} \Bigg(|u-v|\partial_t\phi + \sum_{i=1}^d q_i(u,v)\partial_{x_i}\phi \\
&\qquad + \sum_{i,j=1}^d r_{ij}(u,v)\partial^2_{x_ix_j}\phi\Bigg)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

$$
q_i(u,v) := \mathrm{sign}(u-v)\int_v^u F_i'(\xi)\,\mathrm{d}\xi,\ r_{ij}(u,v) := \mathrm{sign}(u-v)\int_v^u A_{ij}(\xi)\,\mathrm{d}\xi.
$$

How are HJB and CDE connected?

For a.e. $x \in \mathbb{R}^N$ and $t \geq 0$:

$$I := \sum_{i=1}^{d} q_i(u,v)\partial_{x_i}\phi + \sum_{i,j=1}^{d} r_{ij}(u,v)\partial^2_{x_i x_j}\phi = q(u,v)\cdot D\phi + \mathrm{tr}\left(r(u,v)D^2\phi\right)$$

$$\begin{aligned} I &= \mathrm{sign}(u(x,t) - v(x,t))\times \\ &\quad \times \int_{v(x,t)}^{u(x,t)} \left\{F'(\xi)\cdot D\phi(x,t) + \mathrm{tr}\left(A(\xi)D^2\phi(x,t)\right)\right\} \mathrm{d}\xi \\ &\leq |u(x,t) - v(x,t)| \operatorname*{ess\,sup}_{m\leq\xi\leq M} \left\{F'(\xi)\cdot D\phi(x,t) + \mathrm{tr}\left(A(\xi)D^2\phi(x,t)\right)\right\}. \end{aligned}$$

For a.e. $x \in \mathbb{R}^N$ and $t \geq 0$:

$$I := \sum_{i=1}^{d} q_i(u,v)\partial_{x_i}\phi + \sum_{i,j=1}^{d} r_{ij}(u,v)\partial^2_{x_i x_j}\phi = q(u,v)\cdot D\phi + \operatorname{tr}\left(r(u,v)D^2\phi\right)$$

$$\begin{aligned} I &= \operatorname{sign}(u(x,t)-v(x,t))\times \\ &\quad \times \int_{v(x,t)}^{u(x,t)} \left\{F'(\xi)\cdot D\phi(x,t) + \operatorname{tr}\left(A(\xi)D^2\phi(x,t)\right)\right\} \mathrm{d}\xi \\ &\leq |u(x,t)-v(x,t)| \operatorname*{ess\,sup}_{m\leq\xi\leq M} \left\{F'(\xi)\cdot D\phi(x,t) + \operatorname{tr}\left(A(\xi)D^2\phi(x,t)\right)\right\}. \end{aligned}$$

How are HJB and CDE connected?

Going back to (KI), we get

$$\int_{\mathbb{R}^d} |u-v|(x,T)\phi(x,T)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0-v_0|(x)\phi(x,0)\,\mathrm{d}x$$
$$+ \iint_{\mathbb{R}^d\times(0,T)} |u-v| \times$$
$$\times \left(\partial_t\phi + \operatorname*{ess\,sup}_{m\leq\xi\leq M}\left\{F'(\xi)\cdot D\phi(x,t) + \mathrm{tr}\left(A(\xi)D^2\phi(x,t)\right)\right\}\right)\mathrm{d}x\,\mathrm{d}t.$$

We recognize the backward version of the PDE in (HJB) with $\mathcal{E}=[m,M]$, $b=F'$, and $a=A$.

By approximation, we can take $\phi(x,t)=\psi(x,T-t)$ in the above.

BUT if u_0,v_0 are only bounded, then ψ needs to be integrable.

The Cauchy problems

We consider the following Cauchy problem in $\mathbb{R}^N \times (0, \infty)$:

$$\text{(HJB)} \qquad \begin{cases} \partial_t \psi = \sup_{\xi \in \mathcal{E}} \left\{ b(\xi) \cdot D\psi + \mathrm{tr}\left(a(\xi) D^2 \psi \right) \right\}, \\ \psi(\cdot, 0) = \psi_0, \end{cases}$$

where $\psi_0 \in C_b(\mathbb{R}^N) \cap$ "$L^1(\mathbb{R}^N)$" and

$$\text{(H1)} \qquad \begin{cases} \mathcal{E} \text{ is a nonempty set,} \\ b : \mathcal{E} \to \mathbb{R}^d \text{ bounded function,} \\ a = \sigma^a (\sigma^a)^T \text{ for some bounded } \sigma^a : \mathcal{E} \to \mathbb{R}^{d \times K}, \end{cases}$$

with K being a fixed integer.

Background

The problem is often given as

$$\partial_t \psi = H(D\psi, D^2\psi) \quad \text{with} \quad H(p,X) = \sup_{\xi \in \mathcal{E}} \left\{ b(\xi) \cdot p + \operatorname{tr}\left(a(\xi)X\right) \right\}.$$

- It is a fully nonlinear equation in nondivergence form.
- The vector b and the matrix a may degenerate.
- Classical solutions may not exist, and a.e.-solutions may be nonunique.

- The works of Crandall, Lions, Evans, Ishii, Jensen,... suggest that viscosity solutions are indeed the right solution concept: existence, uniqueness and stability in C_b.
- Viscosity solutions are pointwise solutions, and the test function test the equation at local extremal points.

The Cauchy problems

We also consider the following Cauchy problem in $\mathbb{R}^N \times (0, \infty)$:

$$\text{(CDE)} \qquad \begin{cases} \partial_t u + \mathrm{div} F(u) = \mathrm{div}\left(A(u) Du\right), \\ u(\cdot, 0) = u_0, \end{cases}$$

where $u_0 \in L^\infty(\mathbb{R}^N)$ and

$$\text{(H2)} \qquad \begin{cases} F \in W^{1,\infty}_{\text{loc}}(\mathbb{R}, \mathbb{R}^d), \\ A = \sigma^A \left(\sigma^A\right)^T \text{ with } \sigma^A \in L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^{d \times K}). \end{cases}$$

Background

The problem was given as

$$\partial_t u + \mathrm{div} F(u) = \mathrm{div}\left(A(u)Du\right).$$

- It is an equation in divergence form.
- The vector F and the matrix A may degenerate, and we get a mixture of hyperbolic and parabolic equations. Moreover, the diffusion is anisotropic.
- Classical solutions may not exist, and distributional solutions may be nonunique.

- The works of Kružkov, Carrillo, Chen, Perthame,... suggest that entropy solutions are indeed the right solution concept: existence, uniqueness and stability in L^1.
- Entropy solutions are "signed" distributional solutions.

Selective summary of previous results

Well-known that (HJB) is stable in C_b.

M. G. Crandall, H. Ishii, P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

M. Bardi, I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1997.

Well-known that (CDE) is stable in L^1.

J. Carrillo. Entropy solutions for nonlinear degenerate problems. *Arch. Ration. Mech. Anal.*, 147(4):269–361, 1999.

G.-Q. Chen, B. Perthame. Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(4):645–668, 2003.

M. Bendahmane, K. H. Karlsen. Renormalized entropy solutions for quasi-linear anisotropic degenerate parabolic equations. *SIAM J. Math. Anal.*, 36(2):405–422, 2004.

Note that L^1-like stability (even well-posedness) for (HJB) is unusual, and that stability of $\|\cdot\|_{L^\infty}$ is note true for (CDE).

Previously known L^∞-stability for (CDE)

When $A(u) \equiv 0$ in (CDE), we have the classical result

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u(x,t) - v(x,t)| \mathbf{1}_{B(x_0,\,R)}(x) \,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} |u_0(x) - v_0(x)| \mathbf{1}_{B(x_0,\,R+L_F t)}(x) \,\mathrm{d}x.
\end{aligned}$$

Note that $\mathbf{1}_{B(x_0,\,R+L_F t)}$ is a "supersolution" of

$$\begin{cases} \partial_t \psi = L_F |D\psi|, \\ \psi(\cdot, 0) = \mathbf{1}_{B(x_0,\,R)}. \end{cases}$$

S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81(123):228–255, 1970.

Finally, finite speed of propagation is encoded in the estimate.

Previously known L^∞-stability for (CDE)

When $A(u) \equiv 1$ in (CDE) and K is the heat kernel, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} |u(x,t) - v(x,t)| \mathbf{1}_{B(x_0,\,R)}\,\mathrm{d}x\\ &\leq \int_{\mathbb{R}^N} |u_0(x) - v_0(x)| K(\cdot,t) *_x \mathbf{1}_{B(x_0,\,R+L_F t)}(x)\,\mathrm{d}x.\end{aligned}$$

Note that $K(\cdot,t) *_x \mathbf{1}_{B(x_0,\,R+L_F t)}(x)$ is a "supersolution" of

$$\begin{cases}\partial_t \psi = L_F |D\psi| + \Delta\psi,\\ \psi(\cdot,0) = \mathbf{1}_{B(x_0,\,R)}.\end{cases}$$

N. Alibaud. Entropy formulation for fractal conservation laws. *J. Evol. Equ.*, 7(1):145–175, 2007.

Finally, note that we have finite infinite speed of propagation.

Previously known L^∞-stability for (CDE)

When $A(u) = \varphi'(u)I$ in (CDE), we have

$$\int_{\mathbb{R}^N} |u(x,t) - v(x,t)| \mathbf{1}_{B(x_0,\,R)} \, \mathrm{d}x$$
$$\leq \int_{\mathbb{R}^N} |u_0(x) - v_0(x)| \Phi(\cdot, L_\varphi t) *_x \mathbf{1}_{B(x_0,\,R+1+L_F t)}(x) \, \mathrm{d}x.$$

Note that $\Phi(\cdot, L_\varphi t) *_x \mathbf{1}_{B(x_0,\,R+1+L_F t)}(x)$ is a "supersolution" of

$$\begin{cases} \partial_t \psi = L_F |D\psi| + L_\varphi (\Delta \psi)^+, \\ \psi(x,0) = \mathbf{1}_{B(x_0,\,R)}. \end{cases}$$

JE, E. R. Jakobsen. L^1 contraction for bounded (nonintegrable) solutions of degenerate parabolic equations. *SIAM J. Math. Anal.*, 46(6):3957–3982, 2014.

Again, we note the finite infinite speed of propagation.

Previously known L^∞-stability for (CDE)

When $A(u)$ "general" in (CDE), we have

$$\int_{\mathbb{R}^N} |u(x,t) - v(x,t)| e^{-|x|} \, \mathrm{d}x \leq e^{(L_F+L_A)t} \int_{\mathbb{R}^N} |u_0(x) - v_0(x)| e^{-|x|} \, \mathrm{d}x.$$

Note that $e^{(L_F+L_A)t} e^{-|x|}$ is a "supersolution" of

$$\begin{cases} \partial_t \psi = L_F |D\psi| + L_A (\sup_{|h|=1} D^2\psi h \cdot h)^+, \\ \psi(x,0) = e^{-|x|}. \end{cases}$$

G.-Q. Chen, E. DiBenedetto. Stability of entropy solutions to the Cauchy problem for a class of nonlinear hyperbolic-parabolic equations. *SIAM J. Math. Anal.*, 33(4):751–762, 2001.

H. Frid. Decay of Almost Periodic Solutions of Anisotropic Degenerate Parabolic-Hyperbolic Equations. In *Non-linear partial differential equations, mathematical physics, and stochastic analysis*, EMS Ser. Congr. Rep., pages 183–205. Eur. Math. Soc., Zürich, 2018.

L^1-results for (HJB)

A natural question appears:

Can we obtain

$$\|\psi(\cdot,t)-\hat{\psi}(\cdot,t)\|_{L^1} \leq \|\psi_0-\hat{\psi}_0\|_{L^1}$$

where $\psi, \hat{\psi}$ solve (HJB) with initial data $\psi_0, \hat{\psi}_0$?

Not really studied, and only some results for (HJ).

C.-T. Lin, E. Tadmor. L^1-stability and error estimates for approximate Hamilton-Jacobi solutions. *Numer. Math.*, 87(4):701–735, 2001.

What is possible? Initial guess

Consider the eikonal equation

$$\begin{cases} \partial_t \psi = C(|\partial_{x_1}\psi| + |\partial_{x_2}\psi| + \cdots + |\partial_{x_N}\psi|), \\ \psi(\cdot, 0) = \psi_0. \end{cases}$$

Control theory gives the following representation formula:

$$\psi(x,t) = \sup_{x+Ct[-1,1]^N} \psi_0 = \sup_{\overline{Q}_{Ct}(x)} \psi_0.$$

Moreover,

$$\int_{\mathbb{R}^N} \sup_{\overline{Q}_r(x)} \psi(\cdot, t) \,\mathrm{d}x = \int_{\mathbb{R}^N} \sup_{\overline{Q}_{r+Ct}(x)} \psi_0(x) \,\mathrm{d}x \leq \tilde{C}(t) \int_{\mathbb{R}^N} \sup_{\overline{Q}_r(x)} \psi_0 \,\mathrm{d}x.$$

The Banach space L^∞_{int}

We consider the normed space

$$L^\infty_{\text{int}}(\mathbb{R}^N) := \{\psi \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \, : \, \|\psi\|_{L^\infty_{\text{int}}(\mathbb{R}^N)} < \infty\}$$

where

$$\|\psi\|_{L^\infty_{\text{int}}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \operatorname*{ess\,sup}_{\overline{Q}_1(x)} |\psi| \, \mathrm{d}x.$$

Theorem

- *L^∞_{int} is a Banach space.*
- *The space L^∞_{int} is continuously embedded into $L^1 \cap L^\infty$.*
- $\int_{\mathbb{R}^N} \operatorname{ess\,sup}_{\overline{Q}_{r+\varepsilon}(x)} |\psi| \, \mathrm{d}x \leq C_{r,\varepsilon} \int_{\mathbb{R}^N} \operatorname{ess\,sup}_{\overline{Q}_r(x)} |\psi| \, \mathrm{d}x.$

The same for second order equations?

Consider

$$\begin{cases} \partial_t \psi = (\partial^2_{xx}\psi)^+, \\ \psi(\cdot, 0) = \psi_0. \end{cases}$$

For nonnegative solutions, we are able to obtain

$$\psi(\cdot, t) \in L^1 \qquad \Longleftrightarrow \qquad \psi_0 \in L^\infty_{\text{int}}.$$

It seems that L^∞_{int} is a good space for (HJB).

Largest subspace of L^1 stable by the equation (HJB)

Consider a space E such that

$$\begin{cases} E \text{ is a vector subspace of } C_b \cap L^1, \\ E \text{ is a normed space}, \\ E \text{ is continuously embedded into } L^1, \end{cases}$$

and the C_b-semigroup $G(t)$ associated with (HJB) such that

$$G(t) \text{ maps } E \text{ into itself and } G(t) : E \to E \text{ is continuous.}$$

Theorem (Best possible E, [Alibaud & JE & Jakobsen, 2018])

The space $C_b \cap L^\infty_{\text{int}}$ satisfies the above properties. Moreover, any other E satisfying the above properties is continuously embedded into $C_b \cap L^\infty_{\text{int}}$.

L^∞_{int}-stability for (HJB)

Theorem (L^∞_{int}-stability, [Alibaud & JE & Jakobsen, 2018])

Assume (H1). *There exists a modulus of continuity* ω_N *such that, for viscosity solutions* $\psi, \hat{\psi}$ *of* (HJB) *with respective initial data* $\psi_0, \hat{\psi}_0 \in C_b \cap L^\infty_{\text{int}}$, *we have*

$$\|\psi - \hat{\psi}\|_{L^\infty_{\text{int}}} \leq (1 + t|H|_{\text{conv}})^N (1 + \omega_N(t|H|_{\text{diff}})) \|\psi_0 - \hat{\psi}_0\|_{L^\infty_{\text{int}}}.$$

- The modulus of continuity $\omega_N(r)$ will typically be like $\sqrt{r}$.
- The seminorms $|H|_{\text{conv}}, |H|_{\text{diff}}$ measure nonlinearities in (HJB).

L^∞_{int}-quasicontraction for (HJB)

As we saw, we only had a quasicontraction when the diffusion was linear and a contraction when the whole equation was linear.

Theorem (L^∞_{int}-quasicontraction, [Alibaud & JE & Jakobsen, 2018])

Assume (H1). *For viscosity solutions* $\psi, \hat{\psi}$ *of* (HJB) *with respective initial data* $\psi_0, \hat{\psi}_0 \in C_b \cap L^\infty_{\text{int}}$, *we have*

$$\left|\left|\left|\psi - \hat{\psi}\right|\right|\right| \leq e^{t \max\{|H|_{\text{conv}}, |H|_{\text{diff}}\}} \left|\left|\left|\psi_0 - \hat{\psi}_0\right|\right|\right|.$$

- $|||\cdot|||$ is equivalent to $\|\cdot\|_{L^\infty_{\text{int}}}$.
- Remarkably, $|||\cdot|||$ does not depend on the semigroup $\psi_0 \overset{t}{\mapsto} \psi$, but rather a fixed semigroup of some model equation.

L^∞-stability for (CDE)

Let $\mathcal{E} = [m, M]$, $b = F'$, and $a = A$ in (HJB).

Theorem (L^∞-stability, [Alibaud & JE & Jakobsen, 2018])

Assume (H2), u_0, v_0 *take values in* $[m, M]$, *and* $0 \leq \psi_0 \in BLSC$. *Then the associated entropy solutions* u, v *of* (CDE) *and viscosity (minimal) solution* ψ *of* (HJB) *satisfy, for all* $t \geq 0$,

$$\int_{\mathbb{R}^N} |u(x,t) - v(x,t)| \psi_0(x) \, \mathrm{d}x \leq \int_{\mathbb{R}^N} |u_0(x) - v_0(x)| \psi(x,t) \, \mathrm{d}x.$$

- Any other viscosity solution $\hat{\psi}$ of (HJB) will satisfy $\psi \leq \hat{\psi}$. Hence, it includes ALL previous results of this type.
- To make the right-hand side finite, we could require $u_0 - v_0 \in L^1$ or $\psi_0 \in L^\infty_{\text{int}}$.
- When we drop C_b, we might have nonunique solutions, and therefore we consider minimal solutions (unique by definition).

Proof

Recall that we already proved this using the Kato inequality (KI):

$$\int_{\mathbb{R}^d} |u - v|(x, T)\phi(x, T)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0|(x)\phi(x, 0)\,\mathrm{d}x$$

$$+ \iint_{\mathbb{R}^d \times (0,T)} |u - v| \times$$

$$\times \left(\partial_t \phi + \operatorname*{ess\,sup}_{m \leq \xi \leq M} \left\{ F'(\xi) \cdot D\phi(x, t) + \operatorname{tr}\left(A(\xi) D^2 \phi(x, t)\right) \right\} \right) \mathrm{d}x\,\mathrm{d}t.$$

By approximation, we can take $\phi(x, t) = \psi(x, T - t)$ in the above.

A duality result

For the respective unique solutions u, ψ of (CDE),(HJB) define

$$S(t) : u_0 \in L^\infty(\mathbb{R}^d) \mapsto u(\cdot,t) \in L^\infty(\mathbb{R}^d) \quad \forall t \geq 0,$$

$$G_{m,M}(t) : \psi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d) \mapsto \psi(\cdot,t) \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d) \quad \forall t \geq 0.$$

Theorem (Semigroup duality [Alibaud & JE & Jakobsen, 2018])

Assume (H2), *$m < M$, and consider the above semigroups. Then $\{G_{m,M}(t)\}_{t\geq 0}$ is the smallest strongly continuous semigroup on $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ satisfying, for all $t \geq 0$,*

$$\int_{\mathbb{R}^d} |S(t)u_0 - S(t)v_0|\psi_0 \,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0| G_{m,M}(t)\psi_0 \,\mathrm{d}x,$$

for every $u_0, v_0 \in L^\infty(\mathbb{R}^d, [m, M])$, every $0 \leq \psi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$.

Given $S(t)$, then the above inequality characterizes $G_{m,M}(t)$.

A duality result, open problem

Theorem (Semigroup duality [Alibaud & JE & Jakobsen, 2018])

Assume (H2), $m < M$, *and consider the above semigroups. Then* $\{G_{m,M}(t)\}_{t\geq 0}$ *is the smallest strongly continuous semigroup on* $C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$ *satisfying, for all* $t \geq 0$,

$$\int_{\mathbb{R}^d} |S(t)u_0 - S(t)v_0|\psi_0 \,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_0 - v_0| G_{m,M}(t)\psi_0 \,\mathrm{d}x,$$

for every $u_0, v_0 \in L^\infty(\mathbb{R}^d, [m, M])$, *every* $0 \leq \psi_0 \in C_b \cap L^\infty_{\text{int}}(\mathbb{R}^d)$.

The dual question:

Given $G_{m,M}(t)$. Then $S(t)$ is a weak-$\star$ continuous semigroup on L^∞ satisfying the above inequality.

Is $S(t)$ the ONLY such semigroup satisfying such an inequality? If no, which ones do?

Important ingredient in the proofs, L^1-supersolution

Note that

$$\sup_{\mathcal{E}}\{b\cdot D\psi+\mathrm{tr}(aD^2\psi)\}\leq C_b|D\psi|+C_a\Big(\sup_{|h|=1}D^2\psi h\cdot h\Big)^+$$

or

$$\partial_t\psi-\sup_{\mathcal{E}}\{b\cdot D\psi+\mathrm{tr}(aD^2\psi)\}\geq \partial_t\psi-C_b|D\psi|-C_a\Big(\sup_{|h|=1}D^2\psi h\cdot h\Big)^+$$

Let us therefore find an integrable supersolution of

$$\partial_t\psi=C_b|D\psi|+C_a\Big(\sup_{|h|=1}D^2\psi h\cdot h\Big)^+.$$

Important ingredient in the proofs, L^1-supersolution

Note that

$$\sup_{\mathcal{E}}\{b\cdot D\psi+\mathrm{tr}(aD^2\psi)\}\leq C_b|D\psi|+C_a\Big(\sup_{|h|=1}D^2\psi h\cdot h\Big)^+$$

or

$$\partial_t\psi-\sup_{\mathcal{E}}\{b\cdot D\psi+\mathrm{tr}(aD^2\psi)\}\geq \partial_t\psi-C_b|D\psi|-C_a\Big(\sup_{|h|=1}D^2\psi h\cdot h\Big)^+$$

Let us therefore find an integrable supersolution of

$$\partial_t\psi=C_a\Big(\sup_{|h|=1}D^2\psi h\cdot h\Big)^+.$$

Recall that for $\mathrm{tr}(aD^2\psi)=a\Delta\psi$, $\sup_{|h|=1}D^2\psi h\cdot h=\Delta\psi$.

Important ingredient in the proofs, L^1-supersolution

Lemma (Fundamental solution? [Alibaud & JE & Jakobsen, 2018])

Consider solutions ψ_n *of*

$$\begin{cases} \partial_t \psi_n = (\partial^2_{xx}\psi_n)^+, \\ \psi_n(\cdot, 0) = n\omega(n\cdot) \approx \delta_0. \end{cases}$$

Then, for all $(x,t) \in \mathbb{R} \times (0,\infty)$,

$$\lim_{n\to\infty} \psi_n(x,t) = \infty.$$

Important ingredient in the proofs, L^1-supersolution

$$\partial_t \psi = (\Delta\psi)^+ = \begin{cases} \Delta\psi, & \Delta\psi > 0 \\ 0, & \Delta\psi \le 0 \end{cases}$$

So, we want to keep the "convex" regions of the heat equation.

We thus search for a profile of the heat equation with second derivative positive.

Important ingredient in the proofs, L^1-supersolution

Define

$$U(r) := c_0 \int_r^\infty \mathrm{e}^{-\frac{s^2}{4}} \, \mathrm{d}s.$$

Note that U satisfies

$$U(0) = 1, \qquad \frac{r}{2}U' + U'' = 0, \qquad U' < 0, \qquad \text{and} \qquad U'' > 0.$$

Moreover, when $N = 1$, $(x,t) \mapsto U(|x|/\sqrt{t})$ solves

$$\partial_t U = \partial_{xx}^2 U = (\partial_{xx}^2 U)^+ \qquad \text{on} \qquad \mathbb{R} \setminus \{0\} \times (0,\infty).$$

Important ingredient in the proofs, L^1-supersolution

Let us now check that

$$(x,t) \mapsto U((|x|-1)^+/\sqrt{t})$$

is a viscosity supersolution of

$$\partial_t \psi = \Big(\sup_{|h|=1} D^2\psi h \cdot h \Big)^+.$$

We note that we have 3 different regions to check:

1. $\{|x| < 1\}$
2. $\{|x| > 1\}$
3. $\{|x| = 1\}$

•**The region** $\{|x| < 1\}$. Here $U((|x|-1)^+/\sqrt{t}) = U(0) = 1$ for all $t > 0$. And it thus satisfies the equation.

•**The region** $\{|x| > 1\}$. Here
$U((|x|-1)^+/\sqrt{t}) = U((|x|-1)/\sqrt{t})$. Let us check that it satisfies the equation pointwise:

$$\partial_t U((|x|-1)/\sqrt{t}) = -\frac{1}{t}\frac{r}{2}U'(r)$$

and

$$\sum_{i,j} \partial^2_{x_i x_j} U((|x|-1)/\sqrt{t}) h_i h_j$$
$$= (N|x|^2|h|^2 - (x\cdot h)^2)\frac{U'}{|x|^3\sqrt{t}} + (x\cdot h)^2\frac{U''}{|x|^2 t}$$

Important ingredient in the proofs, L^1-supersolution

•**The region** $\{|x| < 1\}$. Here $U((|x|-1)^+/\sqrt{t}) = U(0) = 1$ for all $t > 0$. And it thus satisfies the equation.

•**The region** $\{|x| > 1\}$. Here $U((|x|-1)^+/\sqrt{t}) = U((|x|-1)/\sqrt{t})$. Let us check that it satisfies the equation pointwise:

$$\partial_t U((|x|-1)/\sqrt{t}) = -\frac{1}{t}\frac{r}{2}U'(r)$$

and

$$\begin{aligned}
&\sup_{|h|=1}\sum_{i,j}\partial^2_{x_ix_j}U((|x|-1)/\sqrt{t})h_ih_j \\
&= (N|x|^2 - |x|^2)\frac{U'}{|x|^3\sqrt{t}} + |x|^2\frac{U''}{|x|^2 t} \\
&\leq \frac{U''}{t}
\end{aligned}$$

Important ingredient in the proofs, L^1-supersolution

•**The region** $\{|x| < 1\}$. Here $U((|x|-1)^+/\sqrt{t}) = U(0) = 1$ for all $t > 0$. And it thus satisfies the equation.

•**The region** $\{|x| > 1\}$. Here $U((|x|-1)^+/\sqrt{t}) = U((|x|-1)/\sqrt{t})$. Let us check that it satisfies the equation pointwise:

$$\partial_t U(r) - \Big(\sup_{|h|=1} D^2 U(r)h \cdot h\Big)^+ \geq -\frac{1}{t}\Big(\frac{r}{2}U'(r) + U''(r)\Big) = 0.$$

•**The region** $\{|x| = 1\}$. We need to check the subjet. But it is empty.

Important ingredient in the proofs, L^1-supersolution

Let us now check that

$$(x,t) \mapsto U((|x|-1)^+/\sqrt{t}) \qquad \text{with} \qquad U(r) = c_0 \int_r^\infty e^{-\frac{s^2}{4}}\,\mathrm{d}s.$$

is integrable:

$$\begin{aligned}
&\int U((|x|-1)^+/\sqrt{t})\,\mathrm{d}x \\
&= \int_{|x|\leq 1} U(0)\,\mathrm{d}x + \int_{|x|>1} U((|x|-1)/\sqrt{t})\,\mathrm{d}x \\
&\sim 1 + \int_0^\infty r^{d-1} \int_r^\infty e^{-\frac{s^2}{4}}\,\mathrm{d}s\,\mathrm{d}r \\
&\sim 1 + \int_0^\infty s^d e^{-\frac{s^2}{4}}\,\mathrm{d}s < \infty.
\end{aligned}$$

Thank you for your attention!