# $L^1$ Contraction for Bounded (Nonintegrable) Solutions of Degenerate Parabolic Equations

Published 2014

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12 March 2015

In this talk, we consider the following Cauchy problem:

$$(1)\qquad \begin{cases} \partial_t u + \operatorname{div} f(u) - \mathfrak{L}\,\varphi(u) = g(x,t) & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^d, \end{cases}$$

where $u = u(x,t)$ is the solution. The operator $\mathfrak{L}$ will either be the $x$-Laplacian $\Delta$, or a nonlocal operator $\mathcal{L}^\mu$ defined on $C_c^\infty(\mathbb{R}^d)$ as

$$\mathcal{L}^\mu[\phi](x) := \int_{\mathbb{R}^d\setminus\{0\}} \phi(x+z) - \phi(x) - z\cdot D\phi(x)\mathbf{1}_{|z|\leq 1}\,\mathrm{d}\mu(z),$$

where $\mu$ is a nonnegative Radon measure. Note that $(\mathcal{L}^\mu)^* = \mathcal{L}^{\mu^*}$.

# Assumptions

($A_f$) $f = (f_1, f_2, \dots, f_d) \in W^{1,\infty}_{\text{loc}}(\mathbb{R}, \mathbb{R}^d)$;

($A_\varphi$) $\varphi \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$ and $\varphi$ is non-decreasing ($\varphi' \geq 0$);

($A_g$) $g$ is measurable and $\int_0^T \|g(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \,\mathrm{d}t < \infty$;

($A_{u_0}$) $u_0 \in L^\infty(\mathbb{R}^d)$;

($A_\mu$) $\mu \geq 0$ is a Radon measure on $\mathbb{R}^d \setminus \{0\}$, and $\exists\ M \geq 0$

$$\int_{|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{e}^{M|z|} \,\mathrm{d}\mu(z) < \infty;$$

($A_\mu^+$) Assumption ($A_\mu$) holds with $M > 0$.

We will drop the source term $g$ in (most of) what follows.

## Definition (Entropy solution)

Let $\mathfrak{L} = \mathcal{L}^\mu$. A function $u \in L^\infty(Q_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ is an *entropy subsolution* of (1) if

(i) for all nonnegative $\phi \in C_c^\infty(Q_T)$ and all $k \in \mathbb{R}$

$$\begin{aligned}
&\iint_{Q_T} (u-k)^+ \partial_t \phi + \mathrm{sign}(u-k)^+ [f(u) - f(k)] \cdot D\phi \,\mathrm{d}x\,\mathrm{d}t \\
&+ \iint_{Q_T} (\varphi(u) - \varphi(k))^+ \left( \mathcal{L}_r^{\mu^*}[\phi] + b^{\mu^*,r} \cdot D\phi \right) \\
&\qquad\qquad + \mathrm{sign}(u-k)^+ \mathcal{L}^{\mu,r}[\varphi(u)]\phi \,\mathrm{d}x\,\mathrm{d}t \\
&+ \iint_{Q_T} \mathrm{sign}(u-k)^+ g\,\phi \,\mathrm{d}x\,\mathrm{d}t \geq 0;
\end{aligned}$$

(ii) $u(\cdot,0) \leq u_0(\cdot)$ for a.e. $x \in \mathbb{R}^d$.

A similar definition holds for $\mathfrak{L} = \Delta$.

# History

For the equation

$$\begin{cases} \partial_t u + \mathrm{div} f(u) = 0 & \text{in} \quad Q_T \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^d \end{cases},$$

we have the classical result

$$\int_{B(x_0,\, M)} (u(x,t) - v(x,t))^+ \,\mathrm{d}x \le \int_{B(x_0,\, M + L_f t)} (u_0(x) - v_0(x))^+ \,\mathrm{d}x.$$

# History

For the equation

$$\begin{cases} \partial_t u + \operatorname{div} f(u) + (-\Delta)^{\frac{\alpha}{2}} u = 0 & \text{in} \quad Q_T \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^d \end{cases},$$

Alibaud (2007) obtained the inequality

$$\int_{B(x_0,\,M)} (u(x,t) - v(x,t))^+ \,\mathrm{d}x \le \int_{B(x_0,\,M+L_f t)} \left[\tilde{K}(\cdot,t) * (u_0 - v_0)^+\right](x)\,\mathrm{d}x,$$

where $\tilde{K}$ is a fundamental solution satifying

$$\begin{cases} \partial_t \tilde{K} - \mathfrak{L}^* \tilde{K} = 0 & \text{in} \quad \mathbb{R}^d \times (0,\infty) \\ \tilde{K}(x,0) = \delta_0 & \text{on} \quad \mathbb{R}^d \end{cases},$$

that is, $\tilde{K}(x,t) = \mathcal{F}^{-1}(\mathrm{e}^{-t|2\pi\xi|^\alpha})(x)$ for $\alpha \in (0,2]$.

Now, our main result which is an $L^1$ contraction estimate of the form

$$\begin{aligned} &\int_{B(x_0,\,M)} (u(x,t) - v(x,t))^+ \,\mathrm{d}x \\ &\leq \int_{B(x_0, M+1+L_f t)} \left[\Phi(-\cdot, L_\varphi t) * (u_0 - v_0)^+\right](x)\,\mathrm{d}x, \end{aligned} \tag{2}$$

where $\Phi$ is the (non-smooth viscosity) solution of

$$\begin{cases} \partial_t \Phi - (\mathfrak{L}^* \Phi)^+ = 0 & \text{in} \quad \mathbb{R}^d \times (0, \tilde{T}) \\ \Phi(x,0) = \Phi_0(x) & \text{on} \quad \mathbb{R}^d \end{cases}, \tag{3}$$

for some $0 \leq \Phi_0 \in C_c^\infty(\mathbb{R}^d)$.

### Theorem

*Assume* $(\mathrm{A}_f)$, $(\mathrm{A}_\varphi)$ *hold, and* $\Phi$ *is a integrable viscosity solution of* (3)*. Let* $t \in (0,T)$, $M > 0$, $x_0 \in \mathbb{R}^d$, *and* $u$ *and* $v$ *be entropy sub- and supersolutions of* (1) *with initial data* $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ *and measurable source terms* $g, h$ *satisfying*
$\int_0^T \|g(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} + \|h(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \,\mathrm{d}t < \infty.$

(a) *If* $\mathfrak{L} = \mathcal{L}^\mu$ *and* $(\mathrm{A}_\mu^+)$ *holds, then the* $L^1$ *contraction estimate* (2) *holds.*

(b) *If* $\mathfrak{L} = \Delta$, *then the* $L^1$ *contraction estimate* (2) *holds.*

# There and back again

*Step 1:* Following in Kružkov's footsteps, we will use a doubling of variables technique to obtain a "Kato inequality" or "dual equation" for (1).

*Step 2:* By choosing a certain form for our test function and by a density argument, we will start to see the contours of an $L^1$ contraction.

*Step 3:* Step 2 forces us to solve a special equation.

*Step 4:* The solution of this special equation, and several limit arguments, will prove our result.

# Step 1

Kružkov's doubling of variables technique gives for nonnegative $\psi \in C_c^\infty(Q_T)$

$$\iint_{Q_T} \eta(u(x,t), v(x,t))\partial_t \psi(x,t) + q(u(x,t), v(x,t)) \cdot D\psi(x,t) \,\mathrm{d}x \,\mathrm{d}t$$
$$+ \iint_{Q_T} \eta(\varphi(u(x,t)), \varphi(v(x,t)))\mathfrak{L}^* \psi(x,t) \,\mathrm{d}x \,\mathrm{d}t$$
$$+ \iint_{Q_T} \eta(g(x,t), h(x,t))\psi(x,t) \,\mathrm{d}x \,\mathrm{d}t \geq 0,$$

where $\eta(u,v) = (u-v)^+$ and $q(u,v) = \mathrm{sign}(u-v)^+[f(u) - f(v)]$.

# Step 2

Let $\psi(x,t) = \Gamma(x,t)\Theta(t)$.

If $0 < t < T$, $0 \leq \Gamma \in C_c^\infty(Q_T)$, and $0 \leq \Theta \in C_c^\infty((0,T))$, then

$$
\begin{aligned}
0 \leq & \iint_{Q_T} (u-v)^+(x,t)\Gamma(x,t)\Theta'(t) \,\mathrm{d}x\,\mathrm{d}t \\
& + \iint_{Q_T} \Theta(t)(u-v)^+(x,t)\left[\partial_t\Gamma + L_f|D\Gamma| + L_\varphi\big(\mathfrak{L}^*\Gamma(x,t)\big)^+\right]\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

# Step 2

If

$$0 \leq \Gamma \in C([0,T];L^1(\mathbb{R}^d)) \cap L^1((0,T);W^{2,1}(\mathbb{R}^d)) \cap C^\infty(Q_T) \cap L^\infty(Q_T)$$

satisfies

$$\partial_t \Gamma + L_f |D\Gamma| + L_\varphi \big(\mathfrak{L}^* \Gamma(x,t)\big)^+ \leq 0 \quad \text{in} \quad Q_T,$$

then

$$\int_{\mathbb{R}^d} (u-v)^+(x,T)\Gamma(x,T)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} (u_0 - v_0)^+(x)\Gamma(x,0)\,\mathrm{d}x.$$

# Step 3

We note that if $\phi$ solves

$$\partial_t \phi(x,t) + L_f |D\phi(x,t)| \leq 0 \quad \text{in} \quad Q_T,$$

and $\psi$ solves

$$\partial_t \psi(x,t) + L_\varphi (\mathfrak{L}^* \psi(x,t))^+ \leq 0 \quad \text{in} \quad Q_T,$$

then $\Gamma(x,t) = [\psi(\cdot,t) * \phi(\cdot,t)](x)$ solves

$$\partial_t \Gamma + L_f |D\Gamma| + L_\varphi \big(\mathfrak{L}^* \Gamma(x,t)\big)^+ \leq 0 \quad \text{in} \quad Q_T.$$

# Step 4

Classically we have (see e.g. Kružkov (1970)) that

$$\phi_{\delta,\varepsilon}(x,t) := \left[\mathbf{1}_{(-\infty,R]} * \omega_\varepsilon\right]\left(\sqrt{\delta^2 + |x - x_0|^2} + L_f\, t\right)$$

solves

$$\partial_t \phi_{\delta,\varepsilon}(x,t) + L_f |D\phi_{\delta,\varepsilon}(x,t)| \leq 0 \quad \text{in} \quad Q_T.$$

So, the main difficulty is to solve the other equation.

# Step 4

We have already considered the viscosity solution of

$$\begin{cases} \partial_t \Phi - (\mathfrak{L}^* \Phi)^+ = 0 & \text{in} \quad \mathbb{R}^d \times (0, \tilde{T}) \\ \Phi(x, 0) = \Phi_0(x) & \text{on} \quad \mathbb{R}^d \end{cases},$$

and we see the resemblance to

$$\partial_t \psi(x,t) + L_\varphi(\mathfrak{L}^* \psi(x,t))^+ \leq 0 \quad \text{in} \quad Q_T.$$

But we need a smooth, integrable classical solution, and a viscosity solution is neither smooth (it is $C_b$ though) nor integrable (in the general case)!

**Theorem**

*Let $0 \le \Phi_0 \in C_c^\infty(Q_T)$ (and let all of the assumptions hold). Then there exists a unique viscosity solution $\Phi(x,t)$ of*

$$\begin{cases} \partial_t \Phi - (\mathfrak{L}^* \Phi)^+ = 0 & \textit{in} \quad \mathbb{R}^d \times (0, \tilde{T}) \\ \Phi(x,0) = \Phi_0(x) & \textit{on} \quad \mathbb{R}^d \end{cases},$$

*such that*

$$0 \le \Phi \in C_b(Q_{\tilde{T}}) \cap C([0, \tilde{T}]; L^1(\mathbb{R}^d)).$$

# PROOF:

The existence, uniqueness, and comparison principle are shown in previous papers (see e.g. Jakobsen & Karlsen (2005)), and since the initial data is $C_c^\infty$, we have that $\Phi \in C_b$ by previous results. Moreover, $\Phi \geq 0$ by the comparison principle (the initial data is chosen to be nonnegative).

The tricky, and maybe interesting result, is to show that $\Phi \in C([0, \tilde{T}]; L^1(\mathbb{R}^d))$.

We claim that there are $C > 0$, $k > 0$, $K > 0$, such that for all $|\xi| = 1$,

$$\Phi(x,t) \leq w(x,t) := C e^{Kt} e^{k\xi\cdot x} \quad \text{in } Q_{\tilde{T}}.$$

If this is the case, then $\Phi(x,t) \leq C e^{Kt} e^{-k|x|}$ (take $\xi = -\frac{x}{|x|}$ for $x \neq 0$) and $\Phi \in L^\infty(0, \tilde{T}; L^1(\mathbb{R}^d))$.

Moreover, $\Phi \in C([0, \tilde{T}]; L^1(\mathbb{R}^d))$ since by Lebesgue's dominated convergence theorem (the integrand is dominated by $2Ce^{K\tilde{T}}e^{-k|x|}$),

$$\lim_{h\to 0} \int_{\mathbb{R}^d} |\Phi(x,t+h) - \Phi(x,t)| \,\mathrm{d}x = 0 \qquad \text{for all} \qquad t \in [0, \tilde{T}].$$

To complete the proof, it only remains to prove the claim.

Let $\mathfrak{L}^* = \mathcal{L}^{\mu^*}$ and assume that $(\mathrm{A}_\mu^+)$ holds. Note that $\partial_t w = Kw$ and

$$
\begin{aligned}
&\mathcal{L}^{\mu^*}[w(\cdot,t)](x) \\
&= \int_{|z|>0} w(x+z,t) - w(x,t) - z \cdot Dw(x,t)\mathbf{1}_{|z|\leq 1} \,\mathrm{d}\mu^*(z) \\
&= w(x,t)\Bigg[\int_{0<|z|\leq 1} \mathrm{e}^{k\xi\cdot z} - 1 - k\xi \cdot z \,\mathrm{d}\mu^*(z) \\
&\qquad + \int_{|z|>1} \mathrm{e}^{k\xi\cdot z} - 1 \,\mathrm{d}\mu^*(z)\Bigg] \\
&\leq C_k w(x,t),
\end{aligned}
$$

where we take $k \leq M$ (with $M$ defined in $(\mathrm{A}_\mu^+)$) and $C_k > 0$.

It then follows that

$$\partial_t w - (\mathcal{L}^{\mu^*}[w])^+ = Kw + \min\{-\mathcal{L}^{\mu^*}[w], 0\} \geq w(K - C_k).$$

When $\mathfrak{L}^* = \Delta$, the argument is similar. We take any $k > 0$, and then we observe that

$$\partial_t w - (\Delta w)^+ = w(K - k^2).$$

Now, choose $C$ such that $w(\cdot, 0) \geq \Phi_0$ in both cases, and choose $K$ such that $w$ is a supersolution in both cases. Then we have that $w$ is a classical supersolution, and thus, a viscosity supersolution. By comparison the claim is proved. □

# Step 4

Let us continue by noting that $\Phi_\delta := \Phi * \rho_\delta$ (mollified in both space and time) is a smooth classical solution of

$$\partial_t \Phi_\delta(x,t) - (\mathfrak{L}^* \Phi_\delta(x,t))^+ \geq 0.$$

Moreover, $\Phi_\delta$ satisfies

$$0 \leq \Phi_\delta \in C([0,\tilde{T}]; L^1(\mathbb{R}^d)) \cap C^\infty(Q_{\tilde{T}}) \cap L^\infty(Q_{\tilde{T}}),$$

and

$$\|\Phi_\delta(\cdot,0) - \Phi_0\|_{L^\infty(\mathbb{R}^d)} \leq C\delta,$$

where $C$ is some constant independent of $\delta > 0$.

**Theorem**

*Let* $\tilde{T} = \max\{T, L_\varphi T\}$, $0 < \tau < \tilde{T}$ *and* $0 \leq t \leq \tau$, *and let*

$$K_\delta(x,t) := \Phi_\delta(x, L_\varphi(\tau - t)),$$

*where* $L_\varphi$ *is the Lipschitz constant of* $\varphi$. *Then*

$$0 \leq K_\delta \in C([0,\tilde{T}]; L^1(\mathbb{R}^d)) \cap C^\infty(Q_{\tilde{T}}) \cap L^\infty(Q_{\tilde{T}})$$

*solves*

$$\partial_t K_\delta + L_\varphi(\mathfrak{L}^* K_\delta)^+ \leq 0 \quad \textit{in } Q_{\tilde{T}},$$

*and satisfies*

$$\|K_\delta(\cdot,\tau) - \Phi_0\|_{L^\infty(\mathbb{R}^d)} \leq C\delta,$$

*where* $C$ *is a constant independent of* $\delta > 0$.

# Step 4

Note that

$$\Gamma(x,t) := \big[K_\delta(\cdot,t) * \phi_{\tilde{\delta},\varepsilon}(\cdot,t)\big](x) \qquad \text{for} \qquad 0 \leq t \leq \tau,$$

gives

$$0 \leq \Gamma \in C([0,\tau]; L^1(\mathbb{R}^d)) \cap L^1(0,\tau; W^{2,1}(\mathbb{R}^d)) \cap C^\infty(Q_\tau) \cap L^\infty(Q_\tau),$$

and by Step 2

$$\int_{\mathbb{R}^d} (u-v)^+(x,\tau)\,\Gamma(x,\tau)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} (u_0-v_0)^+(x)\,\Gamma(x,0)\,\mathrm{d}x$$

or

$$\begin{aligned}&\int_{\mathbb{R}^d} (u-v)^+(x,\tau)\,\big[K_\delta(\cdot,\tau) * \phi_{\tilde{\delta},\varepsilon}(\cdot,\tau)\big](x)\,\mathrm{d}x\\ &\leq \int_{\mathbb{R}^d} (u_0-v_0)^+(x)\,\big[K_\delta(\cdot,0) * \phi_{\tilde{\delta},\varepsilon}(\cdot,0)\big](x)\,\mathrm{d}x\end{aligned}$$

# Step 4

Sending $\delta, \tilde{\delta}, \varepsilon \to 0^+$ gives (after numerous Fatou's lemmas and Lebegue's dominated convergence theorems)

$$\int_{B(x_0,M)} (u(x,t) - v(x,t))^+ \,\mathrm{d}x$$
$$\leq \int_{B(x_0,M+1+L_f t)} \left[\Phi(-\cdot, L_\varphi \tau) * (u_0 - v_0)^+\right](x) \,\mathrm{d}x$$

But why do we have $+1$ in the radius of the ball?

# Step 4

After sending $\tilde{\delta}, \delta \to 0^+$, we have

$$\int_{\mathbb{R}^d} (u - v)^+(x, \tau) \left[\Phi_0 * \phi_\varepsilon(\cdot, \tau)\right](x) \, \mathrm{d}x$$
$$\leq \liminf_{\delta \to 0^+} \int_{\mathbb{R}^d} \phi_\varepsilon(x, 0) \left[K_\delta(-\cdot, 0) * (u_0 - v_0)^+\right](x) \, \mathrm{d}x.$$

Now, let $C_c^\infty(\mathbb{R}^d) \ni \Phi_0(\cdot) := \hat{\omega}_{\tilde{\varepsilon}}(\cdot - x_0)$, which is a mollifier in $\mathbb{R}^d$ centered about $x_0$. Note that $[\Phi_0 * \phi_\varepsilon(\cdot, \tau)] \geq 0$ and that $[\Phi_0 * \phi_\varepsilon(\cdot, \tau)]\,(x) = 1$ when $|x - x_0| < R - L_f \tau - \varepsilon - \tilde{\varepsilon}$. Hence, if $\varepsilon + \tilde{\varepsilon} < 1$, then

$$[\Phi_0 * \phi_\varepsilon(\cdot, \tau)]\,(x) \geq \mathbf{1}_{|x - x_0| \leq R - L_f \tau - 1}.$$

# Step 4

So, we get

$$\int_{\mathbb{R}^d} \mathbf{1}_{|x-x_0|\leq R-L_f\tau-1}(u-v)^+(x,\tau)\,\mathrm{d}x$$
$$\leq \int_{\mathbb{R}^d}(u-v)^+(x,\tau)\left[\Phi_0 * \gamma(\cdot,\tau)\right](x)\,\mathrm{d}x.$$

Observe that we cannot send $\tilde{\varepsilon} \to 0^+$ here because this will violate the inequality $w(x,0) \geq \Phi_0$ in the proof that we did earlier, and we would lose the $L^1$ bound on $K_\delta$. Thus, we need to pay the price of $+1$ in the radius of the ball.

## Consequences

Assume ($A_f$) and ($A_\varphi$) hold, ($A_\mu^+$) holds when $\mathfrak{L} = \mathcal{L}^\mu$, and $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. Let $M > 0$, $x_0 \in \mathbb{R}^d$ and $L_f$ and $L_\varphi$ be the Lipschitz constants of $f$ and $\varphi$ respectively.

(a) ($L^1$ contraction). Let $u$ and $v$ be entropy solutions of (1) with initial data $u_0, v_0$ respectively. Then for all $t \in (0, T)$,

$$\begin{aligned} &\|u(\cdot,t) - v(\cdot,t)\|_{L^1(B(x_0,M))} \\ &\leq \|\Phi(-\cdot, L_\varphi t) * |u_0 - v_0|\|_{L^1(B(x_0,M+1+L_f t))} \end{aligned}$$

(b) ($L^1$ bound). Let $u$ be an entropy solution of (1). Then for all $t \in (0, T)$,

$$\|u(\cdot,t)\|_{L^1(B(x_0,M))} \leq \|\Phi(-\cdot, L_\varphi t) * |u_0|\|_{L^1(B(x_0,M+1+L_f t))}$$

# Consequences continued

(c) (Comparison principle). Let $u$ and $v$ be entropy sub- and supersolutions of (1) with initial data $u_0, v_0$ respectively. If $u_0 \leq v_0$ a.e. on $\mathbb{R}^d$, then

$$u(x,t) \leq v(x,t) \qquad \text{a.e. in } Q_T.$$

(d) ($L^\infty$ bound). Let $u$ be an entropy solution of (1), and let $\overline{\psi} := \sup_{x\in\mathbb{R}^d} \psi$ and $\underline{\psi} := \inf_{x\in\mathbb{R}^d} \psi$. Then

$$\underline{u_0}(x) + \int_0^t \underline{g}(x,s)\,\mathrm{d}s \leq u(x,t) \leq \overline{u_0}(x) + \int_0^t \overline{g}(x,s)\,\mathrm{d}s$$

a.e. in $Q_T$.

(e) ($BV$ bound). Let $u$ be an entropy solution of (1) and assume $u_0 \in BV(\mathbb{R}^d)$. Then for all $t \in (0,T)$, $x_0 \in \mathbb{R}^d$, and $M > 0$,

$$\begin{aligned} &|u(\cdot,t)|_{BV(B(x_0,M))} \\ &\leq \sup_{h\neq 0} \frac{\|\Phi(-\cdot, L_\varphi t) * |u_0(\cdot+h) - u_0|\|_{L^1(B(x_0,M+1+L_f t))}}{|h|} \end{aligned}$$

Thank you for your attention!