

Nonlocal nonlinear diffusion equations. Smoothing effects, Green functions, and functional inequalities

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$$(GPME) \quad \begin{cases} \partial_t u + (-\mathfrak{L})[u^m] = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

Think of \mathfrak{L} as Δ or $-(\Delta)^{\frac{\alpha}{2}}$, but it is also more general.

- L^1 - L^∞ -smoothing.
- Their relation with functional inequalities like Gagliardo-Nirenberg-Sobolev:

$$\|f\|_{L^{2^*}(\mathbb{R}^N)} \lesssim Q_{-\mathfrak{L}}[f, f]^{\frac{1}{2}},$$

where

$$2^* > 2 \quad \text{and} \quad Q_{-\mathfrak{L}}[f, g] = \int f(-\mathfrak{L})[g].$$

$\mathfrak{L} = \Delta$ implies that $Q_{-\mathfrak{L}}[f, f] = \|\nabla f\|_{L^2(\mathbb{R}^N)}^2$.



JE, M. BONFORTE. Nonlocal nonlinear diffusion equations. Smoothing effects, Green functions, and functional inequalities. *J. Funct. Anal.*, 284(6):109831, 2023.

Linear case ($m = 1$). Scaling

Consider

$$(HE) \quad \begin{cases} \partial_t u + (-\Delta)[u] = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

Assume that we are searching for an estimate of the form

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\sigma_1} \|u_0\|_{L^1(\mathbb{R}^N)}^{\sigma_2}.$$

What are the only admissible values of σ_1, σ_2 ?

If u solves (HE), then

$$\tilde{u}(x, t) := \kappa u(\Xi x, \Lambda t) \quad \text{for } \kappa, \Xi, \Lambda > 0$$

also solves (HE) with initial data $\tilde{u}_0(x) := \kappa u(\Xi x, 0)$ as long as $\Xi^2 = \Lambda$. Fix \tilde{u} with mass $M = 1$, then $\kappa = M^{-1} \Xi^N$, and $\tilde{u}_0 \in L^1$.

Linear case ($m = 1$). Scaling

Consider

$$(HE) \quad \begin{cases} \partial_t u + (-\Delta)[u] = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

I.e.,

$$\|\tilde{u}(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\sigma_1} \|\tilde{u}_0\|_{L^1(\mathbb{R}^N)}^{\sigma_2}$$

$$\begin{aligned} \iff \|u(\Xi \cdot, \Lambda t)\|_{L^\infty(\mathbb{R}^N)} &\lesssim \kappa^{-1} \Lambda^{\sigma_1} (\Lambda t)^{-\sigma_1} \kappa^{\sigma_2} \|u(\Xi \cdot, 0)\|_{L^1(\mathbb{R}^N)}^{\sigma_2} \\ &= \kappa^{\sigma_2-1} \Lambda^{\sigma_1} (\Lambda t)^{-\sigma_1} \Xi^{-N\sigma_2} \|u_0\|_{L^1(\mathbb{R}^N)}^{\sigma_2} \\ &= \kappa^{\sigma_2-1} \Xi^{2\sigma_1-N\sigma_2} (\Lambda t)^{-\sigma_1} \|u_0\|_{L^1(\mathbb{R}^N)}^{\sigma_2}, \end{aligned}$$

with $\sigma_1 = (N/2)\sigma_2$ and $\sigma_2 = 1$.

Hence,

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\frac{N}{2}} \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Linear case ($m = 1$). Scaling

Consider

$$(HE) \quad \begin{cases} \partial_t u + (-\Delta)[u] = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

i.e.,

$$\begin{aligned} \|\tilde{u}(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} &\lesssim t^{-\sigma_1} \|\tilde{u}_0\|_{L^1(\mathbb{R}^N)}^{\sigma_2} \\ \iff \|u(\Xi \cdot, \Lambda t)\|_{L^\infty(\mathbb{R}^N)} &\lesssim \kappa^{-1} \Lambda^{\sigma_1} (\Lambda t)^{-\sigma_1} \kappa^{\sigma_2} \|u(\Xi \cdot, 0)\|_{L^1(\mathbb{R}^N)}^{\sigma_2} \\ &= \kappa^{\sigma_2-1} \Lambda^{\sigma_1} (\Lambda t)^{-\sigma_1} \Xi^{-N\sigma_2} \|u_0\|_{L^1(\mathbb{R}^N)}^{\sigma_2} \\ &= \kappa^{\sigma_2-1} \Xi^{2\sigma_1-N\sigma_2} (\Lambda t)^{-\sigma_1} \|u_0\|_{L^1(\mathbb{R}^N)}^{\sigma_2}, \end{aligned}$$

with $\sigma_1 = (N/2)\sigma_2$ and $\sigma_2 = 1$.



J. L. VÁZQUEZ. *Smoothing and decay estimates for nonlinear diffusion equations*. Oxford Lecture Series in Mathematics and its Applications, volume 33. Oxford University Press, Oxford, 2006.

Linear case ($m = 1$). Heat kernel

The function

$$u(x, t) = \int_{\mathbb{R}^N} u_0(y) H_{-\Delta}(x - y, t) dy,$$

with

$$H_{-\Delta}(x - y, t) \approx t^{-\frac{N}{2}} \exp\left(-\frac{|x - y|^2}{4t}\right),$$

is the solution of

$$(HE) \quad \begin{cases} \partial_t u + (-\Delta)[u] = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

Since $0 \leq H_{-\Delta}(x - y, t) \lesssim t^{-\frac{N}{2}}$, we immediately have

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\frac{N}{2}} \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Linear case ($m = 1$). Nash inequality

Assume (by L^p -interpo. and the Gagliardo-Nirenberg-Sobolev ineq.)

$$\|f\|_{L^2(\mathbb{R}^N)} \lesssim \|f\|_{L^1(\mathbb{R}^N)}^\vartheta \|\nabla f\|_{L^2(\mathbb{R}^N)}^{1-\vartheta}$$

with

$$\vartheta = \frac{1}{2} \frac{2^* - 2}{2^* - 1} \quad \text{where } 2^* = 2N/(N - 2).$$

Define $Y(t) := \|u(t)\|_{L^2(\mathbb{R}^N)}^2$, and consider

$$\begin{aligned} Y'(t) &= \int \partial_t(u^2) = 2 \int u \partial_t u = -2 \int u(-\Delta)[u] = -2 \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 \\ &\lesssim -Y(t)^{\frac{1}{1-\vartheta}} \|u(t)\|_{L^1(\mathbb{R}^N)}^{-\frac{2\vartheta}{1-\vartheta}} \leq -\|u_0\|_{L^1(\mathbb{R}^N)}^{-2\frac{\vartheta}{1-\vartheta}} Y(t)^{1+\frac{\vartheta}{1-\vartheta}}. \end{aligned}$$

Solving the differential inequality gives

$$\|u(t)\|_{L^2(\mathbb{R}^N)}^2 = Y(t) \lesssim t^{-\frac{1-\vartheta}{\vartheta}} \|u_0\|_{L^1(\mathbb{R}^N)}^2 = t^{-\frac{N}{2}} \|u_0\|_{L^1(\mathbb{R}^N)}^2.$$

Linear case ($m = 1$). Nash inequality

Assume

$$\|f\|_{L^2(\mathbb{R}^N)} \lesssim \|f\|_{L^1(\mathbb{R}^N)}^\vartheta \|\nabla f\|_{L^2(\mathbb{R}^N)}^{1-\vartheta}.$$

We have

$$\|u(t)\|_{L^2(\mathbb{R}^N)} \lesssim t^{-\frac{N}{4}} \|u_0\|_{L^1(\mathbb{R}^N)},$$

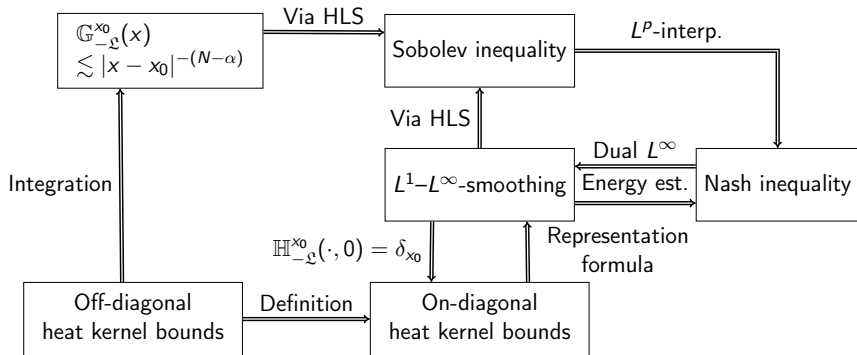
and then, by duality,

$$\begin{aligned} \|u(t)\|_{L^\infty} &= \sup_{\|\phi\|_{L^1}=1} \left| \int u(t)\phi \right| = \sup_{\|\phi\|_{L^1}=1} \left| \int S_t[u_0]\phi \right| \\ &= \sup_{\|\phi\|_{L^1}=1} \left| \int S_{\frac{t}{2}}[S_{\frac{t}{2}}[u_0]]\phi \right| = \sup_{\|\phi\|_{L^1}=1} \left| \int S_{\frac{t}{2}}[u_0]S_{\frac{t}{2}}[\phi] \right| \\ &\leq \sup_{\|\phi\|_{L^1}=1} \|S_{\frac{t}{2}}[u_0]\|_{L^2} \|S_{\frac{t}{2}}[\phi]\|_{L^2} \lesssim t^{-\frac{N}{4}} \|S_{\frac{t}{2}}[u_0]\|_{L^2} \\ &\lesssim t^{-\frac{N}{2}} \|u_0\|_{L^1}. \end{aligned}$$



E. H. LIEB AND M. LOSS. *Analysis. Graduate Studies in Mathematics, volume 14.* American Mathematical Society, Providence, RI, 2001.

Linear case ($m = 1$). Overview



$$\text{(GPME)} \quad \begin{cases} \partial_t u + (-\mathcal{L})[u^m] = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

Cons:

- We do not have a representation formula.
- It is harder to find the correct functional set-up.

Pros:

- We still have scaling (always time-scaling).
- Some estimates are true in the nonlinear, but not true in the linear.

Nonlinear case ($m > 1$). A nice trick

Consider the operator $-\mathfrak{L} \mapsto I - \mathfrak{L}$, i.e.,

$$\partial_t u + (I - \mathfrak{L})[u^m] = 0 \quad \iff \quad \partial_t u + (-\mathfrak{L})[u^m] = -u^m.$$

Then $t \mapsto Y(t)$ solves $Y'(t) = -Y(t)^{1+(m-1)}$, so

$$Y(t) \leq \left(\frac{1}{(m-1)t} \right)^{\frac{1}{m-1}}.$$

Moreover, comparison yields (with $Y(0) = \infty$)

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq Y(t) \leq \left(\frac{1}{(m-1)t} \right)^{\frac{1}{m-1}}.$$

Holds independently of the operator! But needs “good” nonlinearity.

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L. VÉRON. Effets régularisants de semi-groupes non linéaires dans des espaces de Banach. *Ann. Fac. Sci. Toulouse Math.* (5), 1(2):171–200, 1979.

Nonlinear case ($m > 1$). A neat trick

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L. VÉRON. Effets régularisants de semi-groupes non linéaires dans des espaces de Banach. *Ann. Fac. Sci. Toulouse Math.* (5), 1(2):171–200, 1979.

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We need:

- $(-\mathfrak{L})^{-1}$ with kernel $\mathbb{G}_{-\mathfrak{L}}(x-y) = \int_0^\infty H_{-\mathfrak{L}}(x-y, t) dt$.
- Time scaling. $u_\Lambda(x, t) := \Lambda^{\frac{1}{m-1}} u(x, \Lambda t)$ solution when u is.
- Comparison principle.
- L^p -bounds.

$$(GPME) \quad \begin{cases} \partial_t u + (-\mathcal{L})[u^m] = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

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Nonlinear case ($m > 1$). Time-monotonicity

- Time scaling. $u_\Lambda(x, t) := \Lambda^{\frac{1}{m-1}} u(x, \Lambda t)$ solution when u is.
- Comparison principle.

Provides the well-known

$$\partial_t u \geq -\frac{u}{(m-1)t}$$



D. G. ARONSON AND P. BÉNILAN. Régularité des solutions de l'équation des milieux poreux dans \mathbb{R}^N . *C. R. Acad. Sci. Paris Sér. A-B*, 288(2):A103–A105, 1979.



P. BÉNILAN AND M. G. CRANDALL. Regularizing effects of homogeneous evolution equations. In *Contributions to analysis and geometry (Baltimore, Md., 1980)*, pages 23–39. Johns Hopkins Univ. Press, Baltimore, Md., 1981.



M. CRANDALL AND M. PIERRE. Regularizing effects for $u_t + A\varphi(u) = 0$ in L^1 . *J. Funct. Anal.*, 45(2):194–212, 1982.

Nonlinear case ($m > 1$). “Representation formula”

$$\begin{aligned}\partial_t u + (-\mathcal{L})[u^m] = 0 & \iff u^m = -(-\mathcal{L})^{-1}[\partial_t u] \\ & \iff u^m = -\partial_t u *_x \mathbb{G}_{-\mathcal{L}} \\ & \iff u^m \leq \frac{u}{(m-1)t} *_x \mathbb{G}_{-\mathcal{L}}.\end{aligned}$$

Hence,

$$(u(x, t))^m \leq \frac{1}{(m-1)t} \int_{\mathbb{R}^N} u(y, t) \mathbb{G}_{-\mathcal{L}}(x-y) dy.$$

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Hence,

$$\begin{aligned}(u(x, t))^m & \leq \frac{1}{(m-1)t} \int_{\mathbb{R}^N} u(y, t) \underbrace{\mathbb{G}_{-\mathcal{L}}(x-y)}_{= \int_0^\infty H_{-\mathcal{L}}(x-y, t) dt} dy.\end{aligned}$$



M. BONFORTE AND J. L. VÁZQUEZ. A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains. *Arch. Ration. Mech. Anal.*, 218(1):317–362, 2015.

Nonlinear case ($m > 1$). Examples

$$\partial_t u + (-\mathcal{L})[u^m] = 0$$

- $-\mathcal{L} = (-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0, 2]$ gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-N\theta_\alpha} \|u_0\|_{L^1(\mathbb{R}^N)}^{\alpha\theta_\alpha}, \quad \text{where } \theta_\alpha := (\alpha + N(m-1))^{-1}$$

Note that $H_{-\mathcal{L}}(x-y, t) \lesssim t^{-N/\alpha}$.

- $-\mathcal{L} = (\kappa^2 I - \Delta)^{\frac{\alpha}{2}} - \kappa^\alpha I$ with $\kappa > 0$ and $\alpha \in (0, 2)$ gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-N\theta_\alpha} \|u_0\|_{L^1(\mathbb{R}^N)}^{\alpha\theta_\alpha} + t^{-N\theta_2} \|u_0\|_{L^1(\mathbb{R}^N)}^{2\theta_2}.$$

Note that $H_{-\mathcal{L}}(x-y, t) \lesssim t^{-N/\alpha} + t^{-N/2}$.

- $-\mathcal{L} = \sum_{i=1}^N (-\partial_{x_i x_i}^2)^{\frac{\alpha}{2}}$ with $\alpha \in (0, 2)$ gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-1/(m-1)} + \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Strange estimate?

Nonlinear case ($m > 1$). Examples

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Strange estimate? Since $-\mathfrak{L}$ is α -homogeneous, we can use scaling:

Nonlinear case ($m > 1$). Examples

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Nonlinear does *not* imply linear

Consider

$$\partial_t u + \partial_x [u^m] = 0.$$

- If $m = 1$, then this is the transport equation and solutions are as smooth as their initial data.
- If $1 < m \leq 2$, then the Oleinik estimate holds from which we deduce

$$\|u(t)\|_{L^\infty(\mathbb{R})} \lesssim t^{-1/m} \|u_0\|_{L^1(\mathbb{R})}^{1/m}.$$



D. SERRE AND L. SILVESTRE. Multi-dimensional Burgers equation with unbounded initial data: well-posedness and dispersive estimates. *Arch. Ration. Mech. Anal.*, 234(3):1391–1411, 2019.

Nonlinear does *not* imply linear

Consider

$$\partial_t u + (-\mathfrak{L})[u^m] = 0,$$

with

$$-\mathfrak{L}[\psi](x) = \psi(x) - \int_{\mathbb{R}^N} \psi(z) J(x-z) dz = (I - J *_{\times})[\psi](x)$$

where $J \geq 0$ such that $\|J\|_{L^1(\mathbb{R}^N)} = 1$ and $J \in L^p(\mathbb{R}^N)$.

- If $m = 1$, then

$$u(x, t) = u_0(x)e^{-t} + W(x, t),$$

where $W \geq 0$ is some smooth function. Hence, no smoothing.



F. ANDREU-VALLO, J. M. MAZÓN, J. D. ROSSI, J. TOLEDO-MELERO. *Nonlocal diffusion problems*. Mathematical Surveys and Monographs, volume 165. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

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- If $m = 1$, then

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- If $m > 1$, then

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-1/(m-1)} + \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Thank you for your attention!



Nonlinear case ($m > 1$). Why not the Moser iteration?

Idea: $\lim_{p \rightarrow \infty} \|f\|_{L^p} = \|f\|_{L^\infty}$.

The Stroock-Varopoulos inequality gives

$$\begin{aligned} \frac{d}{dt} \|u\|_{L^p} &= \int \partial_t(u^p) = p \int u^{p-1} \partial_t u = -p \int u^{p-1} (-\mathcal{L})[u^m] \\ &= -p Q_{-\mathcal{L}}[u^{p-1}, u^m] \leq -\frac{4mp(p-1)}{(p+m-1)^2} Q_{-\mathcal{L}}[u^{\frac{p+m-1}{2}}]. \end{aligned}$$

L^p -interpo. and the Gagliardo-Nirenberg-Sobolev inequality gives

$$\|f\|_{L^{\tilde{p}}(\mathbb{R}^N)} \leq C \|f\|_{L^{\tilde{q}}(\mathbb{R}^N)}^\vartheta Q_{-\mathcal{L}}[f]^{\frac{1}{2}(1-\vartheta)},$$

where

$$2 \leq \tilde{p} < 2^*, \quad 1 \leq \tilde{q} < \tilde{p}, \quad \vartheta := \frac{\tilde{q} 2^* - \tilde{p}}{\tilde{p} 2^* - \tilde{q}}.$$

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