Numerical solutions of nonlocal (and local) equations of porous medium type

Jørgen Endal
URL: http://folk.ntnu.no/jorgeen

Department of mathematical sciences
NTNU

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F. del Teso and E. R. Jakobsen

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NTNU Workshop on PDEs

Local and nonlocal diffusion

Diffusion is the act of "spreading out" – the movement from areas of high concentration to areas of low concentration.

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$$u(x,t+\tau) = \frac{1}{2}u(x+h,t) + \frac{1}{2}u(x-h,t).$$

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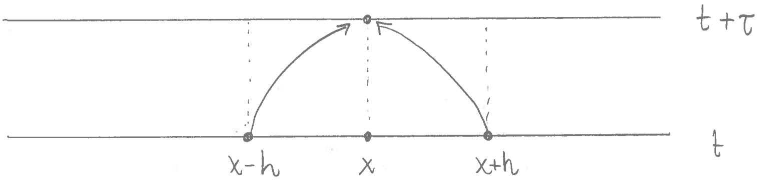

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Choose (the scaling) $\tau = \frac{1}{2}h^2$ and divide by it to obtain

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \frac{u(x+h,t)+u(x-h,t)-2u(x,t)}{h^2}.$$

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$$\partial_t u = \Delta u \qquad \text{in} \qquad \mathbb{R} \times (0,T),$$

that is, u is a solution of the heat equation.

A. Einstein. Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Annalen der Physik* (in German), 322(8): 549–560, 1905.

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We choose a density $K : \mathbb{R} \to [0, \infty)$ up to normalization factors as

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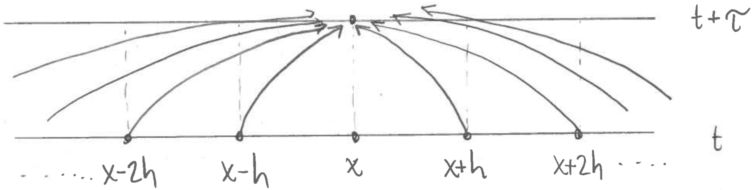

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Then, for the choice (of scaling) $\tau = h^\alpha$,

$$\frac{u(x, t + \tau) - u(x, t)}{\tau} = \sum_{k \in \mathbb{Z} \setminus \{0\}} \big(u(x + hk, t) - u(x, t)\big) K(hk) h.$$

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As $\tau, h \to 0^+$,

$$\begin{aligned}\partial_t u &= \text{P.V.} \int_{|z|>0} \big(u(x+z,t) - u(x,t)\big) \frac{c_{1,\alpha}}{|z|^{1+\alpha}} \,\mathrm{d}z \\ &= -(-\Delta)^{\frac{\alpha}{2}} u \qquad \text{in} \qquad \mathbb{R} \times (0,T)\end{aligned}$$

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where $c_{1,\alpha} > 0$ and $-(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ is the fractional Laplacian. We thus observe that u is a solution of the fractional heat equation.

E. Valdinoci. From the long jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl. SeMA*, (49):33–44, 2009.

Diffusion operators

- $\mathcal{L}^\sigma[\psi] := \text{tr}(\sigma\sigma^T D^2\psi) = \sum_{i=1}^N (\sigma_i \cdot D)^2$ is a possibly degenerate self-adjoint second-order local operator.

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- For general symmetric measures μ,

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which includes the well-known fractional Laplacian by choosing $\mathrm{d}\mu(z) = \frac{c_{N,\alpha}}{|z|^{N+\alpha}}\, \mathrm{d}z$ for some $c_{N,\alpha} > 0$.

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    - Fourier multipliers $\mathcal{F}(\mathcal{L}^\mu[\psi]) = -s_{\mathcal{L}^\mu} \mathcal{F}(\psi)$.

Courrège and the global comparison property

Consider a **linear**, **self-adjoint** Lipschitz map $\mathcal{L} : C_b^2(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ with the property that

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P. Courrège. Sur la forme intégro-différentielle des opérateurs de C_k^∞ dans C satisfaisant au principe du maximum. *Séminaire Brelot-Choquet-Deny. Théorie du Potentiel*, 10(1):1–38, 1965–1966.

Generalized porous medium equations

Let $Q_T := \mathbb{R}^N \times (0, T)$. We consider the following Cauchy problem:

$$\text{(GPME)} \qquad \begin{cases} \partial_t u = \mathcal{L}[\varphi(u)] & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

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- $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and
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Main results:

- Uniqueness for $u_0 \in L^\infty$ with $u - u_0 \in L^1$.
- Convergent numerical schemes in $C([0,T]; L^1(\mathbb{R}^N))$ for $u_0 \in L^1 \cap L^\infty$.

Assumptions

The assumption

$$(\mathrm{A}_\varphi) \qquad \varphi : \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing,}$$

includes nonlinearities of the following kind

- the porous medium $\varphi(u) = u^m$ with $m > 1$,
- fast diffusion $\varphi(u) = u^m$ with $0 < m < 1$, and
- (one-phase) Stefan problem $\varphi(u) = \max\{0, u - c\}$ with $c > 0$.

Local case: $\partial_t u = \Delta u$, $\partial_t u = \Delta u^m$, $\partial_t u = \Delta\varphi(u)$.

- Well-posedness:

J. L. Vázquez. *The porous medium equation. Mathematical theory*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

Selective summary of previous results

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• Numerical results:

Risebro, Karlsen, Bürger, DiBendedetto, Droniou, Eymard, Gallouet, Ebmeyer,. . .

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• Well-posedness when $\mathcal{L}^\mu = -(-\Delta)^{\frac{\alpha}{2}}$:

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• Well-posedness for other $\mathcal{L}^\mu$:

Nonsingular operators

F. Andreu-Vaillo, J. Mazón, J. D. Rossi, and J. J. Toledo-Melero. *Nonlocal diffusion problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

Fractional Laplace like operators (with some x-dependence)

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- Well-posedness for related $\mathcal{L}^\mu$:

G. Karch, M. Kassmann, and M. Krupski. A framework for non-local, non-linear initial value problems. arXiv, 2018.

Nonlocal case: $\partial_t u = \mathcal{L}^{\mu}[\varphi(u)]$.

• Numerical results:
Huang, Oberman, Droniou, Nochetto, Otárola, Salgado, Cifani, Karlsen, Jakobsen, del Teso, La Chioma, Debrabant, Camili, Biswas, . . .

Selective summary of previous results

Previous results (mostly) rely on:

- The porous medium nonlinearity $\varphi(u) = u^m$ with $m > 1$.
- A very restrictive class of Lévy operators.
- The use of L^1-energy solutions.

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- The use of L^1-energy solutions.

In our case:

- Uniqueness is hard to prove because of a very weak solution concept (however, existence is then easier).
- We can handle very weak assumptions on φ and $\mathcal{L}^\mu$.
- Our schemes converge under "rough" conditions.

Distributional solution

Definition

Under the assumptions (A_φ), (A_μ), and $u_0 \in L^\infty(\mathbb{R}^N)$, $u \in L^\infty(Q_T)$ is a **distributional solution** of (GPME) if

$$0 = \int_0^T \int_{\mathbb{R}^N} \Big(u(x,t)\partial_t \psi(x,t) + \varphi(u(x,t))\mathcal{L}[\psi(\cdot,t)](x) \Big) \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} u_0(x)\psi(x,0)\,\mathrm{d}x$$

for all $\psi \in C_c^\infty(\mathbb{R}^N \times [0,T))$.

Uniqueness

Theorem (Preuniqueness, [del Teso&JE&Jakobsen, 2017])

Assume (A_φ) *and* (A_μ). *Let* $u(x,t)$ *and* $\hat{u}(x,t)$ *satisfy*

$$u, \hat{u} \in L^\infty(Q_T),$$

$$u - \hat{u} \in L^1(Q_T),$$

$$\partial_t u - \mathcal{L}[\varphi(u)] = \partial_t \hat{u} - \mathcal{L}[\varphi(\hat{u})] \qquad \text{in} \qquad \mathcal{D}'(Q_T),$$

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_{\mathbb{R}^N} (u(x,t) - \hat{u}(x,t))\psi(x,t)\,\mathrm{d}x = 0 \quad \forall \psi \in C_c^\infty(\mathbb{R}^N \times [0,T)).$$

Then $u = \hat{u}$ *a.e. in* Q_T.

Uniqueness

Corollary (Uniqueness, [del Teso&JE&Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^\infty(\mathbb{R}^N)$. *Then there is at most one distributional solution* u *of* (GPME) *such that* $u \in L^\infty(Q_T)$ *and* $u - u_0 \in L^1(Q_T)$.

Proof: Assume there are two solutions u and $\hat{u}$ with the same initial data u_0. Then all assumptions of Theorem Preuniqueness obviously hold ($\|u-\hat{u}\|_{L^1} \leq \|u-u_0\|_{L^1} + \|\hat{u}-u_0\|_{L^1} < \infty$), and $u = \hat{u}$ a.e. □

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Finite difference discretizations

- **Local operator:**

$$\Delta_h \psi(x) := \frac{1}{h^2} \sum_{i=1}^{N} \big(\psi(x + he_i) + \psi(x - he_i) - 2\psi(x)\big)$$

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- **Nonlocal operator:**

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where $\omega_\beta = \omega_{-\beta} \geq 0$.

Finite difference discretizations, nonlocal operator

Here,

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Better monotonicity if μ abs. cont. and regular (Newton-Cotes).

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Numerical schemes for (GPME)

Recall that our Cauchy problem was given as

$$\text{(GPME)} \qquad \begin{cases} \partial_t u = \mathcal{L}[\varphi(u)] & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N. \end{cases}$$

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Corresponding numerical scheme (NM):

$$\begin{cases} \frac{U^j_\beta - U^{j-1}_\beta}{\Delta t} = \mathcal{L}^{\nu_{h,1}}[\varphi(U^j_\beta)] + \mathcal{L}^{\nu_{h,2}}[\varphi^h(U^{j-1}_\beta)] & \text{in} \quad \Delta x\mathbb{Z}^N \times \Delta t\mathbb{N}, \\ \text{“} U^0_\beta = u_0 \text{”} & \text{in} \quad \Delta x\mathbb{Z}^N, \end{cases}$$

where

$$\mathcal{L}^{\nu_{h,1}} + \mathcal{L}^{\nu_{h,2}} \approx \mathcal{L} = \mathcal{L}^\sigma + \mathcal{L}^\mu$$

$$\varphi^h \approx \varphi$$

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Also:
Explicit methods only works for Lipschitz φ because of CFL. But, in stead of doing implicit methods for "demanding" φ, we can do less costly explicit methods with approximating φ.

Convergence of the numerical schemes

The scheme defined by (NM) is

- monotone,
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For the interpolant $U_{h,\Delta t}$, we have

$$U_{h,\Delta t} \to u \qquad \text{in} \qquad C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N)) \qquad \text{as} \qquad h,\Delta t \to 0^+$$

where $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N))$ is a distributional solution of (GPME).

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Proof of convergence

1. Since the operator and the nonlinearity are x-**independent**, the numerical scheme is equivalent with

$$U^j(x) - \Delta t \mathcal{L}^{\nu_{h,1}}[\varphi(U^j)](x) = U^{j-1}(x) + \Delta t \mathcal{L}^{\nu_{h,2}}[\varphi^h(U^{j-1})](x).$$

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4. To study T_{exp}, the CFL-condition comes naturally

$$\Delta t L_{\varphi^h} \nu_{h,2}(\mathbb{R}^N) \leq 1 \qquad \text{“time derivative} \sim \text{spatial derivatives”}$$

Proof of convergence

5. Both operators T_{imp} and T_{exp} are "well-posed" in $L^1 \cap L^\infty$ and enjoy
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Equitightness, uniform control at infinity

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Recall the definition of distributional solutions, for $\psi \in C_c^\infty(\mathbb{R}^N \times [0,T])$,

$$\begin{aligned}
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&= \int_0^T \int_{\mathbb{R}^N} \Big(u(x,t)\partial_t\psi(x,t) + \varphi(u(x,t))\mathcal{L}[\psi(\cdot,t)](x) \Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^N} u_0(x)\psi(x,0)\,\mathrm{d}x.
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Choose $\psi(x,t) = \mathrm{sign}(u)\mathcal{X}_R(x)$.

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Choose $\mathcal{X}_R(x) \approx \mathbf{1}_{|x|\geq R}$ such that $\mathbf{1}_{|x|\geq R} \leq \mathcal{X}_R(x)$ and $\mathcal{X}_R \to 0$ as $R \to \infty$.

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What can we say about the last term?

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$$\int_0^T \int_{\mathbb{R}^N} |\varphi(u(x,t))||\mathcal{L}[\mathcal{X}_R](x)| \,\mathrm{d}x \,\mathrm{d}t$$
$$\leq TL_\varphi \|u\|_{L^1} \|\Delta \mathcal{X}_R\|_{L^\infty} = TL_\varphi \|u\|_{L^1} \frac{1}{R^2} \|\Delta \mathcal{X}\|_{L^\infty}.$$

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$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^N} |\varphi(u(x,t))||\mathcal{L}[\mathcal{X}_R](x)| \,\mathrm{d}x\,\mathrm{d}t \\ &\leq T|\varphi|_{C^{0,\gamma}}\|u\|_{L^{q\gamma}}\|\Delta\mathcal{X}_R\|_{L^p} = T|\varphi|_{C^{0,\gamma}}\|u\|_{L^{q\gamma}}\frac{1}{R^{-\frac{N}{p}+2}}\|\Delta\mathcal{X}\|_{L^p}.\end{aligned}$$

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- Assume $\mathcal{L} = \Delta$ and φ is γ-Hölder with $\gamma \in (0,1)$. Then

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} |\varphi(u(x,t))||\mathcal{L}[\mathcal{X}_R](x)| \,\mathrm{d}x\,\mathrm{d}t \\ &\leq T|\varphi|_{C^{0,\gamma}}\|u\|_{L^{q\gamma}}\|\Delta\mathcal{X}_R\|_{L^p} = T|\varphi|_{C^{0,\gamma}}\|u\|_{L^{q\gamma}}\frac{1}{R^{-\frac{N}{p}+2}}\|\Delta\mathcal{X}\|_{L^p}. \end{aligned}$$

We have to tune N, γ, p, q such that we have convergence, and we get it when $\frac{\max\{0,N-2\}}{N} < \gamma < 1$.

Equitightness, local diffusion

- Assume $\mathcal{L} = \Delta$ and φ is Lipschitz. Then

$$\lim_{R\to\infty} \int_0^T \int_{\mathbb{R}^N} |\varphi(u(x,t))||\mathcal{L}[\mathcal{X}_R](x)| \,\mathrm{d}x\,\mathrm{d}t = 0.$$

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We have to tune N, γ, p, q such that we have convergence, and we get it when $\frac{\max\{0,N-2\}}{N} < \gamma < 1$. (Extinction when $0 < \gamma < \frac{\max\{0,N-2\}}{N}$.)

Equitightness, local diffusion

Assume $\mathcal{L} = \mathcal{L}^{\mu}$ such that

$$\int_{|z|\leq 1} (\cdots)\,\mathrm{d}\mu(z) \sim \Delta \mathcal{X}_R \quad \text{and} \quad \int_{|z|>R>1} (\cdots)\,\mathrm{d}\mu(z) \sim -(-\Delta)^{\frac{\alpha}{2}}[\mathcal{X}_R]$$

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Convergence, final

Theorem (Convergence, [del Teso&JE&Jakobsen, 2018])

For the interplant $U_{h,\Delta t}$, we have

$$U_{h,\Delta t} \to u \qquad \text{in} \qquad C([0,T]; L^1(\mathbb{R}^N)) \qquad \text{as} \qquad h, \Delta t \to 0^+$$

where $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T]; L^1(\mathbb{R}^N))$ is a distributional solution of (GPME).

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Simulations

- 1D (one-phase) Stefan problem with $\varphi(u) = \max\{0, u - 0.5\}$. Explicit method. $\mathcal{L} = -(-\partial_{xx})^{\frac{\alpha}{2}}$ with $\alpha = 0.5, 1, 1.5$.

- 2D (one-phase) Stefan problem with $\varphi(u) = \max\{0, u - 1\}$. Explicit method. $\mathcal{L} = ((\frac{1}{2}, \frac{47}{100}) \cdot D)^2 + (-\partial_{xx})^{\frac{1}{4}}$.

Thank you for your attention!