Nonlocal nonlinear diffusion equations. Smoothing effects, Green functions, and functional inequalities

Jørgen Endal
URL: https://verso.mat.uam.es/~jorgen.endal
Twitter: @msca_techfront

Departamento de Matemáticas, UAM

04 March 2022

A talk given at UAM in

Seminario de ecuaciones en derivadas parciales

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement no. 839749 "Novel techniques for quantitative behaviour of convection-diffusion equations".

Main results

(GPME) $$\begin{cases} \partial_t u + (-\mathfrak{L})[u^m] = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

Think of $\mathfrak{L}$ as Δ or $-(-\Delta)^{\frac{\alpha}{2}}$, but it is also more general.

- L^1–L^∞-smoothing.
- $(-\mathfrak{L})^{-1}$ with kernel $\mathbb{G}_{-\mathfrak{L}}(x-y) = \int_0^\infty H_{-\mathfrak{L}}(x-y,t)\,\mathrm{d}t$.
- Their relation with functional inequalities like Gagliardo-Nirenberg-Sobolev (and Nash):

$$\|f\|_{L^{2^*}} \lesssim \|(-\mathfrak{L})^{\frac{1}{2}} f\|_{L^2}, \quad \|f\|_{L^p} \lesssim \|f\|_{L^q}^{\vartheta} \|(-\mathfrak{L})^{\frac{1}{2}} f\|_{L^2}^{1-\vartheta},$$

where $2^* > 2$ and $p > q$.

JE, M. Bonforte. Boundedness of solutions of nonlinear and nonlocal diffusion equations driven by a convex nonlinearity of porous medium type. In preparation, 2022.

The linear case

$$\boxed{\partial_t u + (-\mathfrak{L})[u] = 0}$$

Linear case ($m = 1$). Scaling

Consider

$$\text{(HE)} \qquad \begin{cases} \partial_t u + (-\Delta)[u] = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

Assume that we are searching for an estimate of the form

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\sigma_1} \|u_0\|^{\sigma_2}_{L^1(\mathbb{R}^N)}.$$

What are the *unique* admissible values of σ_1, σ_2?

If u solves (HE), then

$$\tilde{u}(x, t) := \kappa u(\Xi x, \Lambda t) \qquad \text{for } \kappa, \Xi, \Lambda > 0$$

also solves (HE) with initial data $\tilde{u}_0(x) := \kappa u(\Xi x, 0)$ as long as $\Xi^2 = \Lambda$. Fix $\tilde{u}$ with mass $M = 1$, then $\kappa = M^{-1}\Xi^N$, and $\tilde{u}_0 \in L^1$.

Linear case ($m = 1$). Scaling

Consider

$$\text{(HE)} \qquad \begin{cases} \partial_t u + (-\Delta)[u] = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

I.e.,

$$\|\tilde{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\sigma_1} \|\tilde{u}_0\|^{\sigma_2}_{L^1(\mathbb{R}^N)}$$

$$\begin{aligned} \iff \|u(\Xi\cdot,\Lambda t)\|_{L^\infty(\mathbb{R}^N)} &\lesssim \kappa^{-1}\Lambda^{\sigma_1}(\Lambda t)^{-\sigma_1}\kappa^{\sigma_2}\|u(\Xi\cdot,0)\|^{\sigma_2}_{L^1(\mathbb{R}^N)} \\ &= \kappa^{\sigma_2-1}\Lambda^{\sigma_1}(\Lambda t)^{-\sigma_1}\Xi^{-N\sigma_2}\|u_0\|^{\sigma_2}_{L^1(\mathbb{R}^N)} \\ &= \kappa^{\sigma_2-1}\Xi^{2\sigma_1-N\sigma_2}(\Lambda t)^{-\sigma_1}\|u_0\|^{\sigma_2}_{L^1(\mathbb{R}^N)}, \end{aligned}$$

with $\sigma_1 = (N/2)\sigma_2$ and $\sigma_2 = 1$.

Hence,

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\frac{N}{2}}\|u_0\|_{L^1(\mathbb{R}^N)}.$$

Consider

$$\text{(HE)} \qquad \begin{cases} \partial_t u + (-\Delta)[u] = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

I.e.,

$$\begin{aligned} \|\tilde{u}(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} &\lesssim t^{-\sigma_1} \|\tilde{u}_0\|^{\sigma_2}_{L^1(\mathbb{R}^N)} \\ \iff \|u(\Xi\cdot, \Lambda t)\|_{L^\infty(\mathbb{R}^N)} &\lesssim \kappa^{-1} \Lambda^{\sigma_1} (\Lambda t)^{-\sigma_1} \kappa^{\sigma_2} \|u(\Xi\cdot, 0)\|^{\sigma_2}_{L^1(\mathbb{R}^N)} \\ &= \kappa^{\sigma_2 - 1} \Lambda^{\sigma_1} (\Lambda t)^{-\sigma_1} \Xi^{-N\sigma_2} \|u_0\|^{\sigma_2}_{L^1(\mathbb{R}^N)} \\ &= \kappa^{\sigma_2 - 1} \Xi^{2\sigma_1 - N\sigma_2} (\Lambda t)^{-\sigma_1} \|u_0\|^{\sigma_2}_{L^1(\mathbb{R}^N)}, \end{aligned}$$

with $\sigma_1 = (N/2)\sigma_2$ and $\sigma_2 = 1$.

J. L. Vázquez. *Smoothing and decay estimates for nonlinear diffusion equations*. Oxford Lecture Series in Mathematics and its Applications, volume 33. Oxford University Press, Oxford, 2006.

Linear case ($m = 1$). Heat kernel

The function

$$u(x,t) = \int_{\mathbb{R}^N} u_0(y) H_{-\Delta}(x-y,t)\,\mathrm{d}y,$$

with

$$H_{-\Delta}(x-y,t) \eqsim t^{-\frac{N}{2}} \exp\Big(-\frac{|x-y|^2}{4t}\Big) \qquad \text{(the constant is } (4\pi)^{-\frac{N}{2}}\text{)},$$

is the solution of

$$\text{(HE)} \qquad \begin{cases} \partial_t u + (-\Delta)[u] = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

Since $0 \leq H_{-\Delta}(x-y,t) \lesssim t^{-\frac{N}{2}}$, we immediately have

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\frac{N}{2}} \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Linear case ($m = 1$). Nash inequality

Assume (by the Gagliardo-Nirenberg-Sobolev ineq.)

$$\|f\|_{L^2(\mathbb{R}^N)} \lesssim \|f\|_{L^1(\mathbb{R}^N)}^{\vartheta} \|\nabla f\|_{L^2(\mathbb{R}^N)}^{1-\vartheta}$$

with

$$\vartheta = \frac{1}{2}\frac{2^* - 2}{2^* - 1} \qquad \text{where } 2^* = 2N/(N-2).$$

Define $Y(t) := \|u(t)\|_{L^2(\mathbb{R}^N)}^2$, and consider

$$\begin{aligned} Y'(t) &= \int \partial_t(u^2) = 2\int u\partial_t u = -2\int u(-\Delta)[u] = -2\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 \\ &\lesssim -Y(t)^{\frac{1}{1-\vartheta}} \|u(t)\|_{L^1(\mathbb{R}^N)}^{-\frac{2\vartheta}{1-\vartheta}} \leq -\|u_0\|_{L^1(\mathbb{R}^N)}^{-2\frac{\vartheta}{1-\vartheta}} Y(t)^{1+\frac{\vartheta}{1-\vartheta}}. \end{aligned}$$

Solving the differential inequality gives

$$\|u(t)\|_{L^2(\mathbb{R}^N)}^2 = Y(t) \lesssim t^{-\frac{1-\vartheta}{\vartheta}} \|u_0\|_{L^1(\mathbb{R}^N)}^2 = t^{-\frac{N}{2}} \|u_0\|_{L^1(\mathbb{R}^N)}^2.$$

Linear case ($m = 1$). Nash inequality

Assume

$$\|f\|_{L^2(\mathbb{R}^N)} \lesssim \|f\|_{L^1(\mathbb{R}^N)}^{\vartheta} \|\nabla f\|_{L^2(\mathbb{R}^N)}^{1-\vartheta}.$$

We have

$$\|u(t)\|_{L^2(\mathbb{R}^N)} \lesssim t^{-\frac{N}{4}} \|u_0\|_{L^1(\mathbb{R}^N)},$$

and then, by duality,

$$\begin{aligned}
\|u(t)\|_{L^\infty} &= \sup_{\|\phi\|_{L^1}=1} \left| \int u(t)\phi \right| = \sup_{\|\phi\|_{L^1}=1} \left| \int S_t[u_0]\phi \right| \\
&= \sup_{\|\phi\|_{L^1}=1} \left| \int S_{\frac{t}{2}}[S_{\frac{t}{2}}[u_0]]\phi \right| = \sup_{\|\phi\|_{L^1}=1} \left| \int S_{\frac{t}{2}}[u_0] S_{\frac{t}{2}}[\phi] \right| \\
&\leq \sup_{\|\phi\|_{L^1}=1} \|S_{\frac{t}{2}}[u_0]\|_{L^2} \|S_{\frac{t}{2}}[\phi]\|_{L^2} \lesssim t^{-\frac{N}{4}} \|S_{\frac{t}{2}}[u_0]\|_{L^2} \\
&\lesssim t^{-\frac{N}{2}} \|u_0\|_{L^1}.
\end{aligned}$$

E. H. Lieb and M. Loss. *Analysis*. Graduate Studies in Mathematics, volume 14. American Mathematical Society, Providence, RI, 2001.

Assume

$$\|f\|_{L^2(\mathbb{R}^N)} \lesssim \|f\|_{L^1(\mathbb{R}^N)}^{\vartheta} \|\nabla f\|_{L^2(\mathbb{R}^N)}^{1-\vartheta}.$$

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&\leq \sup_{\|\phi\|_{L^1}=1} \|S_{\frac{t}{2}}[u_0]\|_{L^2} \|S_{\frac{t}{2}}[\phi]\|_{L^2} \lesssim t^{-\frac{N}{4}} \|S_{\frac{t}{2}}[u_0]\|_{L^2} \\
&\lesssim t^{-\frac{N}{2}} \|u_0\|_{L^1}.
\end{aligned}$$

Note that the symmetry of the semigroup (which is a linear property) is not needed if one applies the Moser iteration instead.

Linear case ($m = 1$). Nash inequality implied by smoothing

Recall

$$\partial_t u + (-\Delta)[u] = 0, \qquad \|u(t)\|_{L^\infty} \leq Ct^{-N/2}\|u_0\|_{L^1}$$

from which we will deduce the Nash inequality

$$\|f\|_{L^2} \lesssim \|f\|_{L^1}^{\vartheta}\|\nabla f\|_{L^2}^{1-\vartheta}.$$

Again, we differentiate the L^2-norm:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} u(t)^2\,\mathrm{d}x = -2\int_{\mathbb{R}^N} |\nabla u(t)|^2\,\mathrm{d}x \geq -\int_{\mathbb{R}^N} |\nabla u_0|^2\,\mathrm{d}x,$$

where the latter follows by

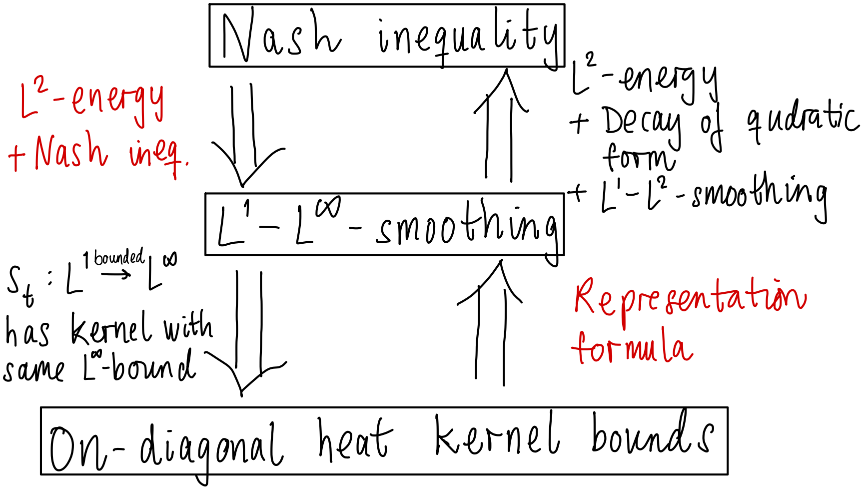
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} |\nabla u(t)|^2\,\mathrm{d}x = 2\int_{\mathbb{R}^N} \nabla u\cdot\nabla(\partial_t u)\,\mathrm{d}x = -2\int_{\mathbb{R}^N} (\Delta u)^2\,\mathrm{d}x \leq 0.$$

Integrating the diff. ineq. on $(0, T)$ and using the smoothing effects we obtain, for all $T > 0$,

$$\|u_0\|_{L^2}^2 \leq T\|\nabla u_0\|_{L^2}^2 + \|u(T)\|_{L^\infty}\|u(T)\|_{L^1} \lesssim T\|\nabla u_0\|_{L^2}^2 + T^{-N/2}\|u_0\|_{L^1}^2.$$

Optimizing in T gives the Nash inequality for $f = u_0$.

Linear case ($m = 1$). Overview 1



Linear case ($m = 1$). Overview, Green function implication

$$0 \le \mathbb{G}^{x_0}_{-\mathcal{L}}(x) \le C|x - x_0|^{-(N-\alpha)}$$

$\Downarrow$ (HLS) $\|(-\Delta)^{-\frac{\alpha}{4}}[f]\|_{L^2} \le C\|f\|_{L^{(2^*)'}}$

$$(DS) \quad \|(-\mathcal{L})^{-\frac{1}{2}}[f]\|_{L^2(\mathbb{R}^N)} \le C\|f\|_{L^{(2^*)'}(\mathbb{R}^N)}$$

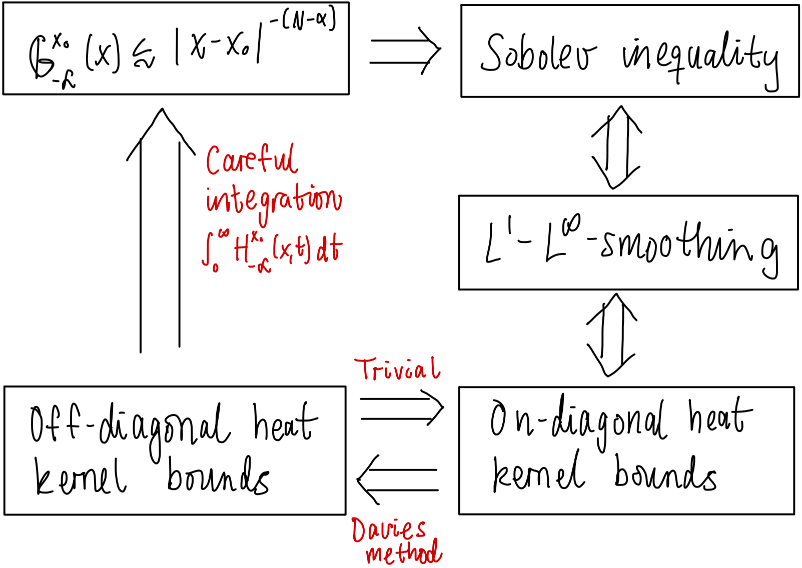
Legendre transform $\Updownarrow$

$$(S) \quad \|f\|_{L^{2^*}(\mathbb{R}^N)} \le C\|(-\mathcal{L})^{\frac{1}{2}}[f]\|_{L^2(\mathbb{R}^N)}$$

Hard, but possible $\Uparrow$ $\qquad$ $\Downarrow$ L^p-interpolation

$$(N) \; \|f\|_{L^2(\mathbb{R}^N)} \le C\|f\|^{\vartheta}_{L^1(\mathbb{R}^N)} \|(-\mathcal{L})^{\frac{1}{2}}[f]\|^{1-\vartheta}_{L^2(\mathbb{R}^N)}$$

Linear case ($m = 1$). Overview 2



The nonlinear case

$$\boxed{\partial_t u + (-\mathfrak{L})[u^m] = 0}$$

Nonlinear case ($m > 1$). Introduction

(GPME) $$\begin{cases} \partial_t u + (-\mathfrak{L})[u^m] = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

Cons:

- We do not have a representation formula.
- It is harder to find the correct functional set-up.

Pros:

- We still have scaling (always time-scaling).
- Some estimates are true in the nonlinear, but not true in the linear.

Nonlinear case ($m > 1$). A neat trick

Consider the operator $-\mathfrak{L} \mapsto I - \mathfrak{L}$, i.e.,

$$\partial_t u + (I - \mathfrak{L})[u^m] = 0 \qquad \Longleftrightarrow \qquad \partial_t u + (-\mathfrak{L})[u^m] = -u^m.$$

Then $t \mapsto Y(t)$ solves $Y'(t) = -Y(t)^{1+(m-1)}$, so

$$Y(t) \leq \left(\frac{1}{(m-1)t}\right)^{\frac{1}{m-1}}.$$

Moreover, comparison (with $Y(0) = \infty$) yields

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq Y(t) \leq \left(\frac{1}{(m-1)t}\right)^{\frac{1}{m-1}}.$$

Holds independently of the operator! But needs "good" nonlinearity.

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L. Véron. Effets régularisants de semi-groupes non linéaires dans des espaces de Banach. *Ann. Fac. Sci. Toulouse Math. (5)*, 1(2):171–200, 1979.

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Nonlinear case ($m > 1$). Why not the Moser iteration?

Idea: $\lim_{p\to\infty}\|f\|_{L^p} = \|f\|_{L^\infty}$.

The Stroock-Varopoulos inequality gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^p}^p = \int \partial_t(u^p) = p\int u^{p-1}\partial_t u = -p\int u^{p-1}(-\mathfrak{L})[u^m]$$
$$= -pQ_{-\mathfrak{L}}[u^{p-1}, u^m] \leq -\frac{4mp(p-1)}{(p+m-1)^2}Q_{-\mathfrak{L}}[u^{\frac{p+m-1}{2}}].$$

The Gagliardo-Nirenberg-Sobolev inequality reads

$$\|f\|_{L^{\tilde{p}}(\mathbb{R}^N)} \leq C\|f\|_{L^{\tilde{q}}(\mathbb{R}^N)}^{\vartheta} Q_{-\mathfrak{L}}[f]^{\frac{1}{2}(1-\vartheta)},$$

where

$$2 \leq \tilde{p} < 2^*, \qquad 1 \leq \tilde{q} < \tilde{p}, \qquad \vartheta := \frac{\tilde{q}}{\tilde{p}}\frac{2^*-\tilde{p}}{2^*-\tilde{q}}.$$

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Nonlinear case ($m > 1$). Tools

$$\text{(GPME)} \qquad \begin{cases} \partial_t u + (-\mathfrak{L})[u^m] = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

We need:

- (Weak dual solutions: $\partial_t U + u^m = 0$, with $U := (-\mathfrak{L})^{-1}[u]$.)
- $(-\mathfrak{L})^{-1}$ with kernel $\mathbb{G}_{-\mathfrak{L}}(x-y) = \int_0^\infty H_{-\mathfrak{L}}(x-y,t)\,\mathrm{d}t$.
- Time scaling. $u_\Lambda(x,t) := \Lambda^{\frac{1}{m-1}} u(x, \Lambda t)$ solution when u is.
- Comparison principle.
- L^p-bounds.

Nonlinear case ($m > 1$). Tools

(GPME) $$\begin{cases} \partial_t u + (-\mathfrak{L})[u^m] = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

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- Comparison principle.
- L^p-bounds.

Nonlinear case ($m > 1$). Time-monotonicity

- Time scaling. $u_\Lambda(x,t) := \Lambda^{\frac{1}{m-1}} u(x, \Lambda t)$ solution when u is.
- Comparison principle.

Provides the well-known *nonlinear* estimate

$$\partial_t u \geq -\frac{u}{(m-1)t}$$

D. G. Aronson and P. Bénilan. Régularité des solutions de l'équation des milieux poreux dans $\mathbf{R}^N$. *C. R. Acad. Sci. Paris Sér. A-B*, 288(2):A103–A105, 1979.

P. Bénilan and M. Crandall. Regularizing effects of homogeneous evolution equations. In *Contributions to analysis and geometry (Baltimore, Md., 1980)*, pages 23–39, Johns Hopkins Univ. Press, Baltimore, Md., 1981.

M. Crandall and M. Pierre. Regularizing effects for $u_t + A\varphi(u) = 0$ in L^1. *J. Funct. Anal.*, 45(2):194–212, 1982.

$$
\begin{aligned}
\partial_t u + (-\mathfrak{L})[u^m] = 0 \qquad &\Longleftrightarrow \qquad u^m = -(-\mathfrak{L})^{-1}[\partial_t u] \\
&\Longleftrightarrow \qquad u^m = -\partial_t u *_x \mathbb{G}_{-\mathfrak{L}} \\
&\Longleftrightarrow \qquad u^m \leq \frac{u}{(m-1)t} *_x \mathbb{G}_{-\mathfrak{L}}.
\end{aligned}
$$

Hence,

$$
\big(u(x,t)\big)^m \leq \frac{1}{(m-1)t} \int_{\mathbb{R}^N} u(y,t) \mathbb{G}_{-\mathfrak{L}}(x-y) \, \mathrm{d}y.
$$

Nonlinear case ($m > 1$). "Representation formula"

$$\partial_t u + (-\mathfrak{L})[u^m] = 0 \qquad \Longleftrightarrow \qquad u^m = -(-\mathfrak{L})^{-1}[\partial_t u]$$
$$\Longleftrightarrow \qquad u^m = -\partial_t u *_x \mathbb{G}_{-\mathfrak{L}}$$
$$\Longleftrightarrow \qquad u^m \leq \frac{u}{(m-1)t} *_x \mathbb{G}_{-\mathfrak{L}}.$$

Hence,

$$\big(u(x,t)\big)^m \leq \frac{1}{(m-1)t} \int_{\mathbb{R}^N} u(y,t) \underbrace{\mathbb{G}_{-\mathfrak{L}}(x-y)}_{=\int_0^\infty H_{-\mathfrak{L}}(x-y,t)\,\mathrm{d}t} \,\mathrm{d}y.$$

M. Bonforte and J. L. Vázquez. A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains. *Arch. Ration. Mech. Anal.*, 218(1):317–362, 2015.

General Green function assumptions

(G_1) For all $R>0$, some $x_0\in\mathbb{R}^N$, and some $\alpha\in(0,2]$,

$$\begin{cases}\int_{B_R(x_0)}\mathbb{G}^{x_0}_{-\mathfrak{L}}(x)\,\mathrm{d}x\leq K_1R^\alpha,\\ \text{for } x\in\mathbb{R}^N\setminus B_R(x_0),\ \mathbb{G}^{x_0}_{-\mathfrak{L}}(x)\leq K_2R^{-(N-\alpha)}.\end{cases}$$

(G_1') For all $R>0$, some $x_0\in\mathbb{R}^N$, and some $\alpha\in(0,2]$,

$$\begin{cases}\int_{B_R(x_0)}\mathbb{G}^{x_0}_{-\mathfrak{L}}(x)\,\mathrm{d}x\leq K_1R^\alpha,\\ \text{for } x\in\mathbb{R}^N\setminus B_R(x_0),\ \mathbb{G}^{x_0}_{-\mathfrak{L}}(x)\leq \max\{K_3,K_1R^{-(N-\alpha)}\}.\end{cases}$$

(G_2) For some $x_0\in\mathbb{R}^N$,

$$\|\mathbb{G}^{x_0}_{-\mathfrak{L}}\|_{L^1(\mathbb{R}^N)}=\|\mathbb{G}^{0}_{-\mathfrak{L}}\|_{L^1(\mathbb{R}^N)}\leq C_1<\infty.$$

Nonlinear case ($m > 1$). Assumption (G_1)

For some fixed $R > 0$, we have

$$\begin{aligned} &u^m(x_0,t)\\ &\le \frac{C(m)}{t}\left(\int_{B_R(x_0)} + \int_{\mathbb{R}^N\setminus B_R(x_0)}\right) u(y,t)\mathbb{G}^{x_0}_{-\mathfrak{L}}(y)\,\mathrm{d}y\\ &\le \|u(t)\|_{L^\infty(\mathbb{R}^N)}\frac{C(m)}{t}K_1R^\alpha + \|u_0\|_{L^1(\mathbb{R}^N)}\frac{C(m)}{t}K_2R^{-(N-\alpha)}. \end{aligned}$$

The Young inequality applied to the first term yields

$$\frac{1}{m}\|u(t)\|^m_{L^\infty(\mathbb{R}^N)} + \frac{m-1}{m}\left(\frac{C(m)}{t}K_1R^\alpha\right)^{\frac{m}{m-1}}.$$

By taking the supremum, with respect to $x_0 \in \mathbb{R}^N$, we get

$$\|u(t)\|^m_{L^\infty(\mathbb{R}^N)} \le \frac{1}{2}\tilde{C}(m)^m\frac{R^{\frac{\alpha m}{m-1}}}{t^{\frac{m}{m-1}}}\left(1 + \frac{t^{\frac{1}{m-1}}\|u_0\|_{L^1(\mathbb{R}^N)}}{R^{\frac{1}{(m-1)\theta_\alpha}}}\right).$$

We then choose $R = \left(t^{\frac{1}{m-1}}\|u_0\|_{L^1(\mathbb{R}^N)}\right)^{(m-1)\theta_\alpha}$.

Nonlinear case ($m > 1$). Green function estimates

For some fixed $R > 0$, we have

$$\begin{aligned} &u^m(x_0,t)\\ &\leq \frac{C(m)}{t}\left(\int_{B_R(x_0)} + \int_{\mathbb{R}^N \setminus B_R(x_0)}\right) u(y,t)\mathbb{G}_{-\mathfrak{L}}^{x_0}(y)\,\mathrm{d}y\\ &\leq \|u(t)\|_{L^\infty(\mathbb{R}^N)}\frac{C(m)}{t}K_1R^\alpha + \|u_0\|_{L^1(\mathbb{R}^N)}\frac{C(m)}{t}K_2R^{-(N-\alpha)}. \end{aligned}$$

The Young inequality applied to the first term yields

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By taking the supremum, with respect to $x_0 \in \mathbb{R}^N$, we get

$$\|u(t)\|^m_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{2}\tilde{C}(m)^m \frac{R^{\frac{\alpha m}{m-1}}}{t^{\frac{m}{m-1}}}\left(1 + \frac{t^{\frac{1}{m-1}}\|u_0\|_{L^1(\mathbb{R}^N)}}{R^{\frac{1}{(m-1)\theta_\alpha}}}\right).$$

I.e., $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-N\theta_\alpha}\|u_0\|^{\alpha\theta_\alpha}_{L^1(\mathbb{R}^N)}$.

General Green function assumptions

(G$_3$) For some $x_0 \in \mathbb{R}^N$ and some $p \in (1, \infty)$,
$\|\mathbb{G}^{x_0}_{I-\mathfrak{L}}\|_{L^p(\mathbb{R}^N)} = \|\mathbb{G}^{0}_{I-\mathfrak{L}}\|_{L^p(\mathbb{R}^N)} \leq C_p < \infty.$

Nonlinear case ($m > 1$). "Representation formula"

$$
\begin{aligned}
&\partial_t u + (-\mathfrak{L})[u^m] = 0 \\
\iff \quad & \partial_t u + (I - \mathfrak{L})[u^m] - u^m = 0 \\
\iff \quad & u^m = (I - \mathfrak{L})^{-1}[-\partial_t u + u^m] \\
\iff \quad & u^m = (-\partial_t u + u^m) *_x \mathbb{G}_{I-\mathfrak{L}} \\
\iff \quad & u^m \leq \left(\frac{u}{(m-1)t} + u^m\right) *_x \mathbb{G}_{I-\mathfrak{L}} \\
\iff \quad & u^m \leq \left(\frac{1}{(m-1)t} + \|u(t)\|_{L^\infty(\mathbb{R}^N)}^{m-1}\right) u *_x \mathbb{G}_{I-\mathfrak{L}}.
\end{aligned}
$$

Hence, we arrive at two cases:

$$
\|u(t)\|_{L^\infty(\mathbb{R}^N)}^{m-1} \leq \frac{1}{(m-1)t} \quad \text{or} \quad \|u(t)\|_{L^\infty(\mathbb{R}^N)}^{m-1} > \frac{1}{(m-1)t}
$$

The second gives

$$
\big(u(x,t)\big)^m \leq 2\|u(t)\|_{L^\infty(\mathbb{R}^N)}^{m-1} \int_{\mathbb{R}^N} u(y,t)\mathbb{G}_{I-\mathfrak{L}}(x-y)\,\mathrm{d}y.
$$

Nonlinear case ($m > 1$). "Representation formula"

$$\partial_t u + (-\mathfrak{L})[u^m] = 0$$

$$\begin{aligned}
&\iff && \partial_t u + (I - \mathfrak{L})[u^m] - u^m = 0\\
&\iff && u^m = (I - \mathfrak{L})^{-1}[-\partial_t u + u^m]\\
&\iff && u^m = (-\partial_t u + u^m) *_x \mathbb{G}_{I-\mathfrak{L}}\\
&\iff && u^m \leq \left(\frac{u}{(m-1)t} + u^m\right) *_x \mathbb{G}_{I-\mathfrak{L}}\\
&\iff && u^m \leq \left(\frac{1}{(m-1)t} + \|u(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)}\right) u *_x \mathbb{G}_{I-\mathfrak{L}}.
\end{aligned}$$

Hence, we arrive at two cases:

$$\|u(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{(m-1)t} \quad \text{or} \quad \|u(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)} > \frac{1}{(m-1)t}$$

The second gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq 2\|u(t)\|_{L^{p'}(\mathbb{R}^N)}\|\mathbb{G}_{I-\mathfrak{L}}\|_{L^p(\mathbb{R}^N)}.$$

Nonlinear case ($m > 1$). "Representation formula"

$$\begin{aligned}
&\partial_t u + (-\mathfrak{L})[u^m] = 0 \\
&\iff \qquad \partial_t u + (I - \mathfrak{L})[u^m] - u^m = 0 \\
&\iff \qquad u^m = (I - \mathfrak{L})^{-1}[-\partial_t u + u^m] \\
&\iff \qquad u^m = (-\partial_t u + u^m) *_x \mathbb{G}_{I-\mathfrak{L}} \\
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&\iff \qquad u^m \leq \left(\frac{1}{(m-1)t} + \|u(t)\|_{L^\infty(\mathbb{R}^N)}^{m-1}\right) u *_x \mathbb{G}_{I-\mathfrak{L}}.
\end{aligned}$$

Hence, we arrive at two cases:

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)}^{m-1} \leq \frac{1}{(m-1)t} \quad \text{or} \quad \|u(t)\|_{L^\infty(\mathbb{R}^N)}^{m-1} > \frac{1}{(m-1)t}$$

The second gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq 2\|\mathbb{G}_{I-\mathfrak{L}}\|_{L^p(\mathbb{R}^N)}\|u_0\|_{L^1(\mathbb{R}^N)}.$$

Nonlinear case ($m > 1$). "Representation formula"

$$
\begin{aligned}
&\partial_t u + (-\mathfrak{L})[u^m] = 0 \\
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\end{aligned}
$$

Hence,

$$
\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-1/(m-1)} + \|u_0\|_{L^1(\mathbb{R}^N)}.
$$

Nonlinear case $(m > 1)$. Examples

$$\partial_t u + (-\mathfrak{L})[u^m] = 0$$

- $-\mathfrak{L} = (-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2]$ gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-N\theta_\alpha} \|u_0\|_{L^1(\mathbb{R}^N)}^{\alpha\theta_\alpha} \qquad \text{where } \theta_\alpha := (\alpha + N(m-1))^{-1}.$$

Note that $H_{-\mathfrak{L}}(x-y,t) \lesssim t^{-N/\alpha}$.

A. de Pablo, F. Quirós, A. Rodríguez, and J. L. Vázquez. A general fractional porous medium equation. *Comm. Pure Appl. Math.*, 65(9):1242–1284, 2012.

Nonlinear case $(m > 1)$. Examples

$$\partial_t u + (-\mathfrak{L})[u^m] = 0$$

- $-\mathfrak{L} = (-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0, 2]$ gives

$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-N\theta_\alpha} \|u_0\|^{\alpha\theta_\alpha}_{L^1(\mathbb{R}^N)}$, where $\theta_\alpha := (\alpha + N(m-1))^{-1}$

Note that $H_{-\mathfrak{L}}(x-y, t) \lesssim t^{-N/\alpha}$.

- $-\mathfrak{L} = (\kappa^2 I - \Delta)^{\frac{\alpha}{2}} - \kappa^\alpha I$ with $\kappa > 0$ and $\alpha \in (0, 2)$ gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-N\theta_\alpha} \|u_0\|^{\alpha\theta_\alpha}_{L^1(\mathbb{R}^N)} + t^{-N\theta_2} \|u_0\|^{2\theta_2}_{L^1(\mathbb{R}^N)}.$$

Note that $H_{-\mathfrak{L}}(x-y, t) \lesssim t^{-N/\alpha} + t^{-N/2}$.

- $-\mathfrak{L} = \sum_{i=1}^N (-\partial^2_{x_i x_i})^{\frac{\alpha}{2}}$ with $\alpha \in (0, 2)$ gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-1/(m-1)} + \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Strange estimate? Let us comment on it.

$$\partial_t u + (-\mathfrak{L})[u^m] = 0$$

- $-\mathfrak{L} = \sum_{i=1}^{N}(-\partial^2_{x_i x_i})^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-1/(m-1)} + \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Strange estimate? Let us comment on it.

Note that $\mathbb{G}^{x_0}_{-\mathfrak{L}}(x) = \infty$ (for some values of α)!

But $\|\mathbb{G}^{x_0}_{I-\mathfrak{L}}\|_{L^p(\mathbb{R}^N)} \leq C_p < \infty$.

$$\partial_t u + (-\mathfrak{L})[u^m] = 0$$

- $-\mathfrak{L} = \sum_{i=1}^{N} (-\partial^2_{x_i x_i})^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-1/(m-1)} + \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Strange estimate? Let us comment on it.

Moreover, $H_{-\mathfrak{L}}(x-y,t) \lesssim t^{-N/\alpha}$, hence, we have

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-N/\alpha} \|u_0\|_{L^1(\mathbb{R}^N)}$$

in the *linear* case. We therefore expect a homogeneous result in the nonlinear case too.

Indeed, $-\mathfrak{L}$ is α-homogeneous, so we can use scaling to get the first estimate on an invariant form:

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-N\theta_\alpha} \|u_0\|^{\alpha\theta_\alpha}_{L^1(\mathbb{R}^N)}, \quad \text{where } \theta_\alpha = (\alpha + N(m-1))^{-1}.$$

Nonlinear does *not* imply linear

Consider

$$\partial_t u + (-\mathfrak{L})[u^m] = 0,$$

with

$$-\mathfrak{L}[\psi](x) = \psi(x) - \int_{\mathbb{R}^N} \psi(z) J(x-z)\,\mathrm{d}z = (I - J*_x)[\psi](x)$$

where $J \geq 0$ such that $\|J\|_{L^1(\mathbb{R}^N)} = 1$ and $J \in L^p(\mathbb{R}^N)$.

- If $m = 1$, then

$$u(x,t) = u_0(x)\mathrm{e}^{-t} + W(x,t),$$

where $W \geq 0$ is some smooth function. Hence, no smoothing.

F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, J. Toledo-Melero. *Nonlocal diffusion problems*. Mathematical Surveys and Monographs, volume 165. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

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- If $m = 1$, then

$$u(x,t) = u_0(x)\mathrm{e}^{-t} + W(x,t),$$

where $W \geq 0$ is some smooth function. Hence, no smoothing.

- If $m > 1$, then

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-1/(m-1)} + \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Thank you for your attention!