Local and nonlocal diffusion

Brownian motion and Lévy flights

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▸ Pollen grains in water

What is diffusion (from a probabilistic viewpoint)?

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Heat equation

$$\partial_t u = \frac{1}{2}\triangle u$$

A. Einstein. Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Annalen der Physik* (in German), 322(8): 549–560, 1905.

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Fractional heat equation

$$\partial_t u = -(-\triangle)^{\frac{\alpha}{2}} u \qquad \text{with} \qquad \alpha \in (0,2)$$

E. Valdinoci. From the long jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl. SeMA*, (49):33–44, 2009.

Ten equivalent definitions of the fractional Laplace operator

Singluar integral definition:

$$-(-\triangle)^{\frac{\alpha}{2}}\psi := c_{N,\alpha}\int_{|z|>0}\Big(\psi(x+z)-\psi(x)-z\cdot D\psi(x)\mathbf{1}_{|z|\leq 1}\Big)\frac{\mathrm{d}z}{|z|^{N+\alpha}}.$$

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Through harmonic extension:

$$\begin{cases}\triangle_x w(x,y)+\alpha^2 c_{\alpha}^{\frac{2}{\alpha}} y^{2-\frac{2}{\alpha}}\partial_y^2 w(x,y)=0 \quad \text{for } y>0,\\ w(x,0)=\psi(x),\\ \partial_y w(x,0)=: -(-\triangle)^{\frac{\alpha}{2}}\psi(x).\end{cases}$$

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M. Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. Preprint, 2015.

General nonlocal diffusion operator

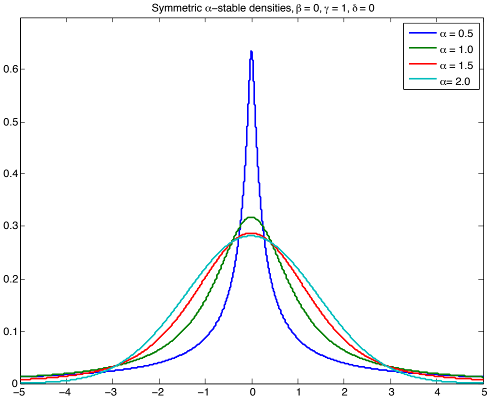
Consider, for $\psi \in C_c^\infty(\mathbb{R}^N)$,

$$\mathcal{L}^\mu[\psi](x) = \int_{|z|>0} \Big(\psi(x+z) - \psi(x) - z \cdot D\psi(x)\mathbf{1}_{|z|\leq 1}\Big)\,\mathrm{d}\mu(z)$$

where $\mu \geq 0$ is a Radon measure ("regular" Borel measure) satisfying

$$\int_{|z|>0} \min\{|z|^2, 1\}\,\mathrm{d}\mu(z) < \infty.$$

Why do we want to study such operators?



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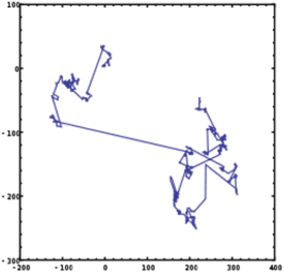

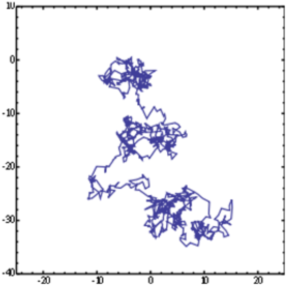

(Pictures taken from Wikipedia.)

Why do we want to study such operators?

Assume $b \in \mathbb{R}^N$ is a given vector, $a = (a_{ij})_{i,j}$ is a nonnegative definite matrix, and $\mu \geq 0$ is a Radon measure satisfying

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Then any Lévy process has a generator given by

$$- b \cdot D\psi(x) + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial_{x_i} \partial_{x_j}} \psi(x)$$
$$+ \int_{|z|>0} \Big(\psi(x+z) - \psi(x) - z \cdot D\psi(x) \mathbf{1}_{|z|\leq 1} \Big) \,\mathrm{d}\mu(z)$$

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and, conversely, for any (b, a, μ), there exists a Lévy process with the above generator.

D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge, UK, 2009.

Assume X is a Lévy process uniquely defined by (b, a, μ). Then X decomposes uniquely as $X = W + Y$, where W is defined by $(b, a, 0)$ and Y by $(0, 0, \mu)$.

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That is, we can decompose the process X into a local and nonlocal part, or rather, into a Gaussian process and a purely discontinuous Lévy process.

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As $\alpha \to 2^-$, the distributional solution of

$$\partial_t u + (-\triangle)^{\frac{\alpha}{2}} u^m = 0$$

converges in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ to the distributional solution of

$$\partial_t u - \triangle u^m = 0.$$

F. del Teso, JE, and E. R. Jakobsen. Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type. *Adv. Math.*, 305:78–143, 2017.

Some surprising nonlocal operators

Recall that

$$\mathcal{L}^{\mu}[\psi](x) = \int_{|z|>0} \Big(\psi(x+z) - \psi(x) - z \cdot D\psi(x)\mathbf{1}_{|z|\leq 1}\Big)\,\mathrm{d}\mu(z).$$

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For the choice

$$\mu_h(z) := \frac{1}{h^2}\left(\delta_h(z) + \delta_{-h}(z)\right),$$

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we have

$$\mathcal{L}^{\mu_h}[\psi](x) = \frac{1}{h^2}\left(\psi(x+h) + \psi(x-h) - 2\psi(x)\right).$$

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Thank you for your attention!