The one-phase fractional Stefan problem

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05 May 2021

A talk given at
Online Analysis and PDE seminar

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement no. 839749 "Novel techniques for quantitative behaviour of convection-diffusion equations".

In collaboration with

Félix del Teso

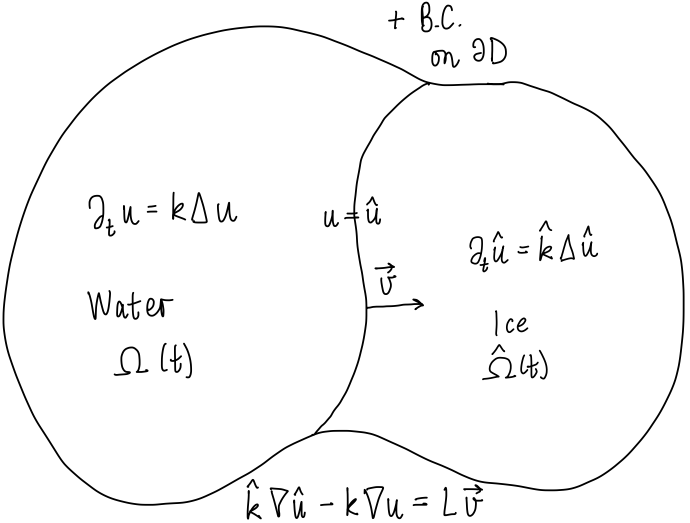
Juan Luis Vázquez

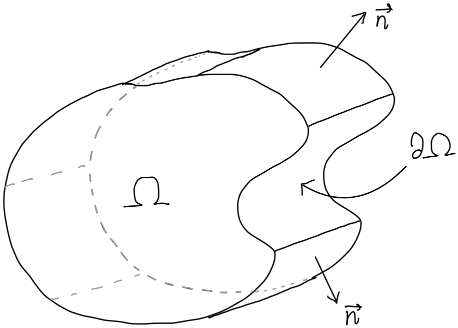
Derivation of the local two-phase model

- We will always think of two phases: water and ice.
- To simplify:
  - Transport of mass plays no role (no convection).
  - The transition region between two phases is an infinitely thin surface.
  - The densities are 1, and the specific heats are also 1.
- The physical quantities that play a role are:
  - Latent heat L (the amount of energy needed to transform one mass unit between phases; melting ice [heat required] versus freezing water [heat released]).
  - Thermal conductivity k (a substance's ability to conduct heat; higher in ice than in water [closeness of atoms])

Derivation of the local two-phase model

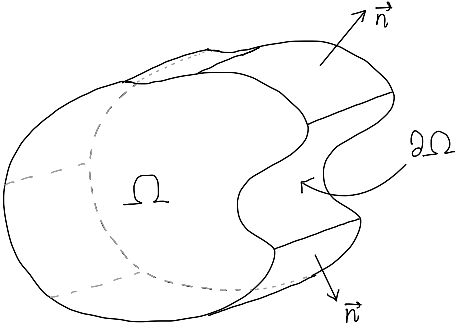
Nonglobal formulation





Let h be enthalpy ("energy") density in $\Omega \subset D$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega h\,\mathrm{d}x = -\int_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S + \int_\Omega f\,\mathrm{d}x.$$



Let h be enthalpy ("energy") density in $\Omega \subset D$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega h\,\mathrm{d}x = -\int_\Omega \mathrm{div}\mathbf{F}\,\mathrm{d}x + \int_\Omega f\,\mathrm{d}x.$$

Derivation of the local two-phase model

In many situations, $\mathbf{F} \sim -Du$ (flow from high to low consentration). By the Fourier law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega h \,\mathrm{d}x = \int_\Omega \mathrm{div}\big(k(u)Du\big)\,\mathrm{d}x + \int_\Omega f\,\mathrm{d}x$$

or

$$\partial_t h = \mathrm{div}(k(u)Du) + f.$$

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or

$$\partial_t h = \mathrm{div}(k(u)Du).$$

Assume:

- $h \in \gamma(u) \implies u = \beta(h)$
- $k(u) = k(\beta(h)) =: K'(\beta(u))$

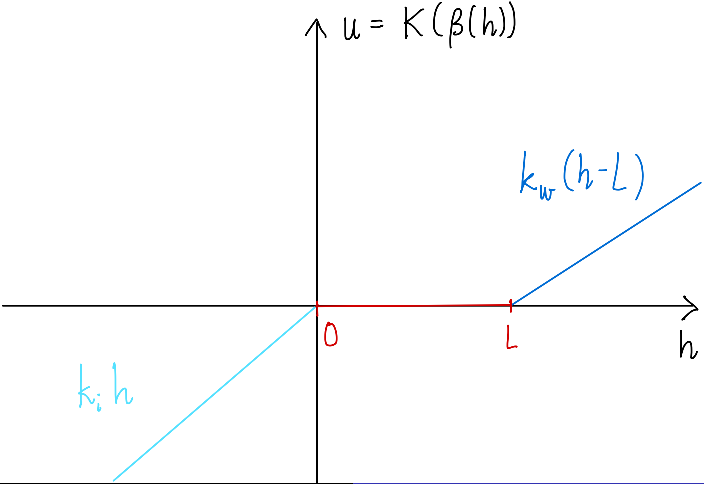
Then

$$\partial_t h = \mathrm{div}(DK(\beta(h))) = \Delta K(\beta(h)).$$

Basically,

$$\boxed{\partial_t h = \Delta K(\beta(h)) =: \Delta\Phi(h)}$$

where $u := \Phi(h) \sim k \times \beta(h)$ is given as

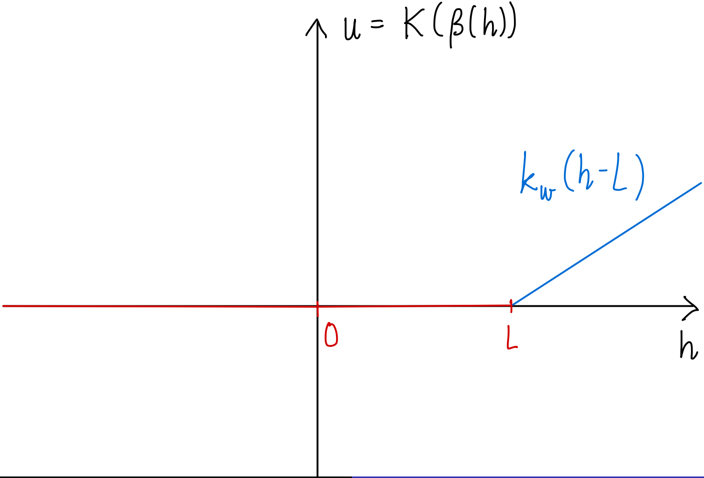


Derivation of the local one-phase model

We keep the ice at critical temperature 0°C. That is, we get

$$\partial_t h = \Delta u$$

where $u := \Phi(h)$ is given as



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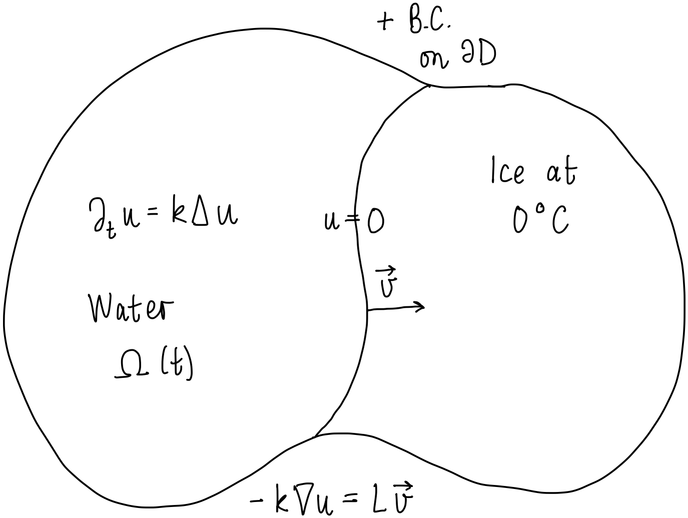

Theory for local one-phase model

- Modeling:

J. Stefan. Über die Theorie der Eisbildung (On the theory of ice formation). *Monatsh. Math. Phys.*, 1(1):1–6, 1890.

- Well-posedness:

S. L. Kamenomostskaja (Kamin). On Stefan's problem. *Mat. Sb. (N.S.)*, 53 (95):489–514, 1961.

- The free boundary is smooth (under certain conditions):

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D. Kinderlehrer and L. Nirenberg. The smoothness of the free boundary in the one phase Stefan problem. *Comm. Pure Appl. Math.*, 31(3):257–282, 1978.

The one-phase Stefan problem can equivalently be expressed as:

- (Nonglobal) The equation $\partial_t u - \Delta u = 0$ in $\{u > 0\}$.
- (Global) The equation $\partial_t h - \Delta u = 0$.
- (Obstacle) The equation $\partial_t U - \Delta U = -1$ in $\{U > 0\}$ where $U(x,t) := \int_0^t u(x,s)\,\mathrm{d}s$.

A. Figalli. Regularity of interfaces in phase transitions via obstacle problems. In *Prooceedings of the International Congress of Mathematicians (ICM 2018)*, Vol. I. Plenary lectures, pp. 225–247. World Sci. Publ., Hackensack, NJ, 2019.

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- Continuity of the temperature (independent of the free boundary):

L. A. Caffarelli and A. Friedman. Continuity of the temperature in the Stefan problem. *Indiana Univ. Math. J.*, 28(1):53–70, 1979.

- The selfsimilar solutions has the form $H(xt^{-1/2})$, and a free boundary given by $x(t) = \xi_0 t^{1/2}$.

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The nonlocal Cauchy problem

We will study the one-phase fractional Stefan problem

$$\text{(FSP)} \qquad \begin{cases} \partial_t h + (-\Delta)^s u = 0 & \text{in} \quad Q_T := \mathbb{R}^N \times (0, T), \\ h(\cdot, 0) = h_0 & \text{on} \quad \mathbb{R}^N, \end{cases}$$

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where $s \in (0,1)$, $h_0 \in L^\infty(\mathbb{R}^N)$ unsigned, and

$$u := \Phi(h) := \max\{h - L, 0\}.$$

Note that Φ is degenerate and Lipschitz, and if $h > L$ then

$$\partial_t u + (-\Delta)^s u = 0.$$

Previous work on one-phase nonlocal Stefan

- Nonsingular spatial-fractional operators:

C. Brändle, E. Chasseigne, and F. Quirós. Phase transitions with midrange interactions: a nonlocal Stefan model. *SIAM J. Math. Anal.*, 44(4):3071–3100, 2012.

- Temporal-fractional operators:

V. R. Voller. Fractional Stefan problems. *International Journal of Heat and Mass Transfer*, 74:269–277, 2014.

- Singular spatial-fractional operators (fractional Laplacian):

Continuity of the temperature:

I. Athanasopoulos and L. A. Caffarelli. Continuity of the temperature in boundary heat control problems. *Adv. Math.*, 224(1):293–315, 2010.

Existence and properties of weak and very weak solutions (e.g.):

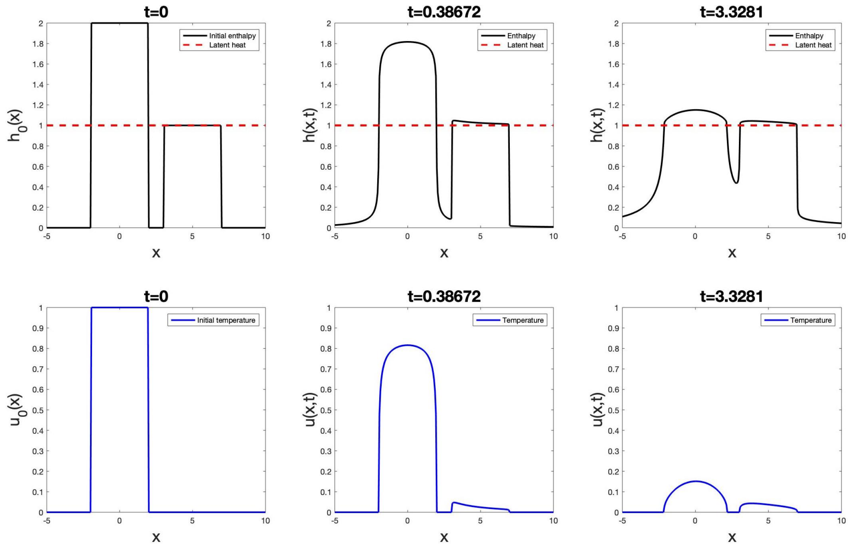
A. de Pablo, F. Quirós, A. Rodríguez and J. L. Vázquez. A general fractional porous medium equation. *Comm. Pure Appl. Math.*, 65(9):1242–1284, 2012.

F. del Teso, JE, and E. R. Jakobsen. Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type. *Adv. Math.*, 305:78–143, 2017.

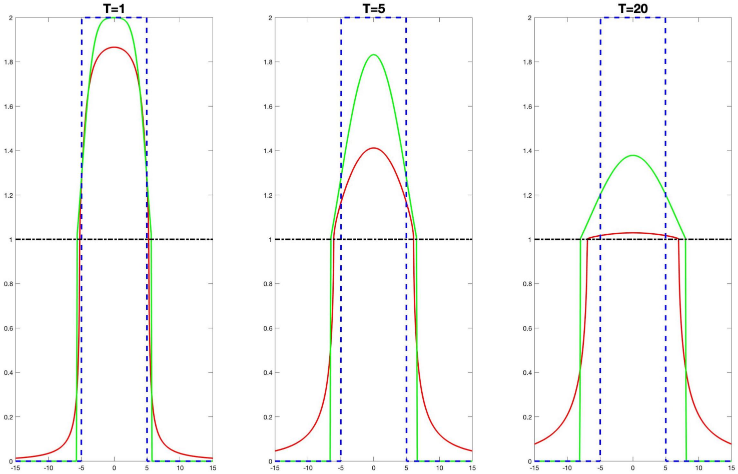
Uniqueness of merely bounded very weak solutions:

G. Grillo, M. Muratori, and F. Punzo. Uniqueness of very weak solutions for a fractional filtration equation. *Adv. Math.*, 365, 107041, 35 pp., 2020.

Still water and ice?



Nonlocal: Initial guesses and thoughts



The numerical solution of the problem

$$\partial_t h + (-\Delta) \setminus (-\Delta)^{\frac{1}{2}} \max\{h-1,0\} = 0.$$

Goals of the talk

- Free boundary of selfsimilar solution given by $x(t) = \xi_0 t^{1/(2s)}$.
- Construct a continuous solution (selfsimilar solution) of (FSP).
- Finite speed of propagation of u, and infinite of h.
- The support of u never recedes.
- Behaviour determined by L.

F. del Teso, JE, and J. L. Vázquez. The one-phase fractional Stefan problem. *Math. Models Methods Appl. Sci.*, 31(1):83–131, 2021.

F. del Teso, JE, and J. L. Vázquez. On the two-phase fractional Stefan problem. *Adv. Nonlinear Stud.*, 20(2):437–458, 2020.

Very weak solutions

Consider very weak solutions of

$$\begin{cases} \partial_t h + (-\Delta)^s u = 0 & \text{in} \quad Q_T := \mathbb{R}^N \times (0,T), \\ h(\cdot,0) = h_0 & \text{on} \quad \mathbb{R}^N. \end{cases}$$

$$\Updownarrow$$

For all $\psi \in C_c^\infty(\mathbb{R}^N \times [0,T))$,

$$\int_0^T \int_{\mathbb{R}^N} \left(h\partial_t \psi - u(-\Delta)^s \psi \right) \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{R}^N} h_0(x)\psi(x,0) \, \mathrm{d}x = 0.$$

Immediate properties

A priori results (dPQuRoVa12, dTEnJa17–19):

- (L^∞-bound) $\|h(\cdot,t)\|_{L^\infty} \leq \|h_0\|_{L^\infty}$
- (Comparison principle) $h_0 \leq \hat{h}_0 \implies h \leq \hat{h}$
- (L^1-contraction) $\int (h(\cdot,t) - \hat{h}(\cdot,t))^+ \leq \int (h_0 - \hat{h}_0)^+$
- (Conservation of mass) $\int h(\cdot,t) = \int h_0$
- (Time regularity) $h \in C([0,T] : L^1_{\text{loc}}(\mathbb{R}^N))$

if $\|h_0(\cdot + \xi) - h_0\|_{L^1(\mathbb{R}^N)} \to 0$ as $|\xi| \to 0^+$

Continuity through approximation (AtCa10):
$u \in C(\mathbb{R}^N \times (0,T))$ with a uniform modulus of continuity for $t \geq \tau > 0$.

OBS: Ok, as long as e.g. $h_0 \in L^\infty$.

Uniqueness (GrMuPu20): If $h_0 \in L^\infty$, then there exists a unique very weak solution h of (FSP) in L^∞.

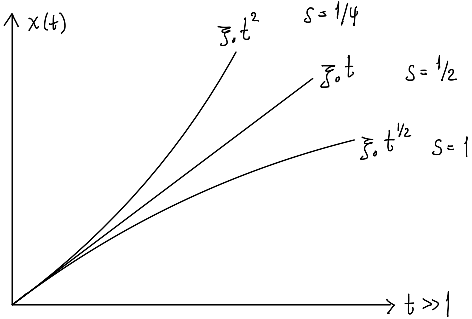
Which special solutions does the equation exhibit?

As the local equation, the nonlocal equation exhibit a special class of solutions of the form

$$H(xt^{-\beta})$$

with $\beta := 1/(2s)$.

Note that $\beta > 1/2$, so that we always have superdiffusion.



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The proof follows from the scaling of the equation:

$$h_0(x) = h_0(ax) \qquad \Longrightarrow \qquad h(x,t) = h(ax, a^{2s}t)$$

for all $a > 0$. In particular for $a = t^{-1/(2s)} > 0$.

Which special solutions does the equation exhibit?

As the local equation, the nonlocal equation exhibit a special class of solutions of the form

$$H(xt^{-\beta}) =: h(xt^{-\beta}, 1)$$

with $\beta := 1/(2s)$.

Note that $\beta > 1/2$, so that we always have superdiffusion.

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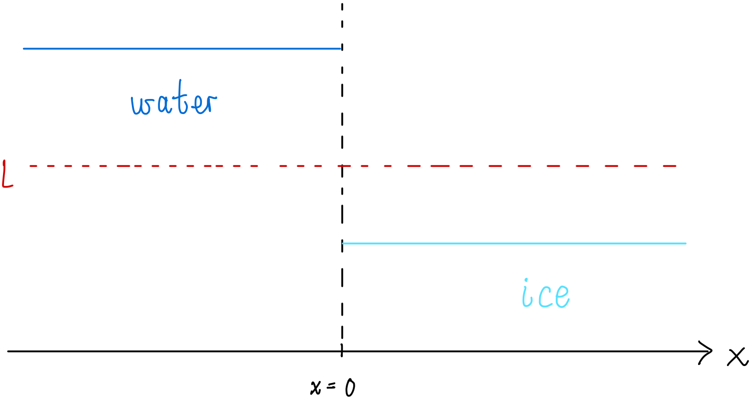
for all $a > 0$. In particular for $a = t^{-1/(2s)} > 0$.

Which solutions do we search for?

When $N = 1$, we can easily choose initial data such that $h_0 = h_0(\cdot a)$. E.g.:



So, we only care about the interphase between water and ice.

Selfsimilar solutions: Elliptic problem in $\mathbb{R}$

For now, fix $N = 1$ and $P_1, P_2 > 0$.

Let h solve (FSP) with initial condition

$$h_0(x) := \begin{cases} L + P_1 & \text{if} \quad x \le 0 \\ L - P_2 & \text{if} \quad x > 0. \end{cases}$$

Then H solves

$$\boxed{-\frac{1}{2s}\xi H'(\xi) + (-\Delta)^s U(\xi) = 0 \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R})}$$

where $U = (H - L)_+$ and $\xi = xt^{-1/(2s)}$.

Immediately, we note that:

- $L - P_2 \le H(\xi) \le L + P_1$ for all $\xi \in \mathbb{R}$.
- $\lim_{\xi \to -\infty} H(\xi) = L + P_1$ and $\lim_{\xi \to +\infty} H(\xi) = L - P_2$.
- H is nonincreasing.

Selfsimilar solutions: Elliptic problem in $\mathbb{R}^N$

In multi-D, we make a constant extension of the 1-D H in the new spatial variables.

So let us focus on the 1-D case.

Goals of the talk

- Free boundary of selfsimilar solution given by $x(t) = \xi_0 t^{1/(2s)}$.
- Construct a continuous solution (selfsimilar solution) of (FSP).
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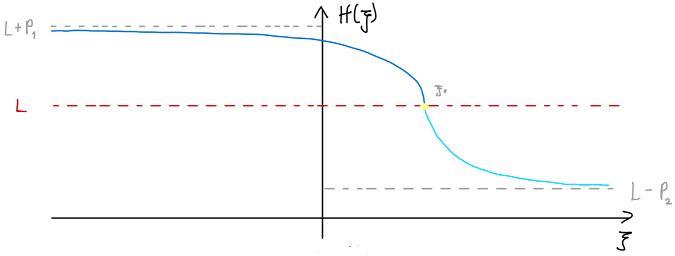
Selfsimilar solutions: Free boundary

Theorem (Free boundary [del Teso & E. & Vázquez, 2021])

There exists a unique finite $\xi_0 > 0$ such that $H(\xi_0^-) = L$. This means that the free boundary of the space-time solution $h(x,t)$ at the level L is given by the curve

$$x(t) = \xi_0 \, t^{\frac{1}{2s}} \qquad \text{for all} \qquad t \in (0, T).$$



Let us argue that H is strictly decreasing in a certain set.

Define

$$D := \{\xi \in \mathbb{R} : H(\xi) \leq L\} = \{\xi \in \mathbb{R} : U(\xi) = 0\}.$$

Assume by contradiction that H is not strictly decreasing in D.

Selfsimilar solutions: Free boundary; Proof

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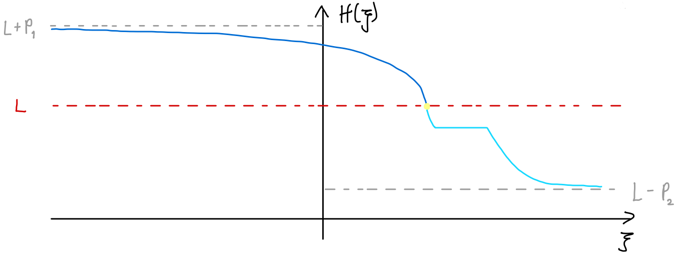


Then H is constant somewhere in D, and $U = 0$ on those parts.

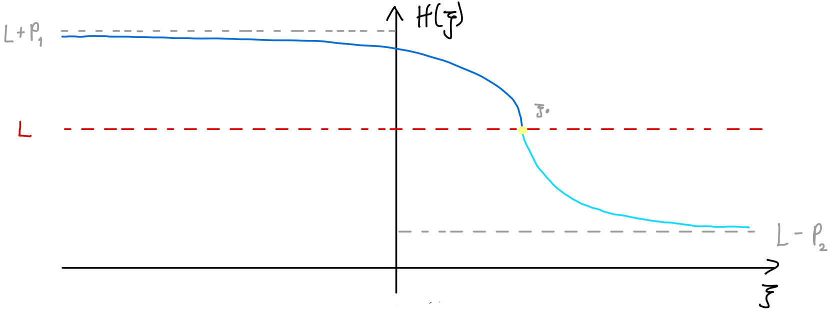
I.e., H, U are regular, $-\frac{1}{2s}\xi H'(\xi) + (-\Delta)^s U(\xi) = 0$, and $H' = 0$.

So, $(-\Delta)^s U = 0$ in "flat part", $U = 0$ in "flat part" to infinity, and $U \geq 0$ and cont.

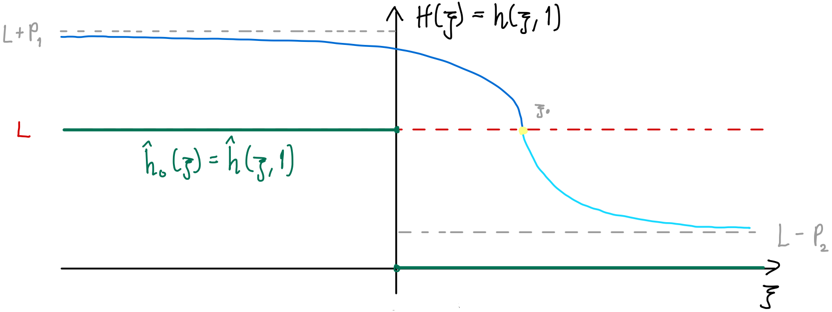
Then $U = 0$ in minus infinity up to "flat part", and $U \equiv 0$.

But $H \to L + P_1$ as $\xi \to -\infty$.

Let us argue that there is a unique interphase point ξ_0.



Let us argue that $\xi_0 \geq 0$.



Selfsimilar solutions: Free boundary; Proof

Let us argue that there is a unique interphase point $\xi_0 > 0$.

Argue by contradiction. If $\xi_0 = 0$, then $U(\xi) \gtrsim |\xi|^s$ for all small enough $\xi < 0$. Which gives H not bounded in $[0, +\infty)$.

Assume that $U(\xi) \gtrsim |\xi|^s$ for $\xi < 0$. Then, for $\xi > 0$,

$$-(-\Delta)^s U(\xi) = c_{1,\alpha} \int_{-\infty}^{0} \frac{U(\eta)}{|\eta - \xi|^{1+2s}} \,\mathrm{d}\eta \gtrsim \int_{-2\xi}^{-\xi} \frac{|\eta|^s}{|\eta - \xi|^{1+2s}} \,\mathrm{d}\eta \sim \frac{1}{|\xi|^s}.$$

Moreover, for $\xi_2 > \xi_1 > 0$, solve $-H'(\xi) = -2s(-\Delta)^s U(\xi)/\xi$:

$$H(\xi_1) = H(\xi_2) + 2s \int_{\xi_1}^{\xi_2} \frac{-(-\Delta)^s U(\eta)}{\eta} \,\mathrm{d}\eta \gtrsim 1 + \int_{\xi_1}^{\xi_2} \frac{\mathrm{d}\eta}{\eta^{1+s}}.$$

Conclusion follows by sending $\xi_1 \to 0^+$.

Selfsimilar solutions: Free boundary; Proof

Let us argue that there is a unique interphase point $\xi_0 > 0$.

Strategy: Argue by contradiction. If $\xi_0 = 0$, then $U(\xi) \gtrsim |\xi|^s$ for all small enough $\xi < 0$. Which gives H not bounded in $[0, +\infty)$:

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Moreover, for $\xi_2 > \xi_1 > 0$, solve $-H'(\xi) = -2s(-\Delta)^s U(\xi)/\xi$:

$$L \geq H(\xi_1) = H(\xi_2) + 2s \int_{\xi_1}^{\xi_2} \frac{-(-\Delta)^s U(\eta)}{\eta} \,\mathrm{d}\eta \gtrsim L - P_2 + \int_{\xi_1}^{\xi_2} \frac{\mathrm{d}\eta}{\eta^{1+s}}.$$

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Selfsimilar solutions: Free boundary; Proof

Let us argue that there is a unique interphase point $\xi_0 > 0$.

If $\xi_0 = 0$, then $U(\xi) \gtrsim |\xi|^s$ for $\xi < 0$.

Fix $\hat{\xi}$, consider $I := [\hat{\xi}, 0]$, and let U^I solve

$$\begin{cases} (-\Delta)^s U^I(\xi) = \frac{1}{2s}\xi H'(\xi) & \text{in} \quad \xi \in I, \\ U^I(\xi) = 0 & \text{in} \quad \xi \in I^c. \end{cases}$$

Since $\xi H' \geq 0$, the Hopf lemma gives

$$U^I(\xi) \gtrsim |\xi|^s \qquad \text{for all} \qquad \xi \in I.$$

X. Ros-Oton. Nonlocal elliptic equations in bounded domains: A survey. *Publ. Mat.*, 60:3–26, 2016.

(The case $U^I \equiv 0$ can be excluded.)

Let us argue that there is a unique interphase point $\xi_0 > 0$.

If $\xi_0 = 0$, then $U(\xi) \geq U^I(\xi) \gtrsim |\xi|^s$ for $\xi < 0$.

$U \geq 0$ and satisfies $(-\Delta)^s U(\xi) = \frac{1}{2s}\xi H'(\xi)$ in $\mathbb{R}$. Then $w := U - U^I$ solves

$$\begin{cases} (-\Delta)^s w(\xi) = 0 & \text{in} \quad \xi \in I, \\ w(\xi) \geq 0 & \text{in} \quad \xi \in I^c, \end{cases}$$

and $w \geq 0$.

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Selfsimilar solutions: Continuity

Theorem (Continuity [del Teso & E. & Vázquez, 2021])

$H \in C_b(\mathbb{R})$. Moreover, $H \in C^{1,\alpha}((-\infty, \xi_0))$ for some $\alpha > 0$, $H \in C^{\infty}((\xi_0, +\infty))$, and

$$(-\Delta)^s U(\xi) = \frac{1}{2s} \xi H'(\xi)$$

is satisfied in the classical sense in $\mathbb{R} \setminus \{\xi_0\}$.

Selfsimilar solutions: Continuity; Proof

- By known results, $H \in C^{1,\alpha}((-\infty, \xi_0))$ for some $\alpha > 0$:

We already know that $U \in C((-\infty, \xi_0)) \cap L^\infty(\mathbb{R}^N)$.
It solves the fractional heat equation.
Then it is C^α away from ξ_0.

L. Silvestre. Hölder estimates for advection fractional-diffusion equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 11(4):843–855, 2012.

Then it is $C^{1,\alpha}$ also. Hence, $H = U + L$ in $(-\infty, \xi_0)$ is $C^{1,\alpha}$.

H. Chang-Lara, G. Dávila. Regularity for solutions of non local parabolic equations. *Calc. Var. Partial Differential Equations*, 49(1–2):139–172, 2014.

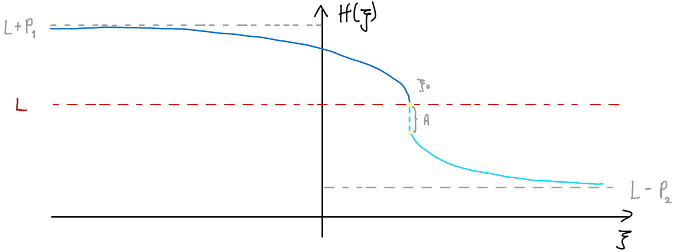
- Let us prove $H \in C^\infty((\xi_0, +\infty))$:

In $[\xi_0, +\infty)$, $U \equiv 0$, and since $0 < U \in L^\infty((-\infty, \xi_0))$, we have $(-\Delta)^s U \in C^\infty((\xi_0, +\infty))$. Then

$$H'(\xi) = 2s\frac{1}{\xi}(-\Delta)^s U(\xi) \qquad \text{holds pointwise in } (\xi_0, +\infty).$$

Selfsimilar solutions: Continuity; Proof

It remains to check that H is continuous at $\xi = \xi_0$.



Let us show that $A = 0$.

Selfsimilar solutions: Continuity; Proof

H is continuous at $\xi = \xi_0 \iff A = 0$.

The equation reads $\xi H'(\xi) = 2s(-\Delta)^s U$ or

$$\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} (H(\xi)\xi)' \, \mathrm{d}\xi = 2s \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} (-\Delta)^s U \, \mathrm{d}\xi + \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} H(\xi) \, \mathrm{d}\xi.$$

The first term is equal to
$H(\xi_0+\varepsilon)(\xi_0+\varepsilon) - H(\xi_0-\varepsilon)(\xi_0-\varepsilon) \to A\xi_0$ as $\varepsilon \to 0^+$.
The third term is bounded by $(L+P_1)2\varepsilon \to 0$ as $\varepsilon \to 0^+$.
We are left with

$$A\xi_0 = 2s \lim_{\varepsilon \to 0^+} \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} (-\Delta)^s U \, \mathrm{d}\xi.$$

Selfsimilar solutions: Continuity; Proof

H is continuous at $\xi = \xi_0 \iff A = 0$.

We are left with

$$A\xi_0 = 2s \lim_{\varepsilon \to 0^+} \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} (-\Delta)^s U \,\mathrm{d}\xi,$$

where (recall that $U = 0$ in $[\xi_0, \infty)$)

$$\begin{aligned}
&\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} (-\Delta)^s U \,\mathrm{d}\xi \\
&= \int_{\xi_0}^{\xi_0+\varepsilon} \int_{-\infty}^{+\infty} \frac{U(\xi) - U(\eta)}{|\xi - \eta|^{1+2s}} \,\mathrm{d}\eta \,\mathrm{d}\xi \\
&\quad + \int_{\xi_0-\varepsilon}^{\xi_0} \int_{-\infty}^{+\infty} \frac{U(\xi) - U(\eta)}{|\xi - \eta|^{1+2s}} \,\mathrm{d}\eta \,\mathrm{d}\xi
\end{aligned}$$

Selfsimilar solutions: Continuity; Proof

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$$\begin{aligned}
&\int_{\xi_0 - \varepsilon}^{\xi_0 + \varepsilon} (-\Delta)^s U \,\mathrm{d}\xi \\
&= \int_{\xi_0}^{\xi_0 + \varepsilon} \int_{-\infty}^{\xi_0} \frac{0 - U(\eta)}{|\xi - \eta|^{1+2s}} \,\mathrm{d}\eta \,\mathrm{d}\xi \\
&\quad + \int_{\xi_0 - \varepsilon}^{\xi_0} \int_{-\infty}^{+\infty} \frac{U(\xi) - U(\eta)}{|\xi - \eta|^{1+2s}} \,\mathrm{d}\eta \,\mathrm{d}\xi
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&\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} (-\Delta)^s U \, \mathrm{d}\xi \\
&= \int_{\xi_0}^{\xi_0+\varepsilon} \int_{-\infty}^{\xi_0} \frac{0 - U(\eta)}{(\xi - \eta)^{1+2s}} \, \mathrm{d}\eta \, \mathrm{d}\xi \\
&\quad + \int_{\xi_0-\varepsilon}^{\xi_0} \int_{-\infty}^{\xi_0-\varepsilon} \frac{U(\xi) - U(\eta)}{(\xi - \eta)^{1+2s}} \, \mathrm{d}\eta \, \mathrm{d}\xi + \int_{\xi_0-\varepsilon}^{\xi_0} \int_{\xi_0-\varepsilon}^{\xi_0} \frac{U(\xi) - U(\eta)}{|\xi - \eta|^{1+2s}} \, \mathrm{d}\eta \, \mathrm{d}\xi \\
&\quad + \int_{\xi_0-\varepsilon}^{\xi_0} \int_{\xi_0}^{+\infty} \frac{U(\xi) - 0}{(\eta - \xi)^{1+2s}} \, \mathrm{d}\eta \, \mathrm{d}\xi
\end{aligned}$$

H is continuous at $\xi = \xi_0 \iff A = 0$.

We are left with

$$A\xi_0 = 2s \lim_{\varepsilon \to 0^+} \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} (-\Delta)^s U \,\mathrm{d}\xi.$$

where (recall that $U = 0$ in $[\xi_0, \infty)$)

$$\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} (-\Delta)^s U \,\mathrm{d}\xi \lesssim \varepsilon^{1-s} + \int_{\xi_0-\varepsilon}^{\xi_0} \int_{\xi_0-\varepsilon}^{\xi_0} \frac{U(\xi) - U(\eta)}{|\xi - \eta|^{1+2s}} \,\mathrm{d}\eta \,\mathrm{d}\xi.$$

Under the assumption $U(z) \lesssim (\xi_0 - z)^s$ when $z \leq \xi_0$.

Selfsimilar solutions: Continuity; Proof

H is continuous at $\xi = \xi_0 \iff A = 0$.

We are left with

$$A\xi_0 = 2s \lim_{\varepsilon \to 0^+} \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} (-\Delta)^s U \,\mathrm{d}\xi.$$

where (recall that $U = 0$ in $[\xi_0, \infty)$)

$$\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} (-\Delta)^s U \,\mathrm{d}\xi \lesssim \varepsilon^{1-s} + \varepsilon^{\alpha+2(1-s)}.$$

Under the assumption $U(z) \lesssim (\xi_0 - z)^s$ when $z \leq \xi_0$.
Under the assumption $U \in C^{1,\alpha}$.

We thus conclude that $A\xi_0 = 0$, i.e., $A = 0$.

Selfsimilar solutions: Continuity; Proof

H is continuous at $\xi = \xi_0 \iff A = 0$.

Why do we have $U(\xi) \lesssim (\xi_0 - \xi)^s$ when $\xi \leq \xi_0$?
Recall that $U(x) = u(x, 1)$ where u satisfies

$$\begin{cases} \partial_t u + (-\Delta)^s u = 0 & \text{in} \quad (-\infty, \xi_0 t^{\frac{1}{2s}}) \times (0,1], \\ u = 0 & \text{in} \quad [\xi_0 t^{\frac{1}{2s}}, +\infty) \times [0,1], \\ u(\cdot, 0) = u_0 & \text{in} \quad (-\infty, \xi_0). \end{cases}$$

Now, if v solves

$$\begin{cases} \partial_t v + (-\Delta)^s v = 0 & \text{in} \quad (-\infty, \xi_0) \times (0,1], \\ v = 0 & \text{in} \quad [\xi_0, +\infty) \times [0,1], \\ v(\cdot, 0) = u_0 & \text{in} \quad (-\infty, \xi_0). \end{cases}$$

Then $0 \leq v(x,t) \lesssim |x - \xi_0|^s$ for $x \leq \xi_0$.

X. Fernández-Real and X. Ros-Oton. Boundary regularity for the fractional heat equation. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 110(1):49–64, 2016.

H is continuous at $\xi = \xi_0 \iff A = 0$.

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To finish, we consider $w = v - u$. It satisfies:

$$\begin{cases} \partial_t w + (-\Delta)^s w \geq 0 & \text{in} \quad (-\infty, \xi_0) \times (0, 1], \\ w = 0 & \text{in} \quad [\xi_0, +\infty) \times [0, 1], \\ w(\cdot, 0) = 0 & \text{in} \quad (-\infty, \xi_0). \end{cases}$$

In $[\xi_0 t^{1/(2s)}, \xi_0] \times (0, 1]$, $u = 0$ and $u \geq 0$ in $\mathbb{R}$ gives $\partial_t u = 0$ and $(-\Delta)^s u \leq 0$ there. Thus, $w \geq 0$.

Goals of the talk

- Free boundary of selfsimilar solution given by $x(t) = \xi_0 t^{1/(2s)}$.
- Construct a continuous solution (selfsimilar solution) of (FSP).
- Finite speed of propagation of u, and infinite of h.
- The support of u never recedes.
- Behaviour determined by L.

F. del Teso, JE, and J. L. Vázquez. The one-phase fractional Stefan problem. *Math. Models Methods Appl. Sci.*, 31(1):83–131, 2021.

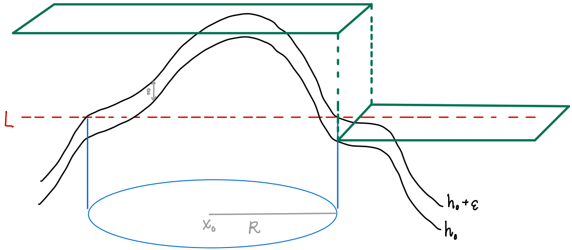
F. del Teso, JE, and J. L. Vázquez. On the two-phase fractional Stefan problem. *Adv. Nonlinear Stud.*, 20(2):437–458, 2020.

Theorem (Finite speed for u, [del Teso & E. & Vázquez, 2021])

Let $h \in L^\infty(Q_T)$ be the very weak solution of (FSP) *with $h_0 \in L^\infty(\mathbb{R}^N)$ as initial data and $u := \Phi(h)$. If $\text{supp}\{\Phi(h_0(x)+\varepsilon)\} \subset B_R(x_0)$ for some $\varepsilon > 0$, $R > 0$, and $x_0 \in \mathbb{R}^N$, then*

$$\text{supp}\{u(\cdot,t)\} \subset B_{R+\xi_0 t^{\frac{1}{2s}}}(x_0) \quad \text{for some } \xi_0 > 0 \text{ and all } t \in (0,T).$$

Proof: Use multi-D special solutions in all directions. Why ε?



Speeds of propagation

Theorem (Finite speed for u, [del Teso & E. & Vázquez, 2021])

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If $\operatorname{supp}\{\Phi(h_0(x)+\varepsilon)\} \subset B_R(x_0)$ for some $\varepsilon > 0$, $R > 0$, and $x_0 \in \mathbb{R}^N$, then

$$\operatorname{supp}\{u(\cdot,t)\} \subset B_{R+\xi_0 t^{\frac{1}{2s}}}(x_0) \quad \text{for some } \xi_0 > 0 \text{ and all } t \in (0,T).$$

Proof: Use multi-D selfsimilar solutions in all directions. Why ε?

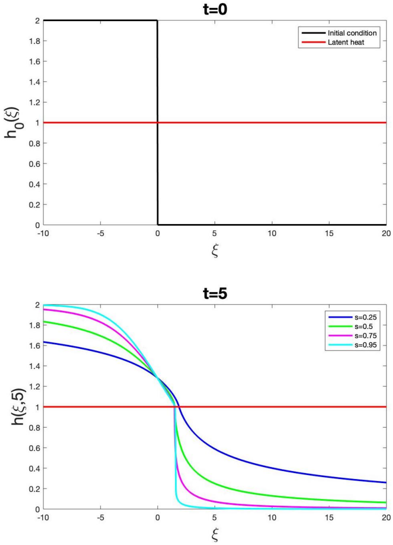
Theorem (Infinite speed for h, [del Teso & E. & Vázquez, 2021])

Let $0 \leq h \in L^\infty(Q_T)$ be the very weak solution of (FSP) *with $0 \leq h_0 \in L^\infty(\mathbb{R}^N)$ as initial data.*
If $h_0 \geq L + \varepsilon > L$ in $B_\rho(x_1)$ for some $\varepsilon > 0$, $\rho > 0$, and $x_1 \in \mathbb{R}^N$, then $h(\cdot,t) > 0$ for all $t \in (0,T)$.

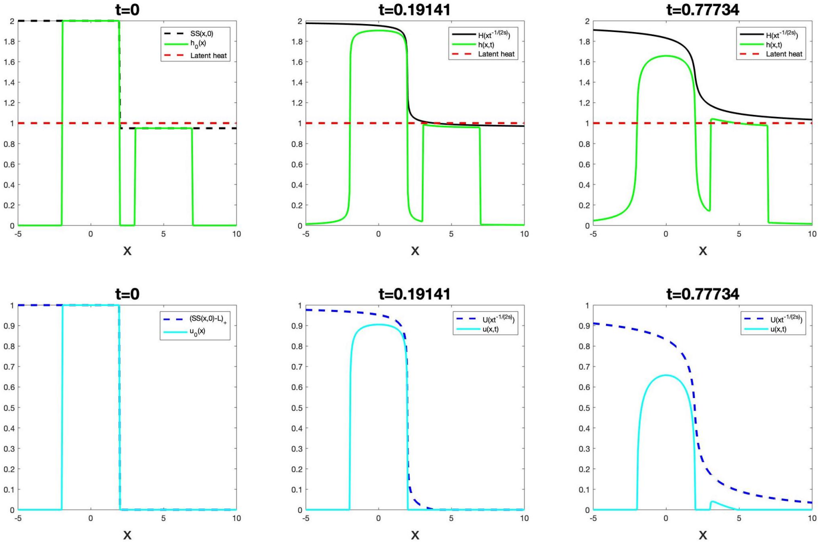
Proof: Show $h(\cdot,t^*) > 0$, then all times $\geq t^*$ by comp.

Speeds of propagation

Free boundary: $x(t) = \xi_0 t^{1/(2s)}$



Speeds of propagation



Goals of the talk

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F. del Teso, JE, and J. L. Vázquez. On the two-phase fractional Stefan problem. Preprint, arXiv:2002.01386v1 [math.AP], 2020.

The support of u never recedes

Theorem (Cons. of positivity, [del Teso & E. & Vázquez, 2021])

If $u(x,t^*) > 0$ in an open set $\Omega \subset \mathbb{R}^N$ for a given time $t^* \in (0,T)$, then

$$u(x,t) > 0 \qquad \text{for all} \qquad (x,t) \in \Omega \times [t^*, T).$$

The same result holds for $t^* = 0$ if $u_0 = \Phi(h_0)$ is continuous in Ω.

Proof: Involved. Use the postive eigenfunction as subsolution.

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Behaviour determined by L

Recall that we solve

$$\partial_t h + (-\Delta)^s \max\{h - L, 0\} = 0.$$

What happens when $L \to 0^+$ or $L \to \infty$?

$L \to 0^+$: It becomes infinitely easy to turn ice into water.

$L \to \infty$: It becomes infinitely hard to turn ice into water.

Behaviour determined by L

Theorem (Limit cases in L, [del Teso & E. & Vázquez, 2021])

Define the initial data

$$h_{0,L} = \begin{cases} L + u_0(x) & \text{in} \quad \Omega, \\ 0 & \text{in} \quad \Omega^c, \end{cases}$$

and let $h_L \in L^\infty(\mathbb{R}^N)$ *be the corresponding very weak solution of* (FSP) *with* $u_L := (h_L - L)_+$.
Then:

- $u_L \to u_{\mathbb{R}^N}$ *pointwise in* Q_T *as* $L \to 0^+$.
- $u_L \to u_\Omega$ *pointwise in* Q_T *as* $L \to +\infty$.
- $u_\Omega \leq u_L \leq u_{\mathbb{R}^N}$.

Thank you for your attention!