On nonlocal (and local) equations of porous medium type

Background, analysis, and numerical simulations

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19 September 2019

In collaboration with
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A talk given at
Seminar, Université de Tours

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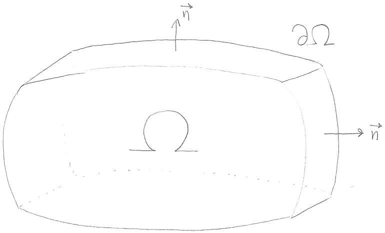
F. del Teso, JE, E. R. Jakobsen. On the well-posedness of solutions with finite energy for nonlocal equations of porous medium type. In *Non-Linear Partial Differential Equations, Mathematical Physics, and Stochastic Analysis. The Helge Holden Anniversary Volume*, pp. 129–168, EMS Series of Congress Reports, 2018.

F. del Teso, JE, E. R. Jakobsen. Robust numerical methods for nonlocal (and local) equations of porous medium type. Part I: Theory. To appear in *SIAM J. Numer. Anal.*, 2019.

F. del Teso, JE, E. R. Jakobsen. Robust numerical methods for nonlocal (and local) equations of porous medium type. Part II: Schemes and experiments. *SIAM J. Numer. Anal.*, 56(6):3611–3647, 2018.

Diffusion is the act of "spreading out" – the movement from areas of high concentration to areas of low concentration.

How do we model this phenomena?



Let u be some heat density inside a region Ω. The rate of change of the total quantity within Ω equals the negative of the net flux through $\partial\Omega$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u \,\mathrm{d}x = -\int_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = -\int_\Omega \mathrm{div}\mathbf{F}\,\mathrm{d}V,$$

or

$$\partial_t u = -\mathrm{div}\mathbf{F}.$$

Introduction: Mathematical modelling

In many situations, $\mathbf{F} \sim Du$, but in the opposite direction (the flow is from high to low consetration):

$$\mathbf{F} = -a(u)Du,$$

and we get

$$\partial_t u = \mathrm{div}(a(u)Du).$$

- **Case 1:** $a(u) = 1$. We obtain the heat equation

$$\partial_t u = \Delta[u]$$

- **Case 2:** $a(u) = u^{m-1}$. We obtain the porous medium equation

$$\partial_t u = \Delta[u^m]$$

J. L. Vázquez. *The porous medium equation. Mathematical theory*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

Introduction: Special case when $m = 6$

It is possible to use

$$\begin{cases} \partial_t u = \Delta[u^6] & \text{in} \quad \mathbb{R}^N \times (0, T), \\ u(x,0) = M\delta_0 & \text{on} \quad \mathbb{R}^N, \end{cases}$$

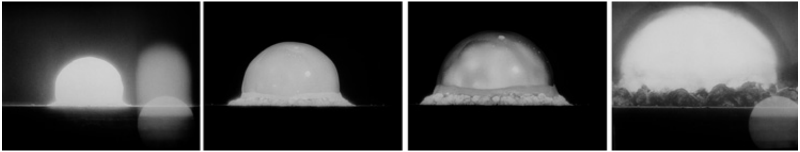
to describe the propagation of heat immediately after a nuclear explosion.

The solution (Barenblatt-solution) will actually be given as

$$t^{-\gamma_1} \max\left\{0, C - k|x|^2 t^{-2\gamma_2}\right\}^{\frac{1}{5}}.$$

G. I. Barenblatt. *Scaling, self-similarity, and intermediate asymptotics*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1996.

Introduction: Special case when $m = 6$



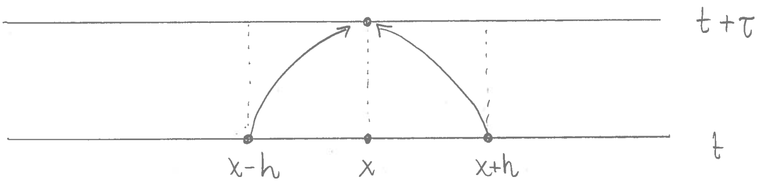
Local vs. nonlocal diffusion, probabilistic view

Let $u(x,t)$ be the probability for a particle to be at discrete $x \in h\mathbb{Z}, t \in \tau\mathbb{N} \cap [0,T]$.

Assume that we are only allowed to jump one point either to the left or to the right, each with probability $\frac{1}{2}$.

The probability of being at point x at time $t+\tau$ is then

$$u(x,t+\tau) = \frac{1}{2}u(x+h,t) + \frac{1}{2}u(x-h,t).$$



Let $u(x,t)$ be the probability for a particle to be at discrete $x \in h\mathbb{Z}$, $t \in \tau\mathbb{N} \cap [0,T]$.

Assume that we are only allowed to jump one point either to the left or to the right, each with probability $\frac{1}{2}$.

Rearrange to get

$$u(x,t+\tau) - u(x,t) = \frac{1}{2}\big(u(x+h,t) + u(x-h,t) - 2u(x,t)\big).$$

Local vs. nonlocal diffusion, probabilistic view

Let $u(x,t)$ be the probability for a particle to be at discrete $x \in h\mathbb{Z}, t \in \tau\mathbb{N} \cap [0,T]$.

Assume that we are only allowed to jump one point either to the left or to the right, each with probability $\frac{1}{2}$.

Choose $\tau = \frac{1}{2}h^2$ and divide by it to obtain

$$\frac{u(x,t+\tau) - u(x,t)}{\tau} = \frac{u(x+h,t) + u(x-h,t) - 2u(x,t)}{h^2}.$$

Let $u(x,t)$ be the probability for a particle to be at discrete $x \in h\mathbb{Z}, t \in \tau\mathbb{N} \cap [0,T]$.

Assume that we are only allowed to jump one point either to the left or to the right, each with probability $\frac{1}{2}$.

As $\tau, h \to 0^+$, we will later see that that u satisfies

$$\partial_t u = \Delta u \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R} \times (0,T)),$$

that is, u is a distributional solution of the heat equation.

A. Einstein. Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Annalen der Physik* (in German), 322(8): 549–560, 1905.

Local vs. nonlocal diffusion, probabilistic view

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

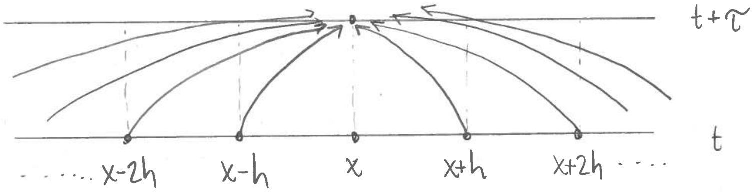
Consider $K : \mathbb{R} \to [0, \infty)$ satisfying

(i) $K(y) = K(-y)$

(ii) $\sum_{k\in\mathbb{Z}} K(k) = 1.$

As before, the probability of being at point x at time $t+\tau$ is

$$u(x, t+\tau) = \sum_{k\in\mathbb{Z}} K(k) u(x+hk, t).$$



Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

Consider $K : \mathbb{R} \to [0, \infty)$ satisfying

(i) $K(y) = K(-y)$

(ii) $\sum_{k\in\mathbb{Z}} K(k) = 1.$

We use property (ii) to obtain

$$u(x, t + \tau) - u(x, t) = \sum_{k\in\mathbb{Z}} K(k)\big(u(x + hk, t) - u(x, t)\big).$$

Local vs. nonlocal diffusion, probabilistic view

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

To continue, we choose K up to normalization factors as

$$K(y) = \begin{cases} \frac{1}{|y|^{1+\alpha}} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

for $\alpha \in (0,2)$.

Divide by τ to obtain

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \sum_{k\in\mathbb{Z}} \frac{K(k)}{\tau}\big(u(x+hk,t)-u(x,t)\big).$$

Local vs. nonlocal diffusion, probabilistic view

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

Choose $\tau = h^\alpha$, and note that

$$\frac{K(k)}{\tau} = \frac{1}{h^\alpha |k|^{1+\alpha}} = \frac{h}{h^{1+\alpha}|k|^{1+\alpha}} = hK(hk).$$

Then

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \sum_{k\in\mathbb{Z}\setminus\{0\}} \big(u(x+hk,t)-u(x,t)\big)K(hk)h.$$

Local vs. nonlocal diffusion, probabilistic view

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

Choose $\tau = h^\alpha$, and note that

$$\frac{K(k)}{\tau} = \frac{1}{h^\alpha |k|^{1+\alpha}} = \frac{h}{h^{1+\alpha}|k|^{1+\alpha}} = hK(hk).$$

Or

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \int_{|z|>0} \big(u(x+z,t)-u(x,t)\big)\,\mathrm{d}\nu_h$$

with the measure $\nu_h(z) := \sum_{k\in\mathbb{Z}\setminus\{0\}} hK(hk)\delta_{hk}(z)$.

Local vs. nonlocal diffusion, probabilistic view

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

As $\tau, h \to 0^+$, we will later see that that u satisfies

$$\partial_t u = \int_{|z|>0} \big(u(x+z,t) - u(x,t) - z\partial_x u(x,t)\mathbf{1}_{|z|\leq 1}\big) \frac{c_{1,\alpha}}{|z|^{1+\alpha}} \, \mathrm{d}z$$
$$= -(-\Delta)^{\frac{\alpha}{2}} u \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R} \times (0,T))$$

where $c_{1,\alpha} > 0$ and $-(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ is the fractional Laplacian. We thus observe that u is a distributional solution of the fractional heat equation.

E. Valdinoci. From the long jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl. SeMA*, (49):33–44, 2009.

Generalized porous medium equations

Let $Q_T := \mathbb{R}^N \times (0, T)$. We consider the following Cauchy problem:

$$\text{(GPME)} \qquad \begin{cases} \partial_t u = \mathcal{L}[\varphi(u)] & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

where

- $$\mathcal{L}[\psi] = \mathcal{L}^\sigma[\psi] + \mathcal{L}^\mu[\psi]$$
  $$= \text{local} + \text{nonlocal} \quad \text{(nonpositive, self-adjoint)}$$
- $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and
- u_0 some rough initial data.

Main results:

- Uniqueness of very weak solutions when $u_0 \in L^\infty$.
- Uniqueness of energy solutions when $\mathcal{L}^\mu$ is nonsymmetric and x-dependent and $u_0 \in L^1 \cap L^\infty$.
- Convergent numerical schemes in $C([0, T]; L^1_{loc}(\mathbb{R}^N))$ when $u_0 \in L^1 \cap L^\infty$.

Assumptions

The assumption

(A_φ) $\qquad \varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing,

includes nonlinearities of the following kind

- linear;
- the porous medium $\varphi(u) = u^m$ with $m > 1$;
- fast diffusion $\varphi(u) = u^m$ with $0 < m < 1$; and
- (one-phase) Stefan problem $\varphi(u) = \max\{0, u - c\}$ with $c > 0$.

Assumptions

The assumption

(A_μ) $\mu \geq 0$ is a symmetric Radon measure on $\mathbb{R}^N \setminus \{0\}$ satisfying

$$\int_{|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} 1 \,\mathrm{d}\mu(z) < \infty.$$

ensures that our $\mathcal{L}^\mu$ includes important examples

- the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$;
- the anisotropic fractional Laplacian $-\sum_{i=1}^N (-\partial^2_{x_i x_i})^{\frac{\alpha_i}{2}}$ with $\alpha_i \in (0,2)$;
- relativistic Schrödinger type operators $m^\alpha I - (m^2 I - \Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ and $m > 0$;
- for the measure ν with $\nu(\mathbb{R}^N) < \infty$, $\mathcal{L}^\nu[\psi](x) = \int_{\mathbb{R}^N} (\psi(x+z) - \psi(x)) \,\mathrm{d}\nu(z)$;
- for the function J with $\int_{\mathbb{R}^d} J(z) \,\mathrm{d}z = 1$, $\mathcal{L}^{J\,\mathrm{d}z}[\psi] = J * \psi - \psi$;
- Fourier multipliers $\mathcal{F}(\mathcal{L}^\mu[\psi]) = -s_{\mathcal{L}^\mu} \mathcal{F}(\psi)$.

Characterization of operators

Theorem

*A **linear**, **self-adjoint** operator which is **translation invariant** and satisfies the **global comparison principle** is of the form* $\mathcal{L} = \mathcal{L}^\sigma + \mathcal{L}^\mu$ *where*

$$\mathcal{L}^\sigma[\psi(x)] := \mathrm{tr}(\sigma\sigma^T D^2\psi(x))$$

$$\mathcal{L}^\mu[\psi(x)] := \mathrm{P.V.}\int_{|z|>0} \big(\psi(x+z) - \psi(x)\big)\,\mathrm{d}\mu(z)$$

Here, $\sigma \in \mathbb{R}^{N\times p}$ *and* $\mu \geq 0$ *is a symmetric Radon measure satisfying*

$$\int \min\{|z|^2, 1\}\,\mathrm{d}\mu(z) < \infty.$$

P. Courrège. Sur la forme intégro-différentielle des opérateurs de C_k^∞ dans C satisfaisant au principe du maximum. *Séminaire Brelot-Choquet-Deny. Théorie du Potentiel*, 10(1):1–38, 1965–1966.

Selective summary of previous results

Local case: $\partial_t u = \Delta u$, $\partial_t u = \Delta u^m$, $\partial_t u = \Delta \varphi(u)$.

J. L. Vázquez. *The porous medium equation. Mathematical theory*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

Nonlocal case: $\partial_t u = \mathcal{L}^{\mu}[\varphi(u)]$.

- Well-posedness when $\mathcal{L}^{\mu} \equiv -(-\Delta)^{\frac{\alpha}{2}}$:

Many people: Vázquez, de Pablo, Quirós, Rodríguez, Brändle, Bonforte, Stan, del Teso, Muratori, Grillo, Punzo, . . .

B. Andreianov and M. Brassart. Uniqueness of entropy solutions to fractional conservation laws with "fully infinite" speed of propagation. Preprint, 2019.

G. Grillo, M. Muratori, and F. Punzo. Uniqueness of very weak solutions for a fractional filtration equation. Preprint, 2019.

Selective summary of previous results

Nonlocal case: $\partial_t u = \mathcal{L}^\mu[\varphi(u)]$.

- Well-posedness for other $\mathcal{L}^\mu$:

Bounded operators

F. Andreu-Vaillo, J. Mazón, J. D. Rossi, and J. J. Toledo-Melero. *Nonlocal diffusion problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

C. Brändle, E. Chasseigne, and F. Quirós. Phase transitions with midrange interactions: a nonlocal Stefan model. *SIAM J. Math. Anal.*, 44(4):3071–3100, 2012.

Fractional Laplace like operators (with some x-dependence)

A. de Pablo, F. Quirós, and A. Rodríguez. Nonlocal filtration equations with rough kernels. *Nonlinear Anal.*, 137:402–425, 2016.

Selective summary of previous results

Previous results (mostly) rely on:

- The porous medium nonlinearity $\varphi(u) = u^m$ with $m > 1$.
- A very restrictive class of Lévy operators.
- The use of L^1-energy solutions.

In our case:

- Uniqueness is hard to prove because of a very weak solution concept and a large class of operators (however, existence is then easier).
- The result we obtain is kind of different since we work in L^∞.
- We can handle very weak assumptions on φ and $\mathcal{L}$.

Distributional (very weak) solution

Definition

Under the assumptions (A_φ), (A_μ), and $u_0 \in L^\infty(\mathbb{R}^N)$, $u \in L^\infty(Q_T)$ is a distributional solution of (GPME) if

$$0 = \int_0^T \int_{\mathbb{R}^N} \Big(u(x,t)\partial_t \psi(x,t) + \varphi(u(x,t))\mathcal{L}^\mu[\psi(\cdot,t)](x) \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}^N} u_0(x)\psi(x,0)\,\mathrm{d}x$$

for all $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^N \times [0,T))$.

Uniqueness

Theorem (Preuniqueness, [del Teso, JE, Jakobsen, 2017])

Assume (A_φ) *and* (A_μ). *Let* $u(x,t)$ *and* $\hat{u}(x,t)$ *satisfy*

$$u, \hat{u} \in L^\infty(Q_T),$$

$$u - \hat{u} \in L^1(Q_T),$$

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = \partial_t \hat{u} - \mathcal{L}^\mu[\varphi(\hat{u})] \qquad \text{in} \qquad \mathcal{D}'(Q_T),$$

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_{\mathbb{R}^N} (u(x,t) - \hat{u}(x,t))\psi(x,t)\,\mathrm{d}x = 0 \quad \forall \psi \in C_c^\infty(\mathbb{R}^N \times [0,T)).$$

Then $u = \hat{u}$ *a.e. in* Q_T.

Uniqueness

Corollary (Uniqueness, [del Teso, JE, Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^\infty(\mathbb{R}^N)$. *Then there is at most one distributional solution* u *of* (GPME) *such that* $u \in L^\infty(Q_T)$ *and* $u - u_0 \in L^1(Q_T)$.

Proof: Assume there are two solutions u and $\hat{u}$ with the same initial data u_0. Then all assumptions of Theorem Preuniqueness obviously hold ($\|u-\hat{u}\|_{L^1} \leq \|u-u_0\|_{L^1} + \|\hat{u}-u_0\|_{L^1} < \infty$), and $u = \hat{u}$ a.e. □

Uniqueness holds for $u_0 \notin L^1$, for example $u_0(x) = c + \phi(x)$ for $c \in \mathbb{R}$ and $\phi \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. However, periodic u_0s are not included because of the assumption $u - u_0 \in L^1$.

Existence

Theorem (Existence, [del Teso, JE, Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then there exists a unique distributional solution* u *of* (GPME) *satisfying*

$$u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N)).$$

Proof: By convergence of numerical solution (as we will see later). □

Note that for the fractional Laplacian/Laplacian, it is possible to obtain existence for pure L^∞-data.

The proof of Theorem Preuniqueness

Based on a proof by Brézis and Crandall.

H. Brézis and M. G. Crandall. Uniqueness of solutions of the initial-value problem for $u_t - \Delta\varphi(u) = 0$. *J. Math. Pures Appl. (9)*, 58(2):153–163, 1979.

1. Define $U := u - \hat{u}$ and $\Phi := \varphi(u) - \varphi(\hat{u})$, then U solves

$$\begin{cases} \partial_t U - \mathcal{L}^\mu[\Phi] = 0 & \text{in} \quad Q_T \\ U(x,0) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases}$$

   Note that $U \in L^1 \cap L^\infty$ and $\Phi \in L^\infty$.

2. Consider

$$\varepsilon v_\varepsilon - \mathcal{L}^\mu[v_\varepsilon] = g \qquad \text{in} \qquad \mathbb{R}^N,$$

   and define $B_\varepsilon^\mu[g] := v_\varepsilon$, that is, $B_\varepsilon^\mu = (\varepsilon I - \mathcal{L}^\mu)^{-1}$ is the resolvent of $\mathcal{L}^\mu$.
   Note that this is a *linear* elliptic equation.

The proof of Theorem Preuniqueness

3. Define

$$h_\varepsilon(t) := \int_{\mathbb{R}^N} U B_\varepsilon^\mu[U] \, \mathrm{d}x = \int_{\mathbb{R}^N} (\varepsilon I - \mathcal{L}^\mu) B_\varepsilon^\mu[U] B_\varepsilon^\mu[U] \, \mathrm{d}x$$
$$= \varepsilon \| B_\varepsilon^\mu[U] \|_{L^2}^2 + \| (-\mathcal{L}^\mu)^{\frac{1}{2}} [B_\varepsilon^\mu[U]] \|_{L^2}^2.$$

4. Show that $h_\varepsilon \to 0$ as $\varepsilon \to 0^+$.
5. $U = (\varepsilon I - \mathcal{L}^\mu) B_\varepsilon^\mu[U] \to 0$ as $\varepsilon \to 0^+$ by Steps 3 and 4.

The proof of Theorem Preuniqueness

The hardest part is to show that $h_\varepsilon \to 0$ as $\varepsilon \to 0^+$. Some important steps:

1. $\varepsilon B^\mu_\varepsilon[U] \to 0$ implies $h_\varepsilon \to 0$ as $\varepsilon \to 0^+$.

We exploit the fact that $h_\varepsilon(t)$ can be written as

$$0 \leq h_\varepsilon(t) = \int_0^t h'_\varepsilon(s)\,\mathrm{d}s$$

where

$$\begin{aligned} h'_\varepsilon(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int U(t)B^\mu_\varepsilon[U(t)]\,\mathrm{d}x = 2\int B^\mu_\varepsilon[U(t)]\partial_t U(t)\,\mathrm{d}x \\ &= 2\int B^\mu_\varepsilon[U(t)]\mathcal{L}^\mu[\Phi(t)]\,\mathrm{d}x = 2\int \mathcal{L}^\mu[B^\mu_\varepsilon[U(t)]]\Phi(t)\,\mathrm{d}x \\ &= 2\int \Big(\varepsilon B^\mu_\varepsilon[U(t)] - U(t)\Big)\Phi(t)\,\mathrm{d}x \leq 2\int \varepsilon B^\mu_\varepsilon[U(t)]\Phi(t)\,\mathrm{d}x. \end{aligned}$$

The proof of Theorem Preuniqueness

The hardest part is to show that $h_\varepsilon \to 0$ as $\varepsilon \to 0^+$. Some important steps:

1. $\varepsilon B^\mu_\varepsilon[U] \to 0$ implies $h_\varepsilon \to 0$ as $\varepsilon \to 0^+$.
2. Enough to prove that $\varepsilon B^\mu_\varepsilon[\gamma] \to 0$ for all $\gamma \in C_\mathrm{c}^\infty(\mathbb{R}^N)$. Note that $\Gamma_\varepsilon := \varepsilon B^\mu_\varepsilon[\gamma]$ solves
$$\varepsilon\Gamma_\varepsilon - \mathcal{L}^\mu[\Gamma_\varepsilon] = \varepsilon\gamma \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^N).$$
3. A priori results and compactness give $\Gamma_\varepsilon \to \Gamma$ as $\varepsilon \to 0^+$.
4. (Liouville) If $\operatorname{supp}\mu \neq \emptyset$, $\Gamma \in C_0$, and $\mathcal{L}^\mu[\Gamma] = 0$ in $\mathcal{D}'$, then $\Gamma \equiv 0$.

Note that a general Liouville result does not hold for $\mathcal{L}^\mu$: Take $\mu(z) = \delta_{2\pi}(z) + \delta_{-2\pi}(z)$, then $\mathcal{L}^\mu[\cos](x) = 0$, but this function is not constant.

Further uniqueness results

By similar methods, we obtain uniqueness in L^∞ for

$$\begin{cases} \partial_t u - \big(\operatorname{tr}(\sigma\sigma^T D^2\varphi(u)) + \mathcal{L}^\mu[\varphi(u)]\big) = 0 & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

and

$$u - \big(\operatorname{tr}(\sigma\sigma^T D^2\varphi(u)) + \mathcal{L}^\mu[\varphi(u)]\big) = g \qquad \text{in} \qquad \mathbb{R}^N.$$

We consider the following Cauchy problem:

(x-GPME)
$$\begin{cases} \partial_t u - A^\lambda[\varphi(u)] = 0 & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

where

- $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and
- A^λ is a x-dependent generalization of $\mathcal{L}^\mu$.

Main results:

- Uniqueness of energy solutions when $u_0 \in L^1 \cap L^\infty$.
- Energy solutions $\iff$ distributional solutions with finite energy.

Uniqueness

Theorem (Uniqueness, [del Teso, JE, Jakobsen, 2018])

Assume (A_φ), $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, *and "*$\lambda$ *satisfies the x-dependent version of* (A_μ)*". Then there is at most one energy solution* u *of* (x-GPME) *in*

$$\{u \in L^1(Q_T) \cap L^\infty(Q_T) \,:\, \varphi(u) \in X\}.$$

Idea of proof: Want $u = v$ when $\varphi(u), \varphi(v) \in \{\text{"finite energy"}\}$ and, for all $\psi \in C_c^\infty(\mathbb{R}^N \times (0,T))$,

$$\int_0^T \left(\int_{\mathbb{R}^N} (u-v)\partial_t \psi \,\mathrm{d}x - \mathcal{E}_\lambda[(\varphi(u)-\varphi(v)),\psi] \right) \mathrm{d}t = 0.$$

The Oleĭnik test function $\int_t^T \big(\varphi(u(x,s)) - \varphi(v(x,s))\big)\,\mathrm{d}s$ ensures that

$$-\int_0^T \mathcal{E}_\lambda[(\varphi(u)-\varphi(v)),\psi]\,\mathrm{d}t \leq 0 \quad \text{i.e.} \quad (u-v)(\varphi(u)-\varphi(v)) = 0.$$

Uniqueness

Theorem (Uniqueness, [del Teso, JE, Jakobsen, 2018])

Assume (A_φ), $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, *and "*$\lambda$ *satisfies the* x*-dependent version of* (A_μ)*". Then there is at most one energy solution* u *of* (x-GPME) *in*

$$\{u \in L^1(Q_T) \cap L^\infty(Q_T) \,:\, \varphi(u) \in X\}.$$

• Under some regularity assumptions on λ, we have

$$X \cap L^2(Q_T) = L^2(Q_T) \cap L^\infty(Q_T) \cap \{\text{"finite energy"}\}.$$

• When λ is bounded above and below by the x-independent measure corresponding to $-(-\Delta)^{\frac{\alpha}{2}}$, then

$$X = L^\infty(Q_T) \cap \{\text{"finite energy"}\}$$

{"finite energy"}

{"finite energy"}

$$:= \left\{ F \, : \, \frac{1}{2}\int_0^T \int_{\mathbb{R}^N} \int_{|z|>0} \left| F(x+z) - F(x) \right|^2 \lambda(x, \mathrm{d}z) \, \mathrm{d}x \, \mathrm{d}t < \infty \right\}$$

The natural energy from the equation which is finite under some assumptions on the initial data.

Special case $\lambda(x, \mathrm{d}z) = \mu(\mathrm{d}z)$

Let us specialize to the case $\lambda(x, \mathrm{d}z) = \mu(\mathrm{d}z)$, that is, $A^\lambda = \mathcal{L}^\mu$.

Theorem (Existence, [del Teso, JE, Jakobsen, 2018])

Assume (A_φ), (A_μ), *and* $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then there exists a distributional solution of* (GPME) *satisfying*

(i) $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$; *and*

(ii) $\varphi(u) \in \{$*"finite energy"*$\}$.

Why?

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}^N} \int_{|z|>0} \left|\varphi(u(x+z)) - \varphi(u(x))\right|^2 \mu(\mathrm{d}z)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \sup_{|r|\leq\|u_0\|_{L^\infty}} \{|\varphi(r)|\} \|u_0\|_{L^1(\mathbb{R}^N)}$$

Final remarks

- Under certain conditions (e.g., $u, u_0 \in L^\infty$), energy solutions $\iff$ distributional solutions with finite energy.
- Energy solutions are distributional solutions with finite energy, and hence, by our first uniqueness result, they are unique without any further requirements on φ.
- Remember that in the second result we needed $\varphi(u) \in X$. Thus, the first result is more robust in the x-independent case, and the second more adapted to the x-dependent case.

$$
\begin{aligned}
\Delta_h \psi(x) &:= \frac{1}{h^2}\Big(\psi(x+h) + \psi(x-h) - 2\psi(x)\Big)\\
&= \int_{\mathbb{R}} \big(\psi(x+z) - \psi(x)\big)\,\mathrm{d}\nu_h(z) =: \mathcal{L}^{\nu_h}[\psi](x)
\end{aligned}
$$

where

$$
\nu_h(z) := \frac{1}{h^2}\Big(\delta_h(z) + \delta_{-h}(z)\Big)
$$

satisfies $\nu_h(\mathbb{R}) < \infty$.

Numerical schemes: Important observations

By now, there exist several spatial discretizations of $\mathcal{L}^\mu$ (e.g. quadrature and spectral methods):

E. R. Jakobsen, K. H. Karlsen, and C. La Chioma Error estimates for approximate solutions to Bellman equations associated with controlled jump-diffusions. *Numer. Math.*, 110(2):221–255, 2008.

Y. Huang and A. Oberman. Finite difference methods for fractional Laplacians. Preprint, arXiv:1611.00164v1 [math.NA], 2016.

Our contribution is to note and exploit that (some of) the discretizations of $\mathcal{L} = \mathcal{L}^\sigma + \mathcal{L}^\mu$ is again a Lévy operator.

Recall that our Cauchy problem was given as

$$\text{(GPME)} \qquad \begin{cases} \partial_t u - \mathcal{L}[\varphi(u)] = 0 & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N. \end{cases}$$

Our numerical scheme can then take the following form

$$\text{(NumGPME)} \quad \begin{cases} \frac{U_\beta^j - U_\beta^{j-1}}{\Delta t} = G_{\Delta x}(U_\beta^j, U_\beta^{j-1}) & \text{in} \quad \Delta x \mathbb{Z}^N \times \Delta t \mathbb{N}, \\ \text{``} U_\beta^0 = u_0 \text{''} & \text{in} \quad \Delta x \mathbb{Z}^N. \end{cases}$$

Numerical schemes for (GPME)

In our most general case, we have that

$$G_{\Delta x}(U_\beta^j, U_\beta^{j-1}) := \mathcal{L}^{\nu_{1,\Delta x}}[\varphi_1(U_\beta^j)] + \mathcal{L}^{\nu_{2,\Delta x}}[\varphi_2(U_\beta^{j-1})]$$

where $\nu_{1,\Delta x}, \nu_{2,\Delta x}$ satisfy $\nu_{1,\Delta x}(\mathbb{R}^N), \nu_{2,\Delta x}(\mathbb{R}^N) < \infty$.

Thus our framework includes

- a mixture of implicit and explicit schemes (θ-methods);
- the possibility of discretizing the singluar and nonsingular parts of $\mathcal{L}^\mu$ in different ways; and
- combinations of the above.

Note that by our previous observations, we are also able to approximate local operators of the form

$$\mathrm{tr}(\sigma\sigma^T D^2 \cdot).$$

Convergence of the numerical schemes

The scheme defined by (NumGPME) is

- monotone,
- (conservative if the φs involved are Lipschitz)
- L^p-stable, and
- consistent.

Theorem (Convergence, [del Teso, JE, Jakobsen, 2018/19])

Assume $\nu_{1,\Delta x}(\mathbb{R}^N), \nu_{2,\Delta x}(\mathbb{R}^N) < \infty$, φ_1, φ_2 satisfy (A_φ)*, and "$U_\beta^0 = u_0$" $\in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then, for the interpolant U, we have*

$$U \to u \qquad \text{in} \qquad C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N)) \qquad \text{as} \qquad \Delta x, \Delta t \to 0^+$$

where $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ is a distributional solution of (GPME).

Numerical simulations: 2D (one-phase) Stefan problem

- 2D (one-phase) Stefan problem with $\varphi(u) = \max\{0, u-1\}$.

Explicit method. $\mathcal{L} = ((\frac{1}{2}, \frac{47}{100}) \cdot D)^2 + (-\partial^2_{xx})^{\frac{1}{4}}$.

Thank you for your attention!