On the duality between Hamilton-Jacobi-Bellmann and convection-diffusion equations

Work in progress

N. Alibaud, J. Endal & E. R. Jakobsen

Department of mathematical sciences
NTNU

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About the speaker

- PhD student (E. R. Jakobsen is my supervisor) at Norwegian University of Science and Technology.
- Currently on a research stay in France (2015/2016).
- URL: `http://www.math.ntnu.no/~jorgeen/`

The equations

$$\text{(CD)} \quad \begin{cases} \partial_t u + \operatorname{div} f(u) = \mathcal{L}^{\mu}[\varphi(u)] & (x,t) \in \mathbb{R}^d \times (0,T) =: Q_T \\ u(x,0) = u_0(x) & x \in \mathbb{R}^d \end{cases}$$

$$\text{(HJB)} \quad \begin{cases} \partial_t \psi = -\operatorname*{ess\,sup}_{\xi \in I} \{ f'(\xi) \cdot D\psi + \varphi'(\xi) \mathcal{L}^{\mu^*}[\psi] \} & \text{in } Q_T \\ \psi(x,T) = \psi_T(x) & \text{in } \mathbb{R}^d \end{cases}$$

Assumptions

- $f = (f_1, f_2, \ldots, f_d) \in W^{1,\infty}_{\text{loc}}(\mathbb{R}; \mathbb{R}^d)$ and $f(0) = 0$;
- $\varphi \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$, nondecreasing ($\varphi' \geq 0$), and $\varphi(0) = 0$;
- $\mathcal{L}^\mu$ is a Lévy operator;
- $u_0 \in L^\infty(\mathbb{R}^d)$ and $\psi_T \in BLSC_c$.

Lévy operator

For $\psi \in C_c^\infty(\mathbb{R}^d)$, we define

$$\mathcal{L}^\mu[\psi](x) := \int_{\mathbb{R}^d \setminus \{0\}} \psi(x+z) - \psi(x) - z \cdot D\psi(x) \mathbf{1}_{|z| \leq 1} \, \mathrm{d}\mu(z),$$

where the measure μ is a nonnegative Radon measure on $\mathbb{R}^d \setminus \{0\}$ which satisfies

$$\int_{|z| \leq 1} |z|^2 \, \mathrm{d}\mu(z) + \int_{|z|>1} 1 \, \mathrm{d}\mu(z) < \infty.$$

Note that when $\frac{\mathrm{d}\mu(z)}{\mathrm{d}z} = \frac{C}{|z|^{d+\alpha}}$ the above operator is the fractional Laplacian.

Notion of solution for (CD)

- If $\mu \equiv 0$ (or $\varphi \equiv 0$), then Kružkov entropy solutions will yield uniqueness; see Kružkov 1970. Note that distributional solutions can develop discontinuities in finite time.
- If $f \equiv 0$, then distributional/weak L^1-energy solutions give uniqueness under some restrictions; see de Pablo, Quirós, Rodríguez & Vázquez 2012, Endal, Jakobsen & del Teso 2015 (preprint), and de Pablo, Quirós & Rodríguez 2015 (preprint).
- However, distributional solutions of the full problem can be nonunique; see Alibaud & Andreianov 2010.
- But Kružkov entropy solutions of the full problem will always be unique; see Alibaud 2007 ($\varphi(u) = u$), Cifani & Jakobsen 2011 ($u_0 \in L^1$), and Endal & Jakobsen 2014 (restrictions on μ).

Notion of solution for (HJB)

Here we assume in addition that $\psi_T \in C_b$.

- Let $\mu^* \equiv 0$.
  - Classical solutions will in general not exist even if ψ_T is smooth (the method of characteristics). Moreover, a.e. solutions may be nonunique.
  - But viscosity solutions do exist and are always unique; see Crandall & Lions 1983.
- When the Lévy operator is replaced by a local second-order operator, viscosity solutions are still unique (under some assumptions); see Jensen 1988 and Ishii 1989.
- Lastly, viscosity solutions for the full problem will always be unique; see Jakobsen & Karlsen 2006 and Barles & Imbert 2008.

Entropy solutions for (CD)

We say that $u \in L^\infty(Q_T) \cap C([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$ is a subsolution (resp. super-) of (CD) if, for all $k \in \mathbb{R}$, $r > 0$ and $0 \leq \psi \in C_c^\infty(Q_T)$,

$$
\begin{aligned}
0 \leq & \iint_{Q_T} (u-k)^\pm \partial_t \psi + q^r_\pm(u,k) \cdot D\psi \,\mathrm{d}x\,\mathrm{d}t \\
& + \iint_{Q_T} \operatorname{sign}^\pm(u-k)\, \mathcal{L}^{\mu,r}[\varphi(u)]\, \psi \,\mathrm{d}x\,\mathrm{d}t \\
& + \iint_{Q_T} (\varphi(u)-\varphi(k))^\pm \mathcal{L}_r^{\mu^*}[\psi] \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where
$q^r_\pm(u,k) := \operatorname{sign}^\pm(u-k)\Big\{ (f(u)-f(k)) + b^{\mu,r}\,(\varphi(u)-\varphi(k)) \Big\}$.

Kato inequality for (CD)

Let u and v be entropy sub- and supersolutions of (CD) with respective initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. Then for a.e. $T > 0$ and all $0 \leq \psi \in C_c^\infty(Q_T)$,

$$0 \leq \iint_{Q_T} (u-v)^+ \partial_t \psi + \mathrm{sign}^+(u-v)\,(f(u)-f(v)) \cdot D\psi \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} (\varphi(u)-\varphi(v))^+ \,\mathcal{L}^{\mu^*}[\psi]\,\mathrm{d}x\,\mathrm{d}t.$$

Main result

Theorem

Let $x_0 \in \mathbb{R}^d$, $R > 0$, $t \in (0, T)$, u, v be entropy solutions of (CD) *with initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ respectively, and Γ be the minimal (discontinuous) viscosity solution of* (HJB) *with terminal data $\Gamma_T(x) = \mathbf{1}_{B(x_0,R)}(x)$. Then*

$$\text{(T-con)} \quad \int_{B(x_0,R)} (u - v)^+(x, t)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} (u_0 - v_0)^+(x)\Gamma(x, 0)\,\mathrm{d}x.$$

Consequences: L^1-contraction, L^1-bound, comparison- and maximum principles, and existence.

Remarks on Γ and (HJB)

NB!! The minimal (discontinuous) viscosity solution Γ is the minimal solution of the best a priori equation (HJB).

Remarks on the main result

- New result for (CD) since it is only assumed that $u_0 \in L^\infty$. Previous assumptions have been $u_0 \in L^\infty \cap L^1$, $u_0 \in L^1$, or $u_0 \in L^\infty$ with restrictions on μ.
- By sending $R \to \infty$ in (T-con) and assuming $(u_0 - v_0)^+ \in L^1$, we get $\|(u-v)^+\|_{L^1(\mathbb{R}^d)} \leq \|(u_0 - v_0)^+\|_{L^1(\mathbb{R}^d)}$.
- We connect the notion of viscosity solutions to the notion of entropy solutions.
- Equation (T-con) is the best possible a priori L^1-contraction estimate.

Plan for the rest of the talk

We obtain the L^1-contraction from the Kato inequality using that Γ is a minimal viscosity solution of (HJB) with terminal data $\mathbf{1}_{B(x_0,R)}$.

Choosing a special test function

Let us recall the Kato inequality

$$0 \le \iint_{Q_T} (u-v)^+ \partial_t \psi + \mathrm{sign}^+(u-v)\,(f(u)-f(v)) \cdot D\psi \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{Q_T} (\varphi(u)-\varphi(v))^+ \,\mathcal{L}^{\mu^*}[\psi]\,\mathrm{d}x\,\mathrm{d}t.$$

Choose $\psi(x,t) = \Theta(t)\Gamma(x,t)$, and insert it into the above to get

$$0 \le \iint_{Q_T} (u-v)^+ \Gamma\Theta' \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{Q_T} \Theta \Big[(u-v)^+ \partial_t \Gamma + \mathrm{sign}^+(u-v)(f(u)-f(v)) \cdot D\Gamma$$
$$+ (\varphi(u)-\varphi(v))^+ \mathcal{L}^{\mu^*}[\Gamma] \Big] \,\mathrm{d}x\,\mathrm{d}t.$$

Choosing a special test function

Let us recall the Kato inequality

$$0 \leq \iint_{Q_T} (u-v)^+ \partial_t \psi + \mathrm{sign}^+(u-v)\left(f(u)-f(v)\right) \cdot D\psi \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{Q_T} (\varphi(u)-\varphi(v))^+ \, \mathcal{L}^{\mu^*}[\psi] \,\mathrm{d}x\,\mathrm{d}t.$$

Choose $\psi(x,t) = \Theta(t)\Gamma(x,t)$, and insert it into the above to get

$$0 \leq \iint_{Q_T} (u-v)^+ \Gamma \Theta' \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{Q_T} \Theta \Big[(u-v)^+ \partial_t \Gamma + \mathrm{sign}^+(u-v)(f(u)-f(v)) \cdot D\Gamma$$
$$+ (\varphi(u)-\varphi(v))^+ \mathcal{L}^{\mu^*}[\Gamma] \Big] \,\mathrm{d}x\,\mathrm{d}t.$$

Rewriting the terms

Note that

$$(\varphi(u)-\varphi(v))^+ = (u-v)^+\int_0^1 \varphi'(v+r(u-v))\,\mathrm{d}r$$

$$\operatorname{sign}^+(u-v)(f(u)-f(v)) = (u-v)^+\int_0^1 f'(v+r(u-v))\,\mathrm{d}r.$$

Hence,

$$0 \le \iint_{Q_T}(u-v)^+\Gamma\Theta' + \iint_{Q_T}\Theta(u-v)^+\times$$

$$\times\left[\partial_t\Gamma + \int_0^1 f'(v+r(u-v))\cdot D\Gamma + \varphi'(v+r(u-v))\mathcal{L}^{\mu^*}[\Gamma]\,\mathrm{d}r\right].$$

And finally, by taking the essential supremum, we get

$$0 \leq \iint_{Q_T} (u-v)^+ \Gamma \Theta' \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \Theta (u-v)^+ \Big[\partial_t \Gamma + \operatorname*{ess\,sup}_{\xi \in I} \{ f'(\xi) \cdot D\Gamma + \varphi'(\xi) \mathcal{L}^{\mu^*}[\Gamma] \} \Big] \,\mathrm{d}x\,\mathrm{d}t.$$

Moreover,

$$I = (\min\{\operatorname{ess\,inf} v, \operatorname{ess\,inf} u\}, \max\{\operatorname{ess\,sup} v, \operatorname{ess\,sup} u\}).$$

Choosing Θ

The point now, is that we, formally, choose Θ as an approximation of a square pulse

$$0 \leq \iint_{Q_T} (u-v)^+ \Gamma \omega_\varepsilon(t-t_1)\,\mathrm{d}x\,\mathrm{d}t - \iint_{Q_T} (u-v)^+ \Gamma \omega_\varepsilon(t-t_2)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{Q_T} \Theta_\varepsilon (u-v)^+ \Big[\partial_t \Gamma + \operatorname*{ess\,sup}_{\xi \in I} \{ f'(\xi) \cdot D\Gamma + \varphi'(\xi) \mathcal{L}^{\mu^*}[\Gamma] \} \Big]\,\mathrm{d}x\,\mathrm{d}t.$$

Choosing Θ

The point now, is that we, formally, choose Θ as an approximation of a square pulse and sending $\varepsilon \to 0^+$ we get Dirac's deltas at, say, $t = t_1$ and $t = t_2$. Then we let $t_1 \to 0^+$ and $t_2 \to T^-$:

$$
\begin{aligned}
0 \leq & \int_{\mathbb{R}^d} (u_0 - v_0)^+ \Gamma(x, 0) \, \mathrm{d}x - \int_{\mathbb{R}^d} (u - v)^+ \Gamma(x, T) \, \mathrm{d}x \\
& + \iint_{Q_T} 1 (u - v)^+ \Big[\partial_t \Gamma + \operatorname*{ess\,sup}_{\xi \in I} \{ f'(\xi) \cdot D\Gamma + \varphi'(\xi) \mathcal{L}^{\mu^*}[\Gamma] \} \Big] \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
$$

Choosing Γ

The point now, is that we, formally, choose Θ as an approximation of a square pulse and sending $\varepsilon \to 0^+$ we get Dirac's deltas at, say, $t = t_1$ and $t = t_2$. Then we let $t_1 \to 0^+$ and $t_2 \to T^-$.

Choose Γ as the minimal viscosity solution of (HJB) with terminal data $\mathbf{1}_{B(x_0,R)}$ to get

$$\int_{B(x_0,R)} (u-v)^+ \,\mathrm{d}x \leq \int_{\mathbb{R}^d} (u_0 - v_0)^+ \Gamma(x,0)\,\mathrm{d}x.$$

Thank you for your attention!