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Walking randomly in the realm of partial differential equations and probability theory

The Laplace operator and Brownian motion

Department of mathematical sciences

12 March 2014

Introduction

We study the following Cauchy problem (heat or diffusion equation)

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u & (x,t) \in \mathbb{R}^d \times (0,\infty) \\ u(x,0) = \delta_0(x) & x \in \mathbb{R}^d \end{cases}, \tag{1}$$

where δ_0 is the Dirac delta centered at the origin. A solution of (1) is called a **fundamental** solution.

Fundamental solution

Let

$$\hat{\phi}(\xi) := \mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} \phi(x) \, \mathrm{d}x,$$

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$$\frac{\partial \hat{u}}{\partial t} = -\frac{1}{2}|\xi|^2 \hat{u}$$

$$\hat{u}(\xi, t) = C e^{-\frac{1}{2}t|\xi|^2},$$

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and using that $\mathcal{F}(\delta_0) = (2\pi)^{-\frac{d}{2}}$ we get

$$\hat{u}(\xi, t) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{2}t|\xi|^2}. \qquad (2)$$

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$$\begin{aligned} u(x,t) =& \mathcal{F}^{-1}((2\pi)^{-\frac{d}{2}} \mathrm{e}^{-\frac{1}{2}t|\cdot|^2})(x) \\ =& (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\xi\cdot x} (2\pi)^{-\frac{d}{2}} \mathrm{e}^{-\frac{1}{2}t|\xi|^2} \, \mathrm{d}\xi \\ =& \frac{1}{(2\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x|^2}{2t}}. \end{aligned} \tag{3}$$

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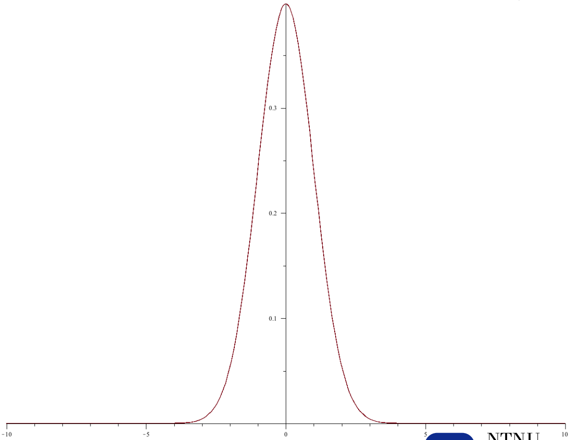
Hopefully this is a well-known function! It is e.g. the probability density function of the normal distribution (with $\mu = 0$ and $\Sigma = tI$).

Normal distribution (or Gaussian distribution) with $\mu = 0$ and $t = 1$



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Family of probability measures

Consider the family $\{Q_t, t \geq 0\}$ where "$Q_t(\mathrm{d}x) = q_t(x)\,\mathrm{d}x$".

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i) $q_t(x)$ is non-negative; and

ii) $\int_{\mathbb{R}^d} q_t(x)\,\mathrm{d}x = 1$,

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and it does. In addition, $q_t(x)$ satisfies

iii) $q_{t+s}(x) = (q_t * q_s)(x)$; and
iv) Q_t is weakly convergent to δ_0.

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(All of these properties can be proved using (3).)

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Think of this as assigning a number to each outcome of an experiment.

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Stochastic process

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A **stochastic process** is parametrized collection of random variables

$$\{X_t\}_{t\in T}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^d$.

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Note that $X_t = X_t(\omega)$ where $\omega \in \Omega$. Here it is useful to think of t as time, and ω as a particle (or an experiment). Then $t \mapsto X_t(\omega)$ would represent the position (or the result) as a function of time t of the particle (experiment) ω.

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Transition probabilities

Definition

For each $0 \leq s \leq t < \infty$, $B \in \mathcal{B}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, we define

$$P_{s,t}(x, B) = P(X_t \in B | X_s = x)$$

as the **transition probability.**

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Note that $P_{s,t}$ gives the probability of going from the point x at time s to the set B at time t.

Chapman-Kolmogorov equations

If we let

$$P_{s,t}(x, \mathrm{d}y) = q_{t-s}(y-x)\,\mathrm{d}y = (2\pi(t-s))^{-\frac{d}{2}} \mathrm{e}^{-\frac{|y-x|^2}{2(t-s)}}\,\mathrm{d}y,$$

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then the transition probabilities satisfies (remember that $q_{t+s} = q_t * q_s$)

$$P_{r,t}(x, B) = \int_{\mathbb{R}^d} P_{s,t}(y, B) P_{r,s}(x, \mathrm{d}y) \tag{4}$$

for all $0 \le r \le s \le t < \infty$, $x \in \mathbb{R}^d$, and $B \in \mathcal{B}(\mathbb{R}^d)$.

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Finite-dimensional distribution

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&= \int_{B_1\times\ldots\times B_n} q_{t_1}(x_1)\cdots q_{t_n-t_{n-1}}(x_n - x_{n-1})\,\mathrm{d}x_1 \ldots \mathrm{d}x_n \\
&= \int_{B_1\times\ldots\times B_n} (2\pi(t_1))^{-\frac{d}{2}} e^{-\frac{|x_1|^2}{2(t_1)}} \\
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\end{aligned} \tag{5}$$

Note that $\nu_{t_1,\ldots,t_n}$ answers the question "what is the probability of $X_{t_1} \in B_1$, and, ..., and $X_{t_n} \in B_n$?". Hence, it is closely related to transition probabilities.

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Kolmogorov's existence theorem

Theorem

Given a family of probability measures $\{\nu_{t_1,\dots,t_n}, t_i \in \mathbb{R}^+ \text{ and } n \in \mathbb{N}\}$ *satisfying the Kolmogorov consistency criteria. Then there exists a probability space* $(\Omega, \mathcal{F}, P)$ *and a stochastic process* $\{X_t\}_{t\geq 0}$ *on* Ω*;* $X_t : \Omega \to \mathbb{R}^d$ *such that*

$$\nu_{t_1,\dots,t_n}(B_1 \times \cdots \times B_n) = P(X_{t_1} \in B_1, \dots, X_{t_n} \in B_n),$$

for all $t_i \in \mathbb{R}^+$*,* $k \in \mathbb{N}$ *and all Borel sets* B_i*.*

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Existence of a stochastic process

We "know" that $\nu_{t_1,\ldots,t_n}$ defined by (5) satisfies Kolmogorov's consistency criteria. Hence, there exists a stochastic process $\{X_t\}_{t\geq 0}$ on Ω such that

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that is, the stochastic process X_t has $\nu_{t_1,\dots,t_n}$ as its finite dimensional distribution.

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Brownian motion

Definition

A stochastic process $B_t : \Omega \times [0, \infty) \to \mathbb{R}^d$ is called **Brownian motion** if

i) $B_0 = 0$

ii) $B_{t_n} - B_{t_{n-1}}$ is $\mathcal{N}(0, (t_n - t_{n-1})I)$

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The process X_t with finite dimensional distribution given by (5) satisfies all of these axioms! That is, we have constructed Brownian motion using the fundamental solution of (1).

Why??

Let us write down the finite dimensional distribution

$$\int_{B_1\times\ldots\times B_n}(2\pi(t_1))^{-\frac{d}{2}}e^{-\frac{|x_1|^2}{2(t_1)}}$$

$$\cdots(2\pi(t_n-t_{n-1}))^{-\frac{d}{2}}e^{-\frac{|x_n-x_{n-1}|^2}{2(t_n-t_{n-1})}}\,\mathrm{d}x_1\ldots\mathrm{d}x_n.$$

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Consider the first axiom; $B_0=0$.

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$$\int_{B_1 \times \ldots \times B_n} (2\pi(t_1))^{-\frac{d}{2}} e^{-\frac{|x_1|^2}{2(t_1)}} \cdots (2\pi(t_n - t_{n-1}))^{-\frac{d}{2}} e^{-\frac{|x_n - x_{n-1}|^2}{2(t_n - t_{n-1})}} \, \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

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Consider the second axiom; $B_{t_n}-B_{t_{n-1}}$ is $\mathcal{N}(0,(t_n-t_{n-1})I)$.

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Consider the third axiom; $B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ is independent.

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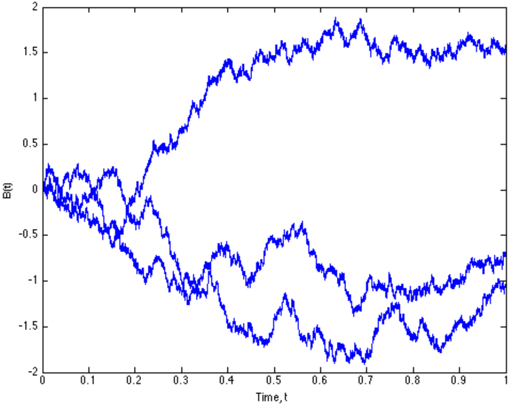
Figure of Brownian motion in $\mathbb{R}^1$



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v) $\frac{1}{c} B_{c^2 t}$ is also a Brownian motion; it is scalar invariant.

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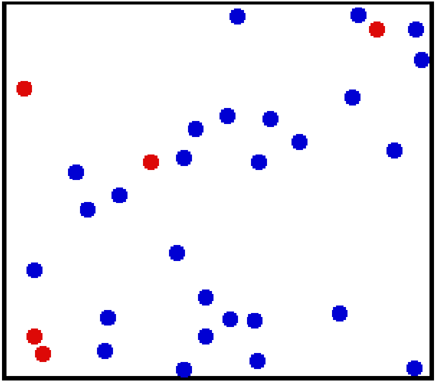
Pollen particles

In 1827, Robert Brown studied pollen grains suspended in water under a microscope.



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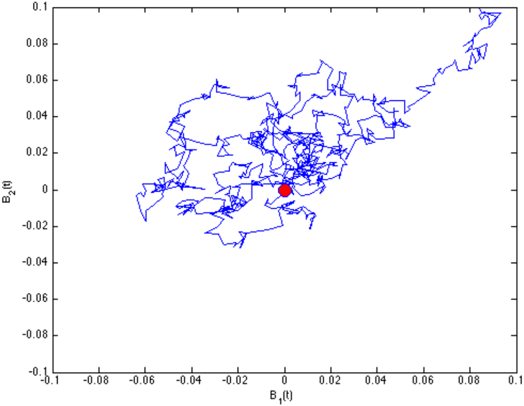
We then know that the path of one of these particles ω will be given by $(B_1(t,\omega), B_2(t,\omega))$; two dimensional Brownian motion.

Pollen particles



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Kolmogorov's forward equation

(Also called the Fokker-Planck equation.)

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Assume that B_t has a nice, smooth transition probability density $p_{s,t}(x,y)$, that is,

$$P(B_t \in B | B_0 = 0) = \int_B p_{s,t}(0,y)\,\mathrm{d}y \qquad \forall B \in \mathcal{B}(\mathbb{R}^d).$$

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Then this density will satisfy

$$\begin{cases} \partial_t p = \frac{1}{2}\Delta_y p & (y,t) \in \mathbb{R}^d \times (0,\infty) \\ p_0(0,y) = \delta_0(y) & (y,t) \in \mathbb{R}^d \times \{0\} \end{cases}.$$

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Fractional Laplace

We turn our attention to the following Cauchy problem (heat or diffusion equation)

$$\begin{cases} \partial_t u = -(-\Delta)^{\frac{\alpha}{2}} u & (x,t) \in \mathbb{R}^d \times (0,\infty) \\ u(x,0) = \delta_0(x) & x \in \mathbb{R}^d \end{cases}, \tag{6}$$

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where δ_0 is the Dirac delta centered at the origin, and $-(-\Delta)^{\frac{\alpha}{2}}$ is defined by the Fourier transform

$$\mathcal{F}(-(-\Delta)^{\frac{\alpha}{2}}\phi)(\xi) = -|\xi|^\alpha \mathcal{F}(\phi)(\xi) \tag{7}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$.

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Lévy processes

By more theoretically advanced theory, we can do similar computations as in the case of the Laplace operator.

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$$u(x,t) = \mathcal{F}^{-1}((2\pi)^{-\frac{d}{2}} e^{-t|\xi|^{\alpha}})(x) \quad \alpha \in (0,2).$$

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Observe that if we take $\alpha = 2$ in the above equation, we get Brownian motion up to some constant (there is a one-half missing).

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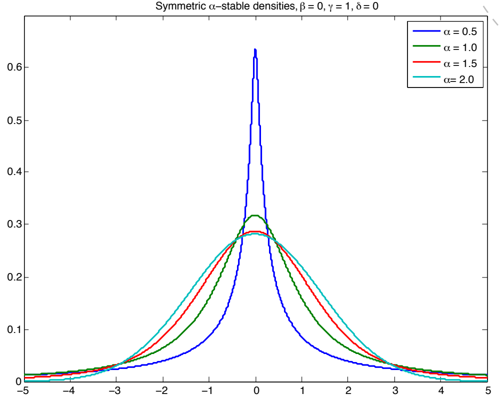
α-stable distributions with $\mu = 0$



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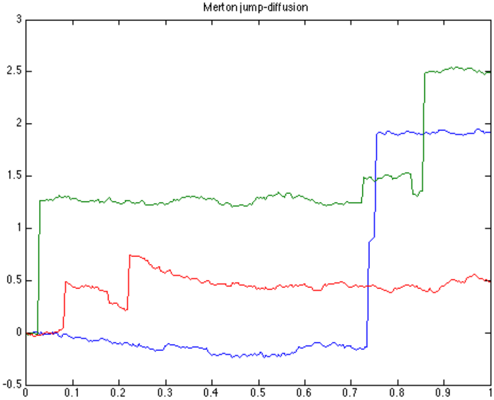
Figures of Lévy processes



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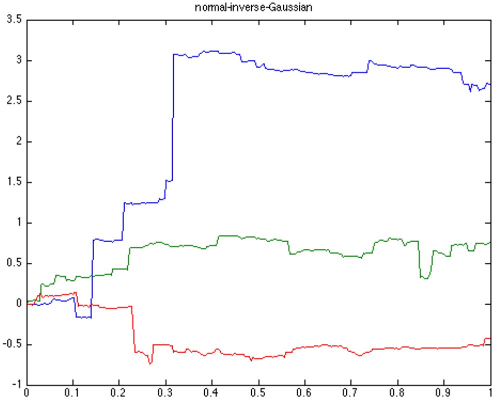

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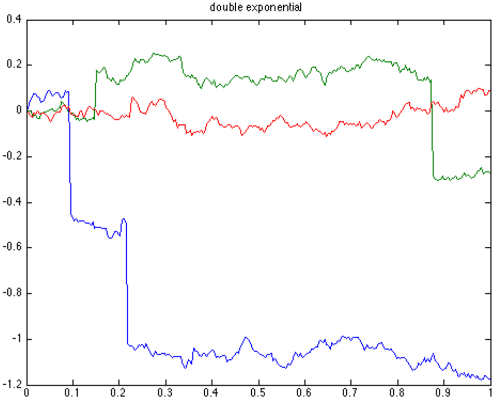

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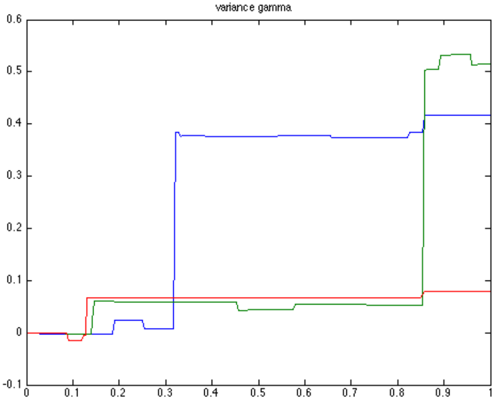

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The Black-Scholes option pricing model

Let us look at an European call option. That is, the right to buy one stock for K NOK at a fixed time $T > t$. We call K the strike price (the agreed price) and S_t the spot price (the price of the stock at time t). If we are lucky we earn $S_T - K$ NOK, so the pay-off is $\max\{S_T - K, 0\}$.

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The problem is how much does this European call option cost? Or, how do we get a good estimate on S_t?

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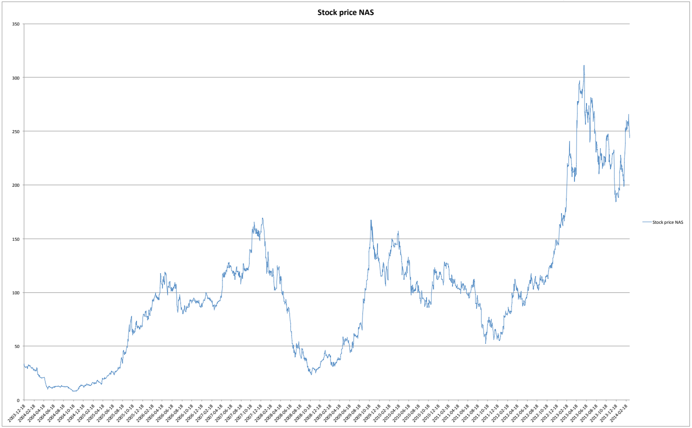

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In 1997, Fischer Black and Myron Scholes won the Nobel Prize in Economics for the equation

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 & t \in [0, T) \\ V(S, T) = \max\{S - K, 0\} & t = T \end{cases},$$

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where $V = V(S, t)$ is the price of the option, r is the risk-free interest rate, and σ is the volatility of the stock.

The solution of this problem is given by the Feynman-Kac formula

$$V(S, t) = E\left[\mathrm{e}^{-r(T-t)} \max\{S_T^{t,x} - K, 0\}\right].$$

The Black-Scholes option pricing model

In the previous slide, S_t was actually modelled as geometric Brownian motion:

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}B_t.$$

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This model has a lot of weaknesses. One crucial example is that Browninan motion has a continuous version; so, this model cannot model sudden jumps in price (which we know occurs in real life). Furthermore, since the tail of a gaussian distribution is very thin, the probability of extreme events is very low.

This is where Lévy processes enter. We allow sudden discontinuous jumps in such processes, and this is a very useful tool when modelling stock prices.

The Black-Scholes option pricing model

As a consequence of this fact, it is common nowadays (at least in finance) to add

$$\int_{|y|>0} u(x+y,t) - u(x,t) - (e^y - 1)\partial_x u(x,t)\nu(\,\mathrm{d}y)$$

to the Black-Scholes equation.

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The solution is still given by a formula similar to Feynman-Kac', but S_t is now a Lévy process.

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