What is a solution of a PDE? Current concepts used in research beyond classical solutions, why are they needed, how are they related.

Trial lecture

Jørgen Endal

Department of mathematical sciences
NTNU

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Nonlinear partial differential equations

All happy families resemble one another, but each unhappy family is unhappy in its own way.

— Tolstoy, *Anna Karenina*

Nonlinear partial differential equations

> *In contrast to the well-understood (and well-studied) classes of linear partial differential equations, each nonlinear equation presents its own particular difficulties.*
>
> — Holden & Risebro, *Front Tracking for Hyperbolic Conservation Laws*

Scalar conservation laws

Consider the Cauchy problem

$$\text{(SCL)} \qquad \begin{cases} \partial_t u + \partial_x(F(u)) = 0 & \text{in} \quad \mathbb{R} \times (0, \infty), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}, \end{cases}$$

where

$$F, u_0 : \mathbb{R} \to \mathbb{R}$$

are functions to be determined.

Contents

Why do we study (SCL)?

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} u(x,t)\,\mathrm{d}x = F(u(x_1,t)) - F(u(x_2,t))$$

What do we want from a solution? Classical well-posedness

Mathematical models of physical phenomena should have properties such that

- a solution exists;
- the solution is unique; and
- the solution's behaviour changes continuously with the initial conditions.

Definition of classical solutions

Definition (0 Classical solutions)

Assume $u_0 \in C^1(\mathbb{R})$ and $F \in C^1(\mathbb{R})$. A function $u \in C^1(\mathbb{R} \times [0, \infty))$ is called a classical solution of (SCL) if it satisfies the problem pointwise everywhere.

Do classical solutions exist?

Method of characteristics: $t = t(\xi,\eta)$, $x = x(\xi,\eta)$, $z = z(\xi,\eta)$

$$\begin{cases} \partial_\xi t = 1, \quad \partial_\xi x = u, \quad \partial_\xi z = 0 \\ t(0,\eta) = 0, \quad x(0,\eta) = \eta, \quad z(0,\eta) = u_0(\eta) \end{cases}$$

$$u(x,t) = z(\xi(x,t),\eta(x,t))$$

Implicit function theorem yields $u_0'(\eta)t + 1 \neq 0 \impliedby u_0' \geq 0$.

$$t^* = -\frac{1}{u_0'(\tilde{x})}$$

Definition of distributional solutions

Definition (1 Distributional solutions)

Assume $u_0 \in L^1_{\text{loc}}(\mathbb{R})$ and $F \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$. A function $u \in L^1_{\text{loc}}(\mathbb{R} \times (0,\infty))$ is called a distributional solution of (SCL) if

$$\partial_t u + \partial_x F(u) = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times [0,\infty)).$$

$$\iint_{\mathbb{R}\times(0,\infty)} u\partial_t \phi + F(u)\partial_x \phi \,\mathrm{d}x\,\mathrm{d}t = 0 \qquad \forall \phi \in C_c^\infty(\mathbb{R} \times (0,\infty))$$

$$x'(t)(u_l - u_r) = F_l - F_r \qquad \text{where} \qquad x(t) \text{ is a shock wave}$$

Nonuniqueness

Consider (SCL) with $F(u) = \frac{1}{2}u^2$ with the initial data

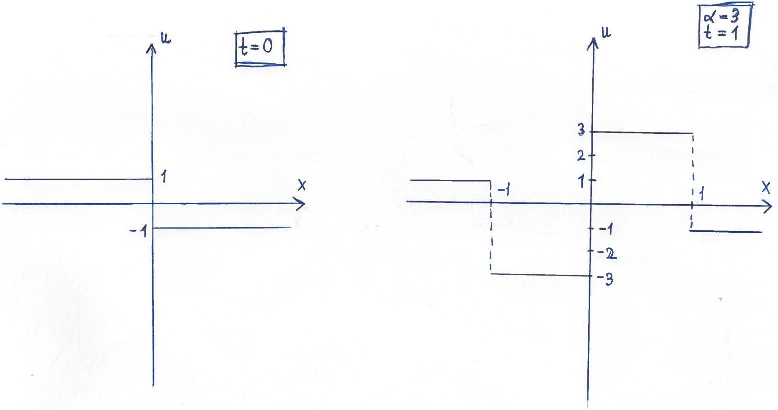
$$u_0(x) = \begin{cases} 1, & x < 0, \\ -1, & x > 0. \end{cases}$$

For each $\alpha \geq 1$,

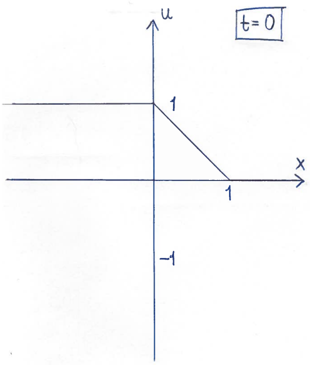
$$u(x,t) = \begin{cases} 1, & 2x < (1-\alpha)t, \\ -\alpha, & (1-\alpha)t < 2x < 0, \\ \alpha, & 0 < 2x < (\alpha-1)t, \\ -1, & (\alpha-1)t < 2x, \end{cases}$$

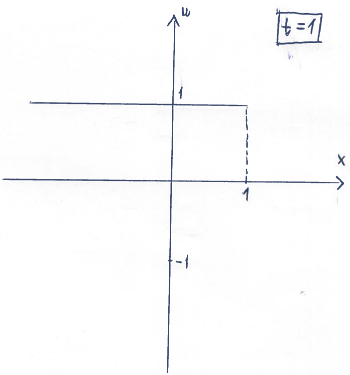
is a distributional solution.

Nonuniqueness



Too many solutions





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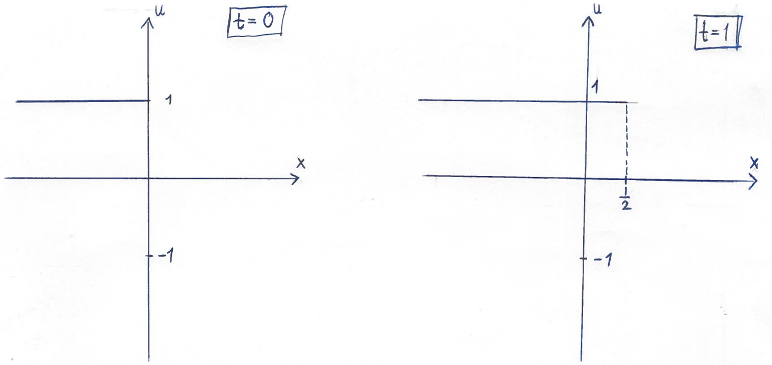

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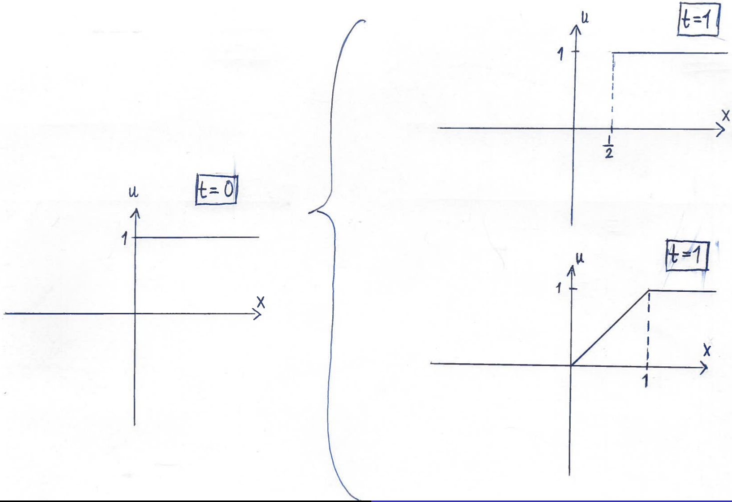

Definition of entropy solutions

Definition (2 Entropy solutions)

Assume $u_0 \in L^\infty(\mathbb{R})$ and $F \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$. A function $u \in L^\infty(\mathbb{R} \times (0,\infty))$ is called an entropy solution of (SCL) if for all entropy-entropy flux pairs (η, q), we have

$$\partial_t \eta(u) + \partial_x q(u) \leq 0 \quad \text{in} \quad \mathcal{D}'_+(\mathbb{R} \times [0,\infty)).$$

Enough to take

$$\eta(u) = |u - k| \qquad \text{and} \qquad q(u) = \text{sign}(u-k)(F(u) - F(k))$$

for all $k \in \mathbb{R}$.

Uniqueness

Theorem (Kružkov, 1970)

Assume $u_0 \in L^\infty(\mathbb{R})$ *and* $F \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$. *Then there exists a unique entropy solution* $u \in L^\infty(\mathbb{R} \times (0,\infty)) \cap C([0,\infty); L^1_{\text{loc}}(\mathbb{R}))$. *Moreover,*

(a) *for all* $R > 0$ *and* $L_F := \operatorname{ess\,sup} |F'|$,

$$\int_{|x|<R} |u(x,t) - v(x,t)| \, \mathrm{d}x \leq \int_{|x|<R+L_F t} |u_0(x) - v_0(x)| \, \mathrm{d}x;$$

(b) *if, in addition,* $u_0 \in L^1(\mathbb{R})$ *(and hence* $u \in L^\infty(\mathbb{R} \times (0,\infty)) \cap C([0,\infty); L^1(\mathbb{R}))$*), then*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}.$$

Is it possible to get a pure L^1-theory for (SCL)?

- Mild solutions
- Renormalized solutions
- Kinetic solutions

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Microscopic level/Boltzmann equation

$$(\mathrm{B}_\varepsilon) \qquad \partial_t f_\varepsilon + \xi \partial_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon, f_\varepsilon)$$

Microscopic level/"Boltzmann equation"

$$(\mathrm{B}'_\varepsilon) \qquad \partial_t f_\varepsilon + F'(\xi)\partial_x f_\varepsilon = \frac{1}{\varepsilon}(\chi(\xi; u_\varepsilon) - f_\varepsilon)$$

where

$$u_\varepsilon(x,t) = \int_{\mathbb{R}} f_\varepsilon(x,t,\xi)\,\mathrm{d}\xi$$

and

$$\chi(\xi; u_\varepsilon) := \begin{cases} 1, & 0 < \xi < u_\varepsilon, \\ -1, & u_\varepsilon < \xi < 0, \\ 0, & (u_\varepsilon - \xi)\xi \leq 0. \end{cases}$$

Macroscopic level/Linear equation

$$(\mathrm{B}') \qquad \partial_t \chi(\xi; u) + F'(\xi)\partial_x \chi(\xi; u) = \partial_\xi m(x, t, \xi)$$

where

$$m \text{ is a nonnegative measure}$$

and

$$\lim_{\varepsilon \to 0^+} u_\varepsilon(x, t) =: u(x, t) = \int_{\mathbb{R}} \chi(\xi; u)\, \mathrm{d}\xi.$$

Entropy solutions revisited

Definition (2' Entropy solutions)

Assume $u_0 \in L^\infty(\mathbb{R})$ and $F \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$. A function $u \in L^\infty(\mathbb{R} \times (0,\infty))$ is called an entropy solution of (SCL) if for all entropy-entropy flux pairs (η, q), we have

$$\partial_t \eta(u) + \partial_x q(u) = -\int_{\mathbb{R}} \eta''(\xi) m \,\mathrm{d}\xi \; (\leq 0) \quad \text{in} \quad \mathcal{D}'_+(\mathbb{R} \times [0,\infty)).$$

Recall that every nonnegative distribution defines a nonnegative Radon measure.

Definition of kinetic solutions

Definition (3 Kinetic solutions)

Assume $u_0 \in L^1(\mathbb{R})$ and $F' \in L^\infty_{\text{loc}}(\mathbb{R})$. A function $f = f(x,t,\xi)$ in $L^\infty_t((0,\infty); L^1_{x,\xi}(\mathbb{R}^2))$ is called a generalized kinetic solution of (SCL) if

$$\begin{cases} \partial_t f + F'(\xi)\partial_x f = \partial_\xi m & \text{in} \quad \mathbb{R}\times(0,\infty) \\ f(x,0,\xi) = \chi(\xi; u_0(x)) & \text{on} \quad \mathbb{R} \end{cases}$$

holds in $\mathcal{D}'(\mathbb{R}\times[0,\infty)\times\mathbb{R})$ for some measure $m \geq 0$, and for some function $\mu(\xi)$ and some measure $\nu \geq 0$, we have

$$\int_0^\infty \int_{\mathbb{R}} m(\,\mathrm{d}x,\,\mathrm{d}t,\xi) \leq \mu(\xi) \in L^\infty_0(\mathbb{R}),$$

$$|f(x,t,\xi)| = \text{sign}(\xi) f(x,t,\xi) \leq 1, \quad \text{and} \quad \partial_\xi f = \delta(\xi) - \nu(x,t,\xi).$$

Remarks on the definition

- Equivalence of entropy and kinetic solutions.
  - For an $L^1 \cap L^\infty$ entropy solution u of (SCL), the function $f(x,t,\xi) = \chi(\xi; u(x,t))$ is a generalized kinetic solution, that is, u is a kinetic solution.
  - If the equation in the definition of kinetic solutions holds for $f(x,t,\xi) = \chi(\xi; u(x,t))$ with $u \in L^1 \cap L^\infty$, then u is an entropy solution of (SCL).
  - Kinetic solutions thus extends, in L^1, entropy solutions, and they coincide for bounded solutions.
- There exists a kinetic solution $u \in C([0,\infty); L^1(\mathbb{R}))$ of (SCL) under the assumptions $u_0 \in L^1, F' \in L^\infty_{\text{loc}}$.
- The properties of the function f in the definition of kinetic solutions are needed to properly characterize limits of sequences $\chi(\xi; u_\varepsilon(x,t))$.

Uniqueness

Theorem (Perthame, 2002)

Assume $u_0 \in L^1(\mathbb{R})$ *and* $F' \in L^\infty_{\text{loc}}(\mathbb{R})$. *Let* $f = f(x,t,\xi)$ *be a generalized kinetic solution of* (SCL). *Then we have*

(a) $f(x,t,\xi) = \chi(\xi; u(x,t))$ *a.e. and* $u(x,t)$ *is a kinetic solution of* (SCL);

(b) $f(x,t,\xi) \to \chi(\xi; u_0(x))$ *in* $L^1(\mathbb{R}^2_{x,\xi})$ *as* $t \to 0^+$; *and*

(c) $\|u(\cdot,t) - v(\cdot,t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}$.

Proof of uniqueness

$$\chi_u = \begin{cases} 1, & 0 < \xi < u \\ -1, & u < \xi < 0 \\ 0, & (u - \xi)\xi \leq 0 \end{cases}$$

$$\chi_u - \chi_v = \begin{cases} 1, & v < \xi < u \\ -1, & u < \xi < v \\ 0, & \text{otherwise} \end{cases}$$

Proof of uniqueness

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Proof of uniqueness

$$\chi_u = \begin{cases} 1, & 0 < \xi < u \\ -1, & u < \xi < 0 \\ 0, & (u-\xi)\xi \leq 0 \end{cases}$$

$$|\chi_u - \chi_v| = \mathbf{1}_{v<\xi<u} + \mathbf{1}_{u<\xi<v}$$

$$\implies \int_{\mathbb{R}} |\chi_u - \chi_v| \, \mathrm{d}\xi = |u - v|$$

Proof of uniqueness

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|u-v|\,\mathrm{d}x \leq 0 \iff \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\int_{\mathbb{R}}|\chi_u-\chi_v|\,\mathrm{d}\xi\,\mathrm{d}x \leq 0$$

$$\iff \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\int_{\mathbb{R}}|\chi_u|+|\chi_v|-2\chi_u\chi_v\,\mathrm{d}\xi\,\mathrm{d}x \leq 0$$